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Temperature Distribution in Manifold of Higher Dimension with Variable Boundary Conditions

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Abstract

In [1], solution of one dimensional heat conduction equation is obtained and its properties were discussed. In [2], solution of two dimensional heat conduction equation were obtained and its properties were discussed. In [3], generalized form of n-dimensional heat flow equation in metal manifold structure of n-dimensions with boundary conditions were discussed. In this paper, we will discuss about study and analysis of temperature distribution in manifold structure of higher dimension with periodic boundary conditions.

Keywords: Heat equations, Partial differential equation Manifold structure.

1. INTRODUCTION

By using partial differential equation and Fourier series the solution of one dimensional heat conduction equation and two dimensional heat conduction equation were obtained.

A **Fourier series** can be expressed in the form: $F(x) = a_o\backslash 2 + \sum a_n \cos nx + \sum b_n \sin nx$

where $a_{o,}$ a_n & b_n are constants and can be expressed as.

$a_o = 1/\pi\int_0^{2\pi}(f(x)\,dx,$ $a_n = 1/\pi\int_0^{2\pi} f(x)\cos nx\ dx$, $b_n = 1/\pi\int_0^{2\pi} f(x)\sin nx\ dx$.

In our next article ,we shall consider the flow of heat and the accompanying variation of temperature with position and with time in conducting solids the following laws are taken as the basis of investigation:

- Heat flows from higher to lower temperature.
- The amount of heat required to produce a given temperature change in a body is proportional to mass of body and to temperature change i.e

  ∆T ⋉ m, ∆T= cm

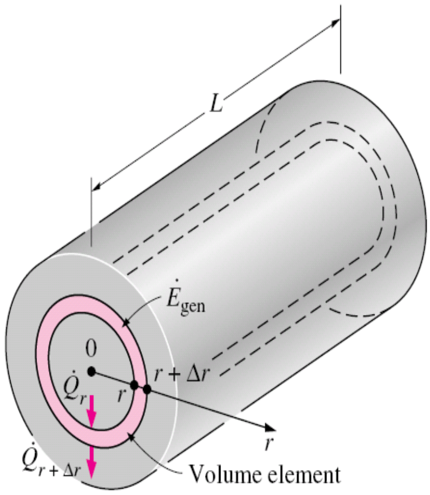
  Where c is called specific heat of material.

- The rate at which heat flows through an area is proportional to the area and to the temperature gradient normal to the area i.e

  ∆T ⋉ A, ∆T= kA

  Where k is called thermal conductivity of material.

Now, temperature distribution in one dimension is discussed below:



Consider a heat conducting homogenous rod of length L , constant density ℓ and with uniform cross section. Considering one end as origin the equation governing distribution is:

$$\frac{1}{c^2}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{1.1}$$

Where $c^2 = K/_{s\rho}$, K is called thermal conductivity of the material, s is specific heat and ρ is density of the material of the rod . It is assumed that loss of heat from the sides by conduction is negligible . The possible solutions of (1.1) by method of separation of variables are given by

$$u(x,t) = (c_1\, e^{kx} + \; c_2\, e^{-kx})\; c_3\, e^{c^2k^2t} \tag{1.2}$$

$$u(x,t) = (c_1 \; \cos kx + c_2 \; \sin kx)\; c_3\, e^{-c^2k^2t} \tag{1.3}$$

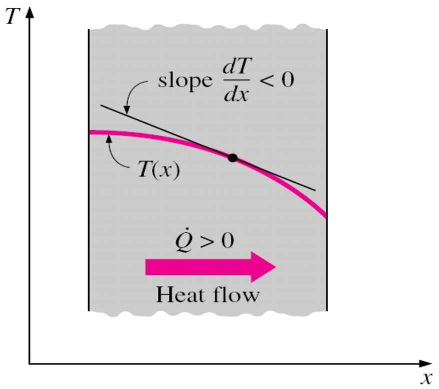
$$u(x,t) = (c_1\, x + \; c_2\,)\; c_3 \tag{1.4}$$

$$\frac{1}{c^2}\frac{\partial u}{\partial t} = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{2.1}$$

Out of above three possible solutions ,We choose that solution which is consistent with the physical nature of the problem . Since according to property of temperature distribution, temperature decreases as time t increases .The best possible solution, suitable for temperature distribution is (1.3) .

2. Now consider two dimensional heat flow in a metal plate. The equation governing the distribution is

Where again $c^2 = {}^{K}\!/_{s\rho}$ .Above equation is transient state equation .



Consider boundary conditions, temperature is zero at edges of the plate (with length & breath a and b respectively) i.e

$u(x,y,t) = 0$ at $x = 0$, $u(x,y,t) = 0$ at $x = a$, $u(x,y,t) = 0$ at $y = 0$

$u(x,y,t) = 0$ at $y = b$ and $u(x,y,t) = g(x,y)$ at $t = 0$.

Solution of (2.1) by method of separation of variable is given by $u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{n\pi x}{a} \; \sin\frac{m\pi y}{b}\, e^{-c^2k_{mn}^2 t}$. (2.2)

Where $k_{mn}^2 = \pi^2\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right)$. This solution can be referred as most general solution . At t=0 ,

$u(x,y,0) = g(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{n\pi x}{a}\ \sin\frac{m\pi y}{b}$. A_{mn} can be found using double Fourier series . Where $A_{mn} = 2\backslash a\ 2\backslash b \int\int sin\ n\pi x\backslash a\ sin\ m\pi y\backslash b\ f(xy)\ dx\ dy$.

3. Before moving towards the generalized form of n-dimensional heat flow equation in metal manifold structure of n-dimensions. Let us first define **Manifold.**

We know that a **smooth manifold** is a subset of Euclidean space which is locally the graph of a smooth curve (perhaps vector- valued function) . A more general topological manifold can be described as a topological space that on a small enough scale resembles the Euclidean space of a specific dimension, called the dimension of the manifold. Examples of one - dimensional manifolds thus are a line, and a circle . Examples of two - dimensional manifolds are a plane and sphere (the surface of a ball) , and so on an high dimensional space . However, the sphere differs from the plane "in the large" .

Now consider n-dimensional heat flow equation in metal manifold structure of n-dimensions with boundary conditions

$$u\ (x_1, x_2, x_3 \ldots\ldots\ldots\ldots x_n\ , t) = 0 \ \text{ at } \ x_1 = 0$$
$$u\ (x_1, x_2, x_3 \ldots\ldots\ldots\ldots x_n\ , t) = 0 \ \text{ at } \ x_2 = a_1$$
$$u\ (x_1, x_2, x_3 \ldots\ldots\ldots\ldots x_n\ , t) = 0 \ \text{ at } \ x_3 = a_2 \ldots\ldots\ldots\ldots$$
$$u\ (x_1, x_2, x_3 \ldots\ldots\ldots\ldots x_n\ , t) = 0 \ \text{ at } \ x_n = a_n \quad .$$

And $u\ (x_1, x_2, x_3 \ldots\ldots\ldots\ldots x_n) = G(x_1, x_2, x_3 \ldots\ldots\ldots\ldots x_n)$ at $t = 0$. Where G is initial temperature function. The equation governing the temperature distribution in such structure is given by: $\frac{1}{c^2}\frac{\partial u}{\partial t} = \left(\frac{\partial^2 u}{\partial {x_1}^2} + \frac{\partial^2 u}{\partial {x_2}^2} + \frac{\partial^2 u}{\partial {x_3}^2} + \cdots \frac{\partial^2 u}{\partial {x_n}^2}\right)$ (3.1)

Where again $c^2 = K/_{s\rho}$. Solution of (3.1) can be given as:

$$u\ (x_1, x_2, x_3 \ldots\ldots\ldots\ldots x_n\ , t) = \sum_{n_{1=1}}^{\infty}\sum_{n_{2=1}}^{\infty} \ldots\ldots\ldots \sum_{n_{n=1}}^{\infty} A_{n_{1}n_{2}\ldots\ldots\ldots n_n}$$

$$sin\ \frac{n_1 \pi x_1}{a_1}\ sin\frac{n_2 \pi x_2}{a_2}\ sin\frac{n_3 \pi x_3}{a_3} \ldots\ldots\ldots sin\frac{n_n \pi x_n}{a_n}\, e^{-c^2 k^2_{n_1 n_2 \ldots\ldots} t}$$

Where $k^2_{n_1 n_2 \ldots\ldots n_n} = \pi^2\left(\frac{{n_1}^2}{{a_1}^2} + \frac{{n_2}^2}{{a_2}^2} + \cdots \frac{n_n^2}{a_n^2}\right)$. Equation (3.2) gives temperature distribution in n-dimensional manifold. Similarly, we can solve for any one dimensional and two dimensional structure. Now consider a rectangular plate summated from legs with variable temperature at ends. Temperature at short edges has been taken into consideration as solution of one dimensional heat conduction equation. Since the surface of plate is two dimensional therefore, In steady state, the temperature u (x , y) at any point P (x , y)

Satisfy the equation $$\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0 \qquad (1)$$

and transient state is governed by $$\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 1/c^2 \frac{\partial u}{\partial t} \qquad (2)$$

Now three possible solutions of (1) are:

$$u = (C_1 e^{px} + C_2 e^{-px})(C_3 \cos py + C_4 \sin py) \qquad (3)$$

$$u = (C_5 \cos px + C_6 \sin px)(C_7 e^{py} + C_8 e^{-py}) \qquad (4)$$

$$u = (C_9 x + C_{10})(C_{11} y + C_{12}) \qquad (5)$$

of these, we have to choose that solution which is consistent with the physical nature of the problem. The solution (3) & (5) cannot satisfy the condition (1) & (2). Thus, only possible solution is (4) i.e, of the form.

$$u(x, y) = (C_1 \cos px + C_2 \sin px)(C_3 e^{py} + C_4 e^{-py}) \qquad (6)$$

consider the boundary condition for the structure as :

$$\{ u(x_1, y_1) = k_1, \ u(x_2, y_2) = k_2, \ u(x_3, y_3) = k_3, \ u(x_4, y_4) = k_4 \} \qquad (7)$$

therefore,

$$\left.\begin{aligned} (C_1 \cos px_1 + C_2 \sin px_1)(C_3 e^{py_1} + C_4 e^{-py_1}) &= k_1 \\ (C_1 \cos px_2 + C_2 \sin px_2)(C_3 e^{py_2} + C_4 e^{-py_2}) &= k_2 \\ (C_1 \cos px_3 + C_2 \sin px_3)(C_3 e^{py_3} + C_4 e^{-py_3}) &= k_3 \\ (C_1 \cos px_4 + C_2 \sin px_4)(C_3 e^{py_4} + C_4 e^{-py_4}) &= k_4 \end{aligned}\right\} \ldots\ldots\ldots \quad (8)$$

The system of equation (8) can be solved by matrix method :

$$\begin{pmatrix} \mathrm{Cos}px_1 & \mathrm{Sin}px_1 \\ \mathrm{Cos}px_2 & \mathrm{Sin}px_2 \\ \mathrm{Cos}px_3 & \mathrm{Sin}px_3 \\ \mathrm{Cos}px_4 & \mathrm{Sin}px_4 \end{pmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} \qquad \ldots\ldots(9)$$

With, $C_3 e^{py_1} + C_4 e^{-py_1} = 1$, $C_3 e^{py_2} + C_4 e^{-py_2} = 1$ ….. (10)

$C_3 e^{py_3} + C_4 e^{-py_3} = 1$, $C_3 e^{py_4} + C_4 e^{-py_4} = 1$

System of equation (9) has possible solution :

$C_3\, e^{py_1} = 0$, $C_4\, e^{-py_1} = 0$ $C_3 = 0$, $C_4 = e^{py_1}$,

Or, $e^{py_1} = 0$ (absurd result) .Therefore, $C_3 = 0$, Therefore, (6) reduces to,

$$u(x, y) = (C_1 \cos px + C_2 \sin px)(C_4\, e^{-py})$$

solution of system of equation of equation (9) has infinite no. of solutions. But if we fix $x_1 = 0$, $x_2 = a$, $x_3 = x$, $x_4 = k$ (constant)

we have, $C_1 = 0$, $p = \frac{n\pi}{a}$, $C_3 = km\pi(k_1 + k_2 + k_3 + k_4) / a$

Equation (6) takes the form

$u(x, y) = \sum_{n=1}^{\infty} bn \sin px\; e^{-py}$ where, $b_n = C_2 C_4$

Value of b_n can be obtained by fourier series,

similarly solution of time dependent heat equation (2) can be written as ,

$u(x, y, t) = (A \cos k_1 x + B \sin k_1 x)(C \cos k_2 y + D \sin k_2 y).\, \varepsilon e^{-c2kt}$

having different boundary condition,

$k_1 = n\pi / x_1$, $k_2 = m\pi/x_2$, $k_3 = (k_1 + k_2)\, n\pi/ x_3$, $k_4 = (k_1 + k_2 + k_3 + k_4)\, m\pi/ x_4$ and,

$u(x, y, t) = A_{mn} \sin n\pi x/x_1 \sin m\pi y/x_2 \sin n\pi x/y_1 \sin m\pi y/y_2\, e(-c^2k^2t)$

where, $k^2 = k_1^2 + k_2^2 + k_3^2 + k_4^2$.

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