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Common fixed theorems satisfying (CLR_{ST}) property in b-metric spaces

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Abstract

In this paper our aim is to prove certain common fixed theorems for four self maps satisfying (CLR_{ST}) property in complete b-metric spaces. Some common fixed point theorems are generalized and unified from our results.

Keywords: b-metric space, (CLR) property, fixed point.

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1. INTRODUCTION AND PRELIMINARIES:

The concept of b−metric spaces was introduced by Bakht [7] in 1989, who used it to prove a generalization of the Banach contraction principle in spaces endowed with such kind of metrics. Since then, this notion has been used by

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many authors to obtain various fixed point theorems. Aydi et al. [4] proved common fixed point results for single valued and multi-valued mappings satisfying a weak $\phi-$contraction in b$-$metric spaces. Roshan et al. [23] used the notion of almost generalized contractive mappings in ordered complete b$-$metric spaces and established some fixed and common fixed point results. Pacurar [19] proved the existence and uniqueness of fixed points of $\phi-$contractions on b-metric spaces. Hussain and Shahin [16] introduced the notion of a cone b$-$metric space, generalizing both notions of b$-$metric spaces and cone metric spaces. Fixed point theorems of contractive mappings in cone b$-$metric spaces without the assumption of the normality of a corresponding cone are proved by Huang and Xu in[17]. Hussain [14] introduced partially ordered b$-$metric space. After that, several interesting results about the existence of a fixed point for single-valued and multi-valued operators in b$-$metric spaces have been obtained ([2], [9–13], [15], [18], [20-23]).

Definition 1.1[24] Let X be a non-empty set and $k \geq 1$ a given real number. A function $d : X \times X \to \mathbf{R}^+$ is a b$-$metric iff for each x, y, z $\in$ X, following conditions are satisfied:

(b1) $d(x, y) = 0$ iff $x = y$,

(b2) $d(x, y) = d(y, x)$,

(b3) $d(x, z) \leq k\, [d(x, y) + d(y, z)]$.

A pair (X, d) is called a b$-$metric space.

It should be noted that the class of b$-$metric spaces is effectively larger than that of metric spaces. Indeed, a b-metric is a metric if and only if $k = 1$.

Example 1.2 [24] Let (X, d) be a metric space and $\rho(x, y) = (d(x, y)^p$ where p>1 is a real number. We show that ρ is a b$-$metric with $k = 2^{p-1}$. Obviously, conditions (b1) and (b2) of definition 1.1 are satisfied. If $1 < p < \infty$, then convexity of the function $f(x) = x^p$ $(x > 0)$ implies that $\left(\frac{a+b}{2}\right)^p \leq \frac{a^p}{2} + \frac{b^p}{2}$ that is, $(a + b)^p \leq 2^{p-1}(a^p+b^p)$ holds.

Thus for each x, y, z ∈ X, we have

$$\rho(x, y) = (d(x, y))^p \leq (d(x, z) + d(z, y))^p$$

$$\leq 2^{p-1}((d(x, z))^p + (d(z, y))^p) = 2^{p-1}(\rho(x, z) + \rho(z, y)).$$

So condition (b3) of definition 1.1 holds and ρ is a b−metric. Note that (X, ρ) is not necessarily a metric space.

For example, if X = R be the set of real numbers and d(x, y) = |x − y| a usual metric, then $\rho(x, y) = (x - y)^2$ is a b−metric on R with k = 2, but not a metric on R, as the triangle inequality for a metric does not hold.

Before stating our results, we present some definitions and propositions in a b−metric space.

Definition 1.3[25] Let (X, d) be a b−metric space. Then a sequence $\{x_n\}$ in X is called:

(a) convergent if and only if there exists x ∈ X such that $d(x_n, x) \to 0$, as

$n \to +\infty$. In this case, we write $\lim_{n\to\infty} x_n = x$

(b) Cauchy if and only if $d(x_n, x_m) \to 0$, as $n, m \to +\infty$.

Proposition 1.4[25] In a b−metric space (X, d) the following assertions hold:

(i) a convergent sequence has a unique limit.

(ii) each convergent sequence is Cauchy,

(iii) in general, a b-metric is not continuous.

Definition 1.5[25] Let (X, d) be a b−metric space. If Y is an on empty subset of X, then the closure $\bar{Y}$ of Y is the set of limits of all convergent sequences of points in Y, i.e., $\bar{Y} = \{x \in X\}$ there exists a sequence $\{x_n\}$ in Y such that $\lim_{n\to\infty} x_n = x$.

Definition 1.6[25] Let (X, d) be a b−metric space. Then a subset Y ⊂X is called closed if and only if for each sequence $\{x_n\}$ in Y which converges to an element x, we have x ∈ Y (i.e. $\bar{Y} = Y$).

Definition 1.7[25] The b−metric space (X, d) is complete if every Cauchy sequence in X converges.

In general a b−metric function d for k>1 is not jointly continuous in all of its two variables. Following is an example of a b−metric which is not continuous.

Example 1.8[26] Let X = N∪{∞} and D : X × X → R defined by:

$$D(m,n)=\begin{cases}0 & \text{if } m=n\\ \left|\frac{1}{m}-\frac{1}{n}\right| & \text{if } m,n \text{ are even or } mn=\infty\\ 5 & \text{if } m,n \text{ are odd and } m\neq n\\ 2 & \text{otherwise}\end{cases}$$

Then it is easy to see that for all $m,n,p\in X$,we have

$$D(m,p)\leq 3(D(m,n)+D(n,p))$$

Thus, (X, D) is a b- metric space with k = 3. If $x_n = 2n$, for each $n\in \mathrm{N}$, the

$$D(2n,\infty)=\frac{1}{2n}\to 0 \; as\; n\to\infty$$

i.e. , $x_n \to\infty$,but $D(x_{2n},1)=2\to D(\infty,1), as\, n\to\infty$

as b-metric is not continuous in general , so we need the following simple Lemma about the b-convergent sequences.

Lemma 1.9 [27] Let (X, d) be a b- metric space with $k\geq 1$.Suppose that $\{x_n\} and \{y_n\}$ are b-convergent to x and y, respectively. Then we have,

$$\frac{1}{k^2}d(x,y)\leq \liminf_{n\to\infty} d(x_n,y_n)\leq \limsup_{n\to\infty} d(x_n,y_n)\leq k^2 d(x,y).$$

In particular, if x = y, then we have, $\lim_{n\to\infty} d(x_n,y_n)=0$. Moreover for each $z\in X$ we have,

$$\frac{1}{k}d(x,z)\leq \liminf_{n\to\infty} d(x_n,z)\leq \limsup_{n\to\infty} d(x_n,z)\leq kd(x,z).$$

Lemma 1.10 Let (X, d) is a b-metric space. If there exists two sequences $\{x_n\} and \{y_n\}$ such that $\lim_{n\to\infty} d(x_n, y_n) = 0$, whenever $\{x_n\}$is a sequence in X such that $\lim_{n\to\infty} x_n = t$ for some $t \in X$ then $\lim_{n\to\infty} y_n = t.$

Proof: By a triangle inequality in b-metric space, we have

$$d(y_n, t) \le k(d(y_n, x_n) + d(x_n, t))$$

Now by taking the upper limit when $n \to \infty$ in the above inequality we get,

$$\limsup_{n\to\infty} d(y_n, t) \le k(\limsup_{n\to\infty} d(x_n, y_n) + \limsup_{n\to\infty} d(x_n, t) = 0$$

Definition 1.11 [28] Let (X, d) be a b-metric space. A pair $\{f, g\}$is said to be compatible if and only if $\lim_{n\to\infty} d(f\, gx_n, g\, fx_n) = 0$, whenever $\{x_n\}$is a sequence in X such that :

$$\lim_{n\to\infty} f\, x_n = \lim_{n\to\infty} g\, x_n = t \text{ for some}, \ t \in X.$$

2. MAIN RESULTS

In this section, we prove some new common fixed point theorems for pair of mappings satisfying (CLR) property in b-metric spaces. The class of b-metric spaces is more general than metric spaces, since every metric space is a b-metric space but converse need not be true. Our results generalize and unify many existing results in the literature in b-metric spaces.

We need the following definition in our main results:

Definition 2.1. Two self maps f and g are said to satisfy b- $(CLRg)$ property if there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} f\{x_n\} = \lim_{n\to\infty} g\{x_n\} = g(x) \text{ for some } x \in X .$$

The following Lemma will be used in the proof of our Main Theorem. In this Lemma, we show that two pairs of self-maps on X satisfy (CLR) property.

Lemma 2.2. Let f, g, S, T be self-mappings of a b-metric space (X, d). Suppose that

(i) The pairs (f, S) and (g, T) satisfies the (CLR_S) and (CLR_T) properties respectively.

(ii) $f\,X \subset T\,X \; and \; g\,X \subset S\,X$,

(iii) SX and TX are closed in X,

(iv) $\{gy_n\}$ Converges for each sequence $\{y_n\}$ in X whenever $\{Ty_n\}$ Converges (respectively fx_n Converges for every sequence $\{x_n\}$ in X whenever $\{Sx_n\}$ converges)

(v) there exits $\varphi \in \phi$ and $\psi \in \Psi$

such that

$$\psi(b^2\, d[fx, gy]) \le \psi[M_b(x,y)] - \phi[M_b(x,y)] \text{..............................(1)}$$

$for\ all\ x, y \in X, where$

$$M_b(x,y) = \max\{\, d\,(Sx,Ty), d\,(fx,Sx), d\,(gy,Ty), \frac{d(fx,Ty) + d(Sx,gy)}{2}\}\,.$$

Then, the pairs (f , S) and (g ,T) share the CLR_{ST} property.

Proof:- If the pair (f ,S) satisfy the (CLRs) property, there exists a sequence $\{x_n\}$ in X such that

$$\lim f\,x_n = \lim S^2 n = p \; where\; p \in SX\,.$$

Now , since $fX \subset TX$, so for each sequence $\{x_n\}$, there exists a sequence $\{y_n\}$ in X such that $f\,x_n = T\,y_n$. But TX is closed, so $\lim Ty_n = \lim fx_n = p$.

So that $p \in TX$ and in all $p \in SX \cap TX$. Thus, we get $f\,x_n \to p$, $S\,x_n \to p$, $T\,y_n \to p$, as $n \to \infty$. Let us show that $g\,y_n \to p$ as $n \to \infty$. On the contrary suppose that $g\,y_n \to q (\ne p)$ as $n \to \infty$.

Putting $x = x_n$ and $y = y_n$ in (1), we get:

$$\psi[d(fx_n, gy_n)] \le \psi(b^2 d(fx_n, gy_n)$$
$$\le \psi[M_b(x_n, y_n) - \phi(M_b(x_n, y_n)]$$
$$\le \psi(M_b(x_n, y_n)).................................(2)$$

where

$$M_b(q, y_n)$$
$$= \lim_{n\to\infty}\left[max\left\{d(Sx_n, Ty_n), d(fx_n, Sx_n), d(Ty_n, gy_n), \frac{d(fx_n, Ty_n) + d(Sx_n, gy_n)}{2b}\right\}\right]$$

$$= \max\left\{0, 0, \frac{d(q,p)}{2b}\right\}$$

$$= d(p, q).$$

$$\psi(d(p,q)) \le \psi(b^2 d(p,q))$$
$$\le \psi(\lim_{n\to\infty} M_b(x_n, y_n))$$
$$\psi(b^2 d(p,q)) < \psi(d(p,q)).$$

From the definition of ψ we get:

$b^2 d(p,q) \le d(p,q)$.

Hence, $d(p,q) = 0$ implies that $q = p$, a contradiction. Hence, $q \to p$ which shows that the pairs (f, S) and (g, T) share the CLR_{ST} property which completes the proof.

Using the above lemma, in the following Theorem, we show thw existence of unique Common fixed point.

Theorem 2.3. Let f, g S and T self-mappings of a b-metric space(X ,d) satisfying (1). If the pairs (f, S) and (g, T) have a point of coincidence. Moreover if (f, S) and (g, T) are weakly compatible the f, g, S and T have a unique Common fixed point.

Proof. Since the pairs (f , S) and (g , T) satisfying the (CLR_{ST}) property, so there exists sequence {x_n} and {y_n} in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Ty_n = p,$$

where $p \in S \times \cap T \times$ Since $SX \in TX$ so $\exists$ a point $q \in X$ Such that Sq=p We show that fq=Sq.

Putting x=q , y=y_n in (1), we get

$$\psi(b^2 d(fq, gy_n)) \leq \psi(M_b(q, y_n)) - \psi(M_b(q, y_n)) \quad \dots\dots\dots \text{(3)}$$

where

$$M_b(q, y_n) = max\left\{d(Sq, Ty_n), d(fq, Sq), d(Ty_n, gy_n), \frac{d(fq, Ty_n) + d(Sq, gy_n)}{2b}\right\}$$

$$= max\left\{d(p, Ty_n), d(fq, Sq), d(Ty_n, gy_n), \frac{d(fq, Ty_n) + d(p, gy_n)}{2b}\right\}.$$

Making n $\to \infty$, we get

$$\lim_{n\to\infty} M_b(q, y_n) = max\left\{d(p,p), d(fq,p), d(p,p), \frac{d(fq,p) + d(p,p)}{2b}\right\} = d(fq,p).$$

Taking limit as n $\to \infty$ in (3) and using the definition of φ,we get $\{\psi(d(fq,p)) \leq \psi(b^2 d(fq,p) \leq \psi(d(fq,p) - \emptyset(d(fq,p)\}$, implies that,$\emptyset(d(fq,p) = 0$. Hence,$(fq = p = sq)$.Therefore, q is the point of coincidence of the pair(f,s).As $p \in TX$, there exists a point $r \in X$ such that $Tr = p$. We assert that $gr = Tr$.

Putting $x = q$ and $y = r$ in eq.(1) we get

$$(\psi(b^2 d(fq, gr) \leq \psi(M_b(q,r) - \emptyset(M_b(q,r)) \quad \dots\dots\dots \text{(4)}$$

where $M_b(q,r) = max\left\{d(Sq,Tr), d(fq,Sq), d(gr,Tr), \frac{d(fq,Tr)+d(Sq,gr)}{2b}\right\}$

$$= max\left\{d(p,p), d(p,p), d(gr,p), \frac{d(p,p) + d(p,gr)}{2b}\right\}$$

$$= d(gr,p).$$

Thus, from (4), and using definition of φ,we get

$$\psi(d(p,gr) \leq \psi\big(b^2 d(fq,qr)\big) = \psi(b^2 d(p,gr) \leq \psi(d(gr,p) - \emptyset(d(gr,p))$$

which gives that,

$\emptyset\big(d(gr,p)\big) = 0, i.e. gr = p = Tr.$

Hence r is the point of coincidence of the pair (g,T).

Since the pair $(f, S)$is weakly compatible and $fq = Sq$. Therefore, $fp = fSq = Sfq = Sp$. Now we show that p is a common fixed point of the pair (f,S).

Putting $x = q, y = r \; in \; eq.(1)$, we get

$$\psi(b^2 d(fp,gr)) \leq \psi(M_b(p,r) - \emptyset(M_b(p,r)) \quad \text{..........(5)}$$

where $M_b(p,r) = max\left\{d(Sp,Tr), d(fp,Sp), d(gr,Tr), \frac{d(fp,Tr)+d(Sp,gr)}{2b}\right\}$

$$= max\left\{d(fp,p), 0,0, \frac{d(fp,p)+d(fp,p)}{2b}\right\}$$

$$= d(fp,p).$$

From (5) and using the property of φ we get,

$$\psi(d(fp,p) \leq \psi(b^2 d(fp,p)) \leq \psi(d(fp,p) - \emptyset(d(fp,p),$$

which gives that,$\emptyset(fp,p) = 0, i.e. fp = p = Sp,$

which shows that p is a common fixed point of the pair (f,s).

Similarly, one can show that $gp = Tp = p$.Hence p is a common fixed point of $f, g, S \; and \; T$.

To prove that p is the unique common fixed point, let t be another common fixed point of $f, g, Sand \; T$.

Using (1), $x = p, y = t, we\ get$

$$\{\psi\big(d(p,t)\big) = \psi(d(fp,gt)) \leq \psi(b^2 d(fp,gt))\} \quad \ldots \ldots .(6)$$

$$\leq \psi\big(M_b(p,t)\big) - \emptyset\big(M_b(p,t)\big),$$

where

$$M_b(p,t) = max\left\{d(Sp,Tt), d(fp,Sp), d(gt,Tt), \frac{d(fp,Tt) + d(Sp,gt)}{2}\right\}$$

$$= max\{d(p,t), 0,0, d(p,t)\}$$

$$= d(p,t)\,.$$

Thus, from equation (6),we get

$$\psi\big(d(p,t)\big) \leq \psi\big(d(p,t)\big) - \emptyset\big(d(p,t)\big), implies\ that\ \emptyset\big(d(p,t)\big) = 0\ i.e. p = t$$

Hence, p is the unique common fixed point of $f, g, S\ and\ T$.

Corollary 2.4. Let f, T be self-mapping of a b-metric space satisfying :

$$\psi(b^2 d(fx,fy) \leq \psi\big(M_b(x,y)\big) - \emptyset\big(M_b(x,y)\big)\ \ for\ all\ x, y$$

where $M_b(x,y) = max\left\{d(x,Ty), d(fx,Tx), (fy,Ty), \frac{d(fx,Ty)+d(Tx,fy)}{2b}\right\}$.

If the pairs $(f,\ T)$ satisfies (CLR_T) property then, $f\ and\ T$ have a common point of coincidence. Moreover if $f\ and\ T$ are weakly compatible, then the pair (f,T) has a unique common fixed point.

Corollary 2.5. Let f, g , S , and T be self - mapping of a b-metric space (X, d) satisfying for all x , y in X.

Where

$$d(fx,gy) \leq M_s(x,y) - \emptyset(M_s(x,y),$$

$$M_b(x,y) = max\left\{d(Sx,Ty), d(fx,Sx), d(gy,Ty), \frac{d(fx,Ty) + d(Sx,gy)}{2b}\right\}.$$

If the pairs (f, S) and (g, T) satisfy the CLR_{ST} property, then (f, S) and (g, T) have a point of coincidence. Moreover, if (f, S) and (g, T) are compatible, then $f, g, S\ and\ T$ have a unique Common fixed point.

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