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Nonlinear Functional Delay Differential Equations Arising from Population Models

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Abstract

In this research, our aim is to use a new variation of parameters formula to analyze the behavior of the purely nonlinear functional delay differential equation that arise from population models

$$x'(t) = g(x(t)) - g(x(t-L)).$$

Our approach will be based on the use of fixed point theory, by constructing suitable mapping on appropriate spaces.

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1 Introduction

In [5], the authors Cooks and Yoke developed biological growth and epidemic models based on the general model of nonlinear functional delay differential equation of the form

$$x'(t) = g(x(t)) - g(x(t-L)), \tag{1.1}$$

where $g : \mathbb{R} \to \mathbb{R}$ and is continuous in x, and the constant delay L is positive, with $\mathbb{R}$ denoting the set of all real numbers. Eqn. (1.1) can be interpreted as follow. Suppose $x(t)$ is the number of individuals in a population at time t. Let the delay L be the life span of each individual. Then the birth rate of the population is some function of $x(t)$, say $g(x(t))$ and the function $g(x(t-L))$ can be thought of as the number of deaths per

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unit time at time t. Then the difference term $g(x(t)) - g(x(t-L))$ represents the net change in population per unit time. This implies that the growth of the population is governed by (1.1).

In [5], the authors used the Lyapunov functional

$$V(x) = -2\int_0^x g(s)ds + \int_{t-L}^t g^2(x(s))ds,$$

to find the maximal interval of existence of the solutions. Recently, in the paper [1], Burton utilized the notion of fixed point arguments and relaxed some of the conditions that Cook and Yorke were faced with, as the result of using Lyapunov functionals. In addition, in [1] the author noticed that every constant is a solution of (1.1) and utilized this idea to show that as $t \to \infty$ solutions approach a pre-determined constant. Even though every constant c is a solution (1.1), Cook and York claim that to have a correct biological interpretation we must have $c = 0$. For this reason we will only address the stability of its zero solution. Moreover, we will show that every solution approaches zero as t approaches infinity, when $g(0) \neq 0$. For more on the studies of stability and periodicity in discrete systems, we refer the reader to [7, 8, 10]. For comprehensive reference on functional difference equations, we refer to the book [9]. This paper is motivated by the papers of [1]– [4].

In addition to considering (1.1), we analyze the existence of a unique periodic solution of the functional model

$$x'(t) = g(t, x(t)) - g(t, x(t-L)), \quad g(t+L, x) = g(t, x) \tag{1.2}$$

We note that every constant is a solution of equations (1.1) and (1.2).
Let $\mathcal{C}$ denote the set of all continuous functions $\phi : [-L, \infty) \to \mathbb{R}$. For $x \in \mathcal{C}$, we say $x(t) := x(t, 0, \psi)$ where $\psi : [-L, 0] \to \mathbb{R}$ is a given continuous and bounded initial function is a solution of (1.1) if $x(t, 0, \psi) = \psi(t)$ on $[-L, 0]$ and $x(t, 0, \psi)$ satisfies (1.1) for $t \geq 0$.

In [6, 9], the author considered discrete variation forms of (1.1) and (1.2) and analyzed the stability of the pre-determined constant solution. In this work, our study is totally different in the analysis that we consider and in the formulation of new variation of parameters.

2 Stability and Boundedness

In this section we qualitatively study the boundedness of solutions and the stability of the zero solution of (1.1). However, our work relies on a new variation of parameters formula and the application of the contraction mapping principle. We follow the same procedure of inversion as in [2]. We begin with the following lemma, in which we develop a new variation of parameters formula.

Lemma 2.1. *Suppose $\psi : [-L, 0] \to \mathbb{R}$ is a given continuous and bounded initial function. If $x(t)$ is a solution of* (1.1) *on an interval $[0, T]$ and satisfies $x(t) = \psi(t)$ for $t \in [-L, 0]$, then $x(t)$ is a solution of the integral equation*

$$\begin{aligned} x(t) &= \Big(\psi(0) - \int_{-L}^{0} v(s)\psi(s)ds\Big)e^{-\int_0^t v(s)ds} - e^{-\int_0^t v(s)ds}\int_{-L}^{0} g(\psi(s))ds \\ &+ \int_{t-L}^{t} v(s)x(s)ds - \int_0^t \int_{u-L}^{u} g(x(s))dsv(u)e^{-\int_u^t v(s)ds}du \\ &+ \int_{t-L}^{t} g(x(s))ds - \int_0^t e^{-\int_u^t v(s)ds}v(u-L)x(u-L)du. \end{aligned} \tag{2.1}$$

on $[0, T]$, where $v : [0, \infty) \to \mathbb{R}^+$. Conversely, if $x(t)$ is continuous and satisfies (2.1) *on an interval $[0, \gamma]$ such that $x(t) = \psi(t)$ for $t \in [-L, 0]$ then $x(t)$ is a solution of* (1.1).

Proof. First we rewrite (1.1) in the form

$$x'(t) = \frac{d}{dt}\int_{t-L}^{t} g(x(s))ds. \tag{2.2}$$

Multiply both sides of (2.2) by $e^{\int_0^t v(s)ds}$ and then integrate from 0 to t. That is

$$\int_0^t e^{\int_0^u v(s)ds}\, x'(u)du = \int_0^t \Big(\frac{d}{du}\int_{u-L}^{u} g(x(s))ds\Big)e^{\int_0^u v(s)ds}du.$$

Using integration by parts, the left hand side of the above expression is equal to

$$x(t)e^{\int_0^t v(s)ds} - \psi(0) - \int_0^t x(u)v(u)e^{\int_0^u v(s)ds}du.$$

As for the right hand side , we let $U = e^{\int_0^u v(s)ds}$, and $dV = \dfrac{d}{du}\displaystyle\int_{u-L}^{u} g(x(s))ds$, then

$$\begin{aligned} \int_0^t \Big(\frac{d}{du}\int_{u-L}^{u} g(x(s))ds\Big)e^{\int_0^u v(s)ds}du &= e^{\int_0^u v(s)ds}\int_{u-L}^{u} g(x(s))ds\Big|_{u=0}^{t} \\ &- \int_0^t \Big(\int_{u-L}^{u} g(x(s))ds\Big)v(u)e^{\int_0^u v(s)ds}du \\ &= e^{\int_0^t v(s)ds}\int_{t-L}^{t} g(x(s))ds - \int_{-L}^{0} g(x(s))ds \\ &- \int_0^t \Big(\int_{u-L}^{u} g(x(s))ds\Big)v(u)e^{\int_0^u v(s)ds}du. \end{aligned}$$

Setting the above two expressions equal to each others and then dividing by $\int_0^t e^{\int_0^t v(s)ds}$ leads to

$$\begin{aligned} x(t) &= \Big(\psi(0) - \int_{-L}^{0} g(\psi(s))ds\Big)e^{-\int_0^t v(s)ds} + \int_{t-L}^{t} g(x(s))ds \\ &+ \int_0^t e^{-\int_u^t v(s)ds} v(u)x(u)du \\ &- \int_0^t \int_{u-L}^{u} e^{-\int_u^t v(s)ds} v(u)g(x(s))dsdu. \end{aligned} \quad (2.3)$$

We pause here to make the important observation that using (2.3) to define a suitable mapping to obtain results concerning boundedness or stability is not very helpful when the delay depends on time. To be precise, we let K be a positive constant and define the set

$$\mathcal{M}_b = \{\phi : [-L, \infty) \to \mathbb{R}/\phi(t) = \psi(t), t \in [-L, 0], \phi \in \mathcal{C}, |\phi(t)| \le K\}.$$

Use (2.3) to define a map $\mathcal{P} : \mathcal{M} \to \mathcal{M}$. Thus, for $\phi \in \mathcal{M}$ we have that

$$\begin{aligned} \Big|\int_0^t e^{-\int_u^t v(s)ds} v(u)x(u)du\Big| &\le K\int_0^t e^{-\int_u^t v(s)ds} v(u)du \\ &= Ke^{-\int_u^t v(s)ds}\big|_{u=0}^{t} \\ &= K\big(1 - e^{-\int_0^t v(s)ds}\big) \le K. \end{aligned}$$

This term alone will make it impossible for $\mathcal{P}$ to map $\mathcal{M}$ into itself, since more positive terms will be added to it. Now we go back to the proof of (2.1). We notice that

$$\begin{aligned} \int_0^t e^{-\int_u^t v(s)ds} v(u)x(u)du &= \int_0^t e^{-\int_u^t v(s)ds} d\Big(\int_{u-L}^{u} v(s)x(s)ds\Big) \\ &- \int_0^t e^{-\int_u^t v(s)ds} v(u-L)x(u-L)du. \end{aligned}$$

Performing an integration by parts on the first term on the right side we get,

$$\begin{aligned} &\int_0^t e^{-\int_u^t v(s)ds} d\Big(\int_{u-L}^{u} v(s)x(s)ds\Big) \\ &= e^{-\int_u^t v(s)ds}\int_{u-L}^{u} v(s)x(s)ds\Big|_{u=0}^{u=t} \\ &- \int_0^t v(u)e^{-\int_u^t v(s)ds}\int_{u-L}^{u} v(s)x(s)dsdu \\ &= \int_{t-L}^{t} v(s)x(s)ds - e^{-\int_0^t v(s)ds}\int_{-L}^{0} v(s)x(s)ds. \end{aligned}$$

A substitution into (2.3) leads to (2.1), which is a new variation of parameters formula. □

For our next theorem we assume the existence of a continuous initial function ψ and we define the set

$$\mathcal{M}_b = \{\phi : [-L, \infty) \to \mathbb{R}/\phi(t) = \psi(t), t \in [-L, 0], \phi \in \mathcal{C}, |\phi(t)| \leq 1\}.$$

Then $(\mathcal{M}_b, \rho)$ is a complete metric space under the metric

$$\rho(x, y) = \sup_{t \geq 0}\{|x(t) - y(t)|\}.$$

For the remainder of the paper we assume the function g is Lipschitz. That is, for x and $y \in \mathcal{M}_b$ there exists a constant $q > 0$ such that

$$|g(x) - g(y)| \leq q|x - y|. \tag{2.4}$$

Finally we let $L_1 = \max\{|g(x)| : x \in \mathcal{M}_b\}$.

Theorem 2.2. *Assume* (2.4) *and there exists a constant* $\alpha \in (0, 1)$ *such that*

$$\begin{aligned} LL_1 e^{-\int_0^t v(s)ds} &+ \int_{t-L}^t v(s)ds + L_1 L + \int_0^t e^{-\int_u^t v(s)ds} v(u - L)du \\ &+ L_1 L \int_0^t v(u)e^{-\int_u^t v(s)ds}du \leq \alpha, \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} qL + \int_{t-L}^t v(s)ds &+ qL \int_0^t v(u)e^{-\int_u^t v(s)ds}du \\ &+ \int_0^t e^{-\int_u^t v(s)ds} v(u - L)du \leq \alpha. \end{aligned} \tag{2.6}$$

If ψ *is a given continuous initial function which is sufficiently small, then there is a solution* $x(t, 0, \psi)$ *of* (1.1) *with* $|x(t, 0, \psi)| < 1,$ *for all* $t \geq 0$.

Proof. First we define a mapping $\mathcal{P} : \mathcal{M}_b \to \mathcal{M}_b$ using (2.3) so that for $\phi \in \mathcal{M}_b$ we have

$$(\mathcal{P}\phi)(t) = \psi(t), \quad -L \leq t \leq 0,$$

and for $t \geq 0$

$$\begin{aligned} (\mathcal{P}\phi)(t) &= \Big(\psi(0) - \int_{-L}^0 v(s)\psi(s)ds\Big)e^{-\int_0^t v(s)ds} - e^{-\int_0^t v(s)ds}\int_{-L}^0 g(\psi(s))ds \\ &+ \int_{t-L}^t v(s)\phi(s)ds - \int_0^t \int_{u-L}^u g(\phi(s))dsv(u)e^{-\int_u^t v(s)ds}du \end{aligned}$$

$$+ \int_{t-L}^{t} g(\phi(s))ds - \int_{0}^{t} e^{-\int_u^t v(s)ds} v(u-L)\phi(u-L)du. \tag{2.7}$$

Now for $\phi \in \mathcal{M}_b$, we have from (2.7) that

$$\begin{aligned}
|(\mathcal{P}\phi)(t)| &\leq ||\psi||\Big(1 + \int_{-L}^{0} v(s)ds\Big)e^{-\int_0^t v(s)ds} + e^{-\int_0^t v(s)ds}\int_{-L}^{0} |g(\psi(s))|ds \\
&+ \int_{t-L}^{t} v(s)|\phi(s)|ds + \int_0^t \int_{u-L}^{u} \big|g(\phi(s))\big|ds v(u)e^{-\int_u^t v(s)ds}du \\
&+ \int_{t-L}^{t} |g(\phi(s))|ds + \int_0^t e^{-\int_u^t v(s)ds} v(u-L)|\phi(u-L)du \\
&\leq ||\psi||\Big(1 + \int_{-L}^{0} v(s)ds\Big)e^{-\int_0^t v(s)ds} + LL_1 e^{-\int_0^t v(s)ds} \\
&+ \int_{t-L}^{t} v(s)ds + L_1 L + \int_0^t e^{-\int_u^t v(s)ds} v(u-L)du \\
&+ L_1 L \int_0^t v(u)e^{-\int_u^t v(s)ds} \\
&\leq ||\psi||\Big(1 + \int_{-L}^{0} v(s)ds\Big)e^{-\int_0^t v(s)ds} + \alpha < 1
\end{aligned} \tag{2.8}$$

provided that $||\psi||$ is sufficiently small. Next, we show the mapping $\mathcal{P}$ defines a contraction. Let $\phi_1, \phi_2 \in \mathcal{M}_b$. Then, using (2.7) we arrive at

$$\begin{aligned}
\Big|(\mathcal{P}\phi_1)(t) - (\mathcal{P}\phi_2)(t)\Big| &\leq \Big|\int_{t-L}^{t} g(\phi_1(s))ds - \int_{t-L}^{t} g(\phi_2(s))ds\Big| \\
&+ \Big|\int_{t-L}^{t} v(s)\phi_1(s)ds - \int_{t-L}^{t} v(s)\phi_2(s)ds\Big| \\
&+ \Big|\int_0^t v(u)e^{-\int_u^t v(s)ds}\int_{u-L}^{u} g(\phi_1(s))dsdu \\
&- \int_0^t v(u)e^{-\int_u^t v(s)ds}\int_{u-L}^{u} g(\phi_2(s))dsdu\Big| \\
&+ \Big|\int_0^t e^{-\int_u^t v(s)ds} v(u-L)\phi_1(u-L)du \\
&- \int_0^t e^{-\int_u^t v(s)ds} v(u-L)\phi_2(u-L)du\Big|.
\end{aligned} \tag{2.9}$$

Next, we perform some calculations to simplify expression (2.9):

$$\Big|\int_{t-L}^{t} g(\phi_1(s))ds - \int_{t-L}^{t} g(\phi_2(s))ds\Big| \leq \int_{t-L}^{t} \big|g(\phi_1(s)) - g(\phi_2(s))\big|ds$$

$$\leq \quad qL||\phi_1 - \phi_2||.$$

Similarly,

$$\Big|\int_0^t v(s)\phi_1(u)ds - \int_0^t v(s)\phi_2(s)ds\Big| \leq \int_0^t v(s)ds||\phi_1 - \phi_2||.$$

Moreover,

$$\Big|\int_0^t v(u)e^{-\int_u^t v(s)ds}\int_{u-L}^u g(\phi_1(s))dsdu - \int_0^t v(u)e^{-\int_u^t v(s)ds}\int_{u-L}^u g(\phi_2(s))dsdu\Big|$$
$$\leq qL\int_0^t v(u)e^{-\int_u^t v(s)ds}du||\phi_1 - \phi_2||.$$

Finally,

$$\Big|\int_0^t e^{-\int_u^t v(s)ds}v(u-L)\phi_1(u-L)du - \int_0^t e^{-\int_u^t v(s)ds}v(u-L)\phi_2(u-L)du\Big|$$
$$\leq \int_0^t e^{-\int_u^t v(s)ds}v(u-L)du||\phi_1 - \phi_2||.$$

A substitution of the above four inequalities into expression (2.9) leads to

$$\Big|(\mathcal{P}\phi_1)(t) - (\mathcal{P}\phi_2)(t)\Big| \leq \alpha||\phi_1 - \phi_2||.$$

Thus by the contraction mapping principle, (2.7) has a unique solution that solves (1.1) and is bounded. □

Theorem 2.3. *Assume the hypothesis of Theorem 2.2. In addition, we assume the function* $v(t) \geq 0, 0 \leq t < \infty$, *and*

$$g(0) = 0. \tag{2.10}$$

Then the zero solution of (1.1) *is stable. Moreover, if we assume that*

$$\int_0^\infty v(s)ds = \infty, \tag{2.11}$$

then the zero solution of (1.1) *is asymptotically stable.*

Proof. Due to conditions (2.4) and (2.10) we have that for all $x \in \mathcal{M}_b$, $|g(x)| \leq q|x|$. Moreover, condition (2.6) automatically follows from (2.5) with $q = L_1$. Let J be a positive constant such that $e^{-\int_0^t v(s)ds} \leq J$. Let the mapping $\mathcal{P}$ be given by (2.7). Then we have that for all $x \in \mathcal{M}_b$,

$$|(\mathcal{P}x)(t)| \quad \leq \quad ||\psi||\Big(1 + \int_{-L}^0 v(s)ds + Lq\Big)J + \Big\{\int_{t-L}^t v(s)ds + Lq$$

$$\begin{aligned} &+ \int_0^t e^{-\int_u^t v(s)ds} v(u-L)du + Lq\int_0^t v(u)e^{-\int_u^t v(s)ds}\}||x|| \\ &\leq ||\psi||\Big(1+\int_{-L}^0 v(s)ds + Lq\Big)J + \alpha||x||. \end{aligned}$$

Let $Q = \big(1+\int_{-L}^0 v(s)ds + Lq\big)J$. Then for any $\epsilon > 0$, take $\delta = \dfrac{\epsilon(1-\alpha)}{Q}$. Since x is a fixed point, we have from the above inequality that for $||\psi|| < \delta$,

$$(1-\alpha)||x|| < \delta Q,$$

or

$$||x|| < \epsilon.$$

This shows the zero solution is stable. To show $(\mathcal{P}\phi)(t) \to 0,$ as $t \to \infty$, we slightly modify $\mathcal{M}_b$ and define the new set

$$\begin{aligned} \mathcal{M}_s &= \{\phi : [-L,\infty) \to \mathbb{R}/\phi(t) = \psi(t), t \in [-L,0], \\ &\quad \phi \in \mathcal{C}, |\phi(t)| \leq 1,\ \phi(t) \to 0 \text{ as } t \to \infty\}. \end{aligned}$$

Then $(\mathcal{M}_s, \rho)$ is a complete metric space under the metric

$$\rho(x,y) = \sup_{t\geq 0}\{|x(t)-y(t)|\}.$$

Use (2.7) to define the mapping $\mathcal{P} : \mathcal{M}_s \to \mathcal{M}_s$. Thus for $\phi \in \mathcal{M}_s$, we have by (2.7) that the first two terms on the right side of (2.7) go to zero as $t \to \infty$. Let $u = s-t$.Then

$$\int_{t-L}^t g(\phi(s))ds = \int_{-L}^0 g(\phi(t+u))du \to \int_{-L}^0 g(0)du = Lg(0) \text{ as } t \to \infty.$$

Next we concentrate on the sixth term of the right side (2.7). As $\phi(t) \to 0,$ as $t \to \infty$, for each $\epsilon > 0$, there exists a $T_1 > 0$ such that $u \geq T_1$ implies that $|\phi(u-L)| < \epsilon$. Thus, for $t \geq T_1$, we have

$$\begin{aligned} \Big|\int_0^t e^{-\int_u^t v(s)ds} v(u-L)\phi(u-L)du\Big| &\leq \int_0^{T_1} e^{-\int_u^t v(s-L)ds} v(u-L)|\phi(u)|du \\ &+ \int_{T_1}^t e^{-\int_u^t v(s-L)ds} |v(u-L)||\phi(u)|du \\ &\leq \sup_{\eta\geq 0}|\phi(\eta)|\int_0^{T_1} e^{-\int_u^t v(s)ds} v(u-L)du \\ &+ \epsilon\int_{T_1}^t e^{-\int_u^t v(s)ds} v(u-L)du \end{aligned}$$

$$\leq \epsilon\alpha + \sup_{\eta\geq 0}|\phi(\eta)| \int_0^{T_1} e^{-\int_u^t v(s)ds} v(u-L)du.$$

By (2.11), there exists $t_2 > T_1$ such that $t \geq t_2$ implies

$$\begin{aligned} &\sup_{\eta\geq 0}|\phi(\eta)| \int_0^{T_1} e^{-\int_u^t v(s)ds} v(u-L)du \\ \leq\ & \sup_{\eta\geq 0}|\phi(\eta)| e^{-\int_{T_1}^t v(s)ds} \int_0^{T_1} e^{-\int_u^{T_1} v(s)ds} v(u-L)du < \epsilon. \end{aligned}$$

Thus, we have

$$\left| \int_0^t e^{-\int_u^t v(s)ds} v(u)\phi(u)du \right| \leq \epsilon + \alpha\epsilon < 2\epsilon.$$

Similarly, we can show that the third and fifth terms in (2.7) approach zero as $t \to \infty$. Next we show that the fourth term in expression (2.7) approaches $-Lg(0)$ as $t \to \infty$. Let $\epsilon > 0$ be given and for a fixed $\phi \in \mathcal{M}_b$, find $T > 0$ such that for $t \geq t - L$ implies that $|Lg(\phi(t)) - Lg(0)| < \epsilon$. Hence

$$\begin{aligned} \int_0^t v(u)e^{-\int_u^t v(s)ds} \int_{u-L}^u g(\phi(s))dsdu \ \leq\ & \int_0^T v(u)e^{-\int_u^t v(s)ds} \int_{u-L}^u g(\phi(s))dsdu \\ +\ & \int_T^t v(u)e^{-\int_u^t v(s)ds} \int_{u-L}^u g(\phi(s))dsdu \end{aligned}$$

The first term on the right hand side tends to zero as $t \to \infty$.

$$\begin{aligned} & \int_T^t v(u)e^{-\int_u^t v(s)ds} \int_{u-L}^u g(\phi(s))dsdu \\ \leq\ & \int_T^t v(u)e^{-\int_u^t v(s)ds} \int_{u-L}^u |g(\phi(s)) - Lg(0)|dsdu \\ +\ & \left| \int_T^t v(u)e^{-\int_u^t v(s)ds} \int_{u-L}^u g(0)dsdu \right| \\ \leq\ & \epsilon \int_T^t v(u)e^{-\int_u^t v(s)ds}du \\ +\ & \left| \int_T^t v(u)e^{-\int_u^t v(s)ds} \int_{u-L}^u g(0)dsdu \right| \\ <\ & \epsilon + \left| \int_T^t v(u)e^{-\int_u^t v(s)ds} \int_{u-L}^u g(0)dsdu \right|. \end{aligned}$$

Note that

$$\int_T^t v(u)e^{-\int_u^t v(s)ds} \int_{u-L}^u g(0)dsdu \ =\ Lg(0)\big[1 - e^{-\int_T^t v(s)ds}\big]$$

$$\to \quad Lg(0).$$

Thus, for $\phi \in \mathcal{M}_b$, we have shown that

$$(\mathcal{P}\phi)(t) \to Lg(0) - Lg(0) = 0, \text{ as } t \to \infty. \tag{2.12}$$

By the contraction mapping principle,$\mathcal{P}$ has a unique fixed point x in $\mathcal{M}_b$, which is a solution of (1.1) with $x(s) = \psi(s)$ on $[-L, 0]$ and $x(t) = x(t, 0, \psi) \to 0$ as $t \to \infty$. □

Note that from inequality (2.12) $(\mathcal{P}\phi)(t) \to 0$ without the requirement $g(0) = 0$. The requirement that $g(0) = 0$ was only to obtain stability of the zero solution, and hence the asymptotic stability. In the next corollary, we show if $x(t) = x(t, 0, \psi)$ is a solution of (1.1) and $x \in \mathcal{M}_b$, then $x(t) = x(t, 0, \psi) \to 0$ as $t \to \infty$.

Corollary 2.4. *Assume the hypothesis of Theorem 2.2 along with* (2.11)*. If $x(s) = \psi(s)$ on $[-L, 0]$ and $x(t) = x(t, 0, \psi)$ is a solution of* (1.1)*, then $x(t) = x(t, 0, \psi) \to 0$ as $t \to \infty$.*

Proof. Let $\phi \in \mathcal{M}_b$ and $(\mathcal{P}\phi)(t)$ be given by (2.7). Then by Theorem 2.2, $\mathcal{P}$ has a unique fixed point that solve (1.1). Inequality (2.12) is still valid here and hence $(\mathcal{P}\phi)(t) \to Lg(0) - Lg(0) = 0, \text{ as } t \to \infty.$ □

Example 2.5. Consider the delay differential equation

$$x'(t) = \frac{x(t)}{1 + x^2(t)} - \frac{x(t-L)}{1 + x^2(t-L)}, \ t \geq 0 \tag{2.13}$$

where L is a positive constant. Then clearly $g(x) = \dfrac{x}{1+x^2}$ is continuous and Lipschitz with Lipschitz constant $q = 1/2$ with $g(0) = 0$ and $|g(x)| \leq \dfrac{1}{2}|x|$. Hence Theorem 2.3 is applicable. Let $v(s) = \dfrac{1}{s^2+1}$. Then for $x \in \mathcal{M}_b$ we have that

$$\begin{aligned}
\int_0^t e^{-\int_u^t v(s)ds} v(u-L)du &= \int_0^t e^{-\int_u^t \frac{1}{s^2+1}ds} \frac{1}{(u-L)^2+1}du \\
&\leq \int_0^t e^{-\int_u^t \frac{1}{(s-L)^2+1)}ds} \frac{1}{(u-L)^2+1}du \\
&= e^{-\int_u^t \frac{1}{(s-L)^2+1}ds}\Big|_{u=0}^t \\
&= 1 - e^{-\int_0^t \frac{1}{(s-L)^2+1}ds} \\
&= 1 - e^{-\arctan(t-L)+\arctan(L)} \\
&\leq 1 - e^{-\pi/2+\arctan(L)}.
\end{aligned}$$

Similarly,

$$\int_{t-L}^t v(s)ds \leq \int_{t-L}^t \frac{1}{s^2+1}ds \leq \int_{t-L}^t \frac{1}{2}ds = \frac{L}{2}.$$

Finally,

$$\begin{aligned}
\int_0^t e^{-\int_u^t v(s)ds} v(u)du &= \int_0^t e^{-\int_u^t \frac{1}{s^2+1}ds} \frac{1}{(u^2+1}du \\
&= e^{-\int_u^t \frac{1}{s^2+1}ds}\Big|_{u=0}^t \\
&= 1 - e^{-\int_0^t \frac{1}{s^2+1}ds} \\
&= 1 - e^{-\arctan(t)} \\
&\leq 1 - e^{-\pi/2}.
\end{aligned}$$

For sufficiently small delay L

$$\frac{L}{2} + \frac{L}{2} + \frac{L}{2}\big(1 - e^{-\pi/2}\big) + 1 - e^{-\pi/2+\arctan(L)} \leq \alpha, \tag{2.14}$$

is satisfied. This satisfies condition (2.6). Thus if ψ is a given continuous initial function which is sufficiently small, then there is a solution $x(t, 0, \psi)$ of (2.13) with $|x(t, 0, \psi)| < 1$ for all $t \geq 0$ and its zero solution is stable by Theorem 2.3.

In Theorems 2.2 , 2.3 and Corollary 2.4 we constructed our map using (2.1) which has advantage when the equation of interest has a time delay, instead of a constant delay. In the next two corollaries we use (2.3) to construct the suitable map and then apply them to our previous example. Just for the record, to consider (1.1) with time delay is another big problem that the author might consider taking on later.

Corollary 2.6. *Assume* (2.4) *and there exists a constant* $\alpha \in (0, 1)$ *such that*

$$LL_1 + \big(1 + L_1 L\big) \int_0^t v(u)e^{-\int_u^t v(s)ds}du \leq \alpha, \tag{2.15}$$

and

$$qL + \big(1 + qL\big) \int_0^t v(u)e^{-\int_u^t v(s)ds}du \leq \alpha. \tag{2.16}$$

If ψ *is a given continuous initial function which is sufficiently small, then there is a solution* $x(t, 0, \psi)$ *of* (1.1) *with* $|x(t, 0, \psi)| < 1,$ *for all* $t \geq 0.$

Proof. Define a map $\mathcal{P}$ using (2.3) and then imitate the proof of Theorem 2.2. □

Corollary 2.7. *Assume the hypothesis of Corollary 2.6. In addition, we assume the function* $v(t) \geq 0, 0 \leq t < \infty$, *and*

$$g(0) = 0. \tag{2.17}$$

Then the zero solution of (1.1) *is stable. Moreover, if we assume that*

$$\int_0^\infty v(s)ds = \infty, \tag{2.18}$$

then the zero solution of (1.1) *is asymptotically stable.*

Proof. Define a map $\mathcal{P}$ using (2.3) and then imitate the proof of Corollary 2.6. □

If we consider (2.13), then (2.16) corresponds to

$$\frac{L}{2}+\left(1+\frac{L}{2}\right)\left(1-e^{-\pi/2}\right)<1,$$

for $L=0.1$. On the other hand, (2.14) does not hold for $L=0.1$.

3 Periodic Solutions

Our new inversion techniques leads to an improved results concerning the existence of periodic solutions. To be specific and to motivate the reader we begin by considering the nonlinear differential equation and show the existence of periodic solutions without the requirement of some classic conditions. To better illustrate our approach, we consider the nonlinear differential equation

$$x'=a(t)x(t)+f(t,x), \tag{3.1}$$

where f is continuous in x. For $L\in R$, we assume the periodicity condition

$$a(t+L)=a(t),\ \text{and}\ f(t+L,\cdot)=f(t,\cdot). \tag{3.2}$$

Let BC be the space of continuous and bounded functions $\phi:\mathbb{R}\to\mathbb{R}$ with the maximum norm $||\cdot||$. Define $P_L=\{\phi\in BC,\phi(t+L)=\phi(t)\}$.Then P_L is a Banach space when it is endowed with the maximum norm

$$\|x\|=\max_{t\in[0,L]}|x(t)|.$$

Also, we assume that

$$e^{\int_0^L a(s)ds}\neq 1. \tag{3.3}$$

Throughout this section we assume that $a(t)\neq 0$ for all $t\in[0,L]$. Then Eqn. (3.1) is equivalent to

$$\left[x(t)e^{-\int_0^t a(s)ds}\right]'=f(t,x(t))e^{-\int_0^t a(s)ds}. \tag{3.4}$$

Integrating equation (3.4) from $t-L$ to t and using the fact that $x(t-L)=x(t)$, gives

$$x(t)=\left(1-e^{\int_0^L a(s)ds}\right)^{-1}\int_{t-L}^{t}f(u,x(u))e^{\int_u^t a(s)ds}du. \tag{3.5}$$

On the other hand, if we take a function $v(t)$ as before and assume $v\in P_L$ with $v(t)\neq 0,\forall t\in[0,L]$, then we obtain the following variation of parameters formula ,

$$x(t)\ =\ \left(1-e^{\int_0^L v(s)ds}\right)^{-1}\int_{t-L}^{t}\Big[x(u)v(u)+x(u)a(u)$$

$$+ \quad f(u,x(u))e^{\int_u^t a(s)ds}du\Big]. \tag{3.6}$$

We note that (3.6) can now handle equations of the form

$$x'(t) = \cos(t)x(t) + f(t,x(t)).$$

We can easily see that

$$1 - e^{\int_0^L a(s)ds} = 1 - e^{\int_0^{2\pi} \cos(s)ds} = 0,$$

and hence (3.5) can not be used. We end this paper by adding an $L-$periodic function to the right hand side of Equation (1.2). Particularly, we consider

$$x'(t) = g(t,x(t)) - g(t,x(t-L)) + h(t),\ t \geq 0, \tag{3.7}$$

where

$$g(t+L,x) = g(t,x) \text{ and } h(t+L) = h(t). \tag{3.8}$$

First, we rewrite (3.7) in the form

$$x'(t) = \frac{d}{dt}\int_{t-L}^{t} g(x(s))ds + h(t). \tag{3.9}$$

Choose a function $v(t)$ such that $v \in P_L$ with $v(t) \neq 0, \forall t \in [0,L]$. Multiply both sides of (3.9) by $e^{\int_0^t v(s)ds}$ and then integrate from $t-T$ to t to have

$$\begin{aligned}
&\int_{t-T}^{t} e^{\int_0^u v(s)ds}\, x'(u)du \\
= &\int_{t-T}^{t} \Big(\frac{d}{du}\int_{u-L}^{u} g(x(s))ds\Big)e^{\int_0^u v(s)ds}du + \int_{t-T}^{t} e^{\int_0^u v(s)ds}h(u)du.
\end{aligned}$$

Then by similar calculations as before, one can easily arrive at

$$\begin{aligned}
x(t) \;=\; &\Big(1 - e^{-\int_0^L v(s)ds}\Big)^{-1}\Big[\int_{t-L}^{t} x(u)v(u)e^{-\int_u^t v(s)ds}du + \int_{t-L}^{t} g(s,x(s))ds \\
- \;& e^{-\int_0^L v(s)ds}\int_{t-L}^{t} g(s,x(s))ds + \int_{t-L}^{t} e^{-\int_u^t v(s)ds}h(u)du \\
+ \;& \int_{t-L}^{t}\int_{u-L}^{u} e^{-\int_u^t v(s)ds}v(u)g(s,x(s))dsdu\Big].
\end{aligned} \tag{3.10}$$

Let

$$\eta = \max\Big(1 - e^{-\int_0^L v(s)ds}\Big)^{-1}.$$

Also, we assume the function g is Lipschitz. That is, for x and $y \in P_L$ and for all $t \in [0,L]$, there exists a constant $q > 0$ such that

$$|g(t,x) - g(t,y)| \leq q|x-y|. \tag{3.11}$$

Theorem 3.1. *Assume* (3.8), (3.11) *and there exists a constant* $\alpha \in (0,1)$ *such that*

$$\eta\Big[\int_{t-L}^{t} e^{-\int_u^t v(s)ds} v(u)du \quad + \quad qL + qLe^{-\int_0^L v(s)ds} du \\ + \quad qL\int_{t-L}^{t} e^{-\int_u^t v(s)ds} v(u)du\Big] \le \alpha. \qquad (3.12)$$

Then (3.9) *has a unique* L*-periodic solution.*

Proof. For $\phi \in P_L$, use (3.10) to define the map $\mathcal{P}$. It can be easily shown that $\mathcal{P}$ is continuous and $(\mathcal{P}\phi)(t+L) = (\mathcal{P}\phi)(t)$. Hence $\mathcal{P} : P_L \to P_L$. Moreover, it follows from (3.12) that the map $\mathcal{P}$ is a contraction. Hence the result. □

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