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On Three General q,ω-Apostol Polynomials and their Connection to q,ω-Power Sums

Thomas Ernst
Uppsala University
Department of Mathematics
P.O. Box 480, SE-751 06 Uppsala, Sweden

Abstract

The purpose of this article is to continue our study of q,ω-special functions, in particular three general q,ω-Apostol polynomials, which contain an extra parameter λ. We find complementary argument theorems, multiplication formulas and recurrence relations for the corresponding numbers. Furthermore, an explicit formula for the multiple alternating q,ω-power sum is found. These formulas are then specialized to q,ω-Apostol–Bernoulli and Euler polynomials.

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Contents

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1 Introduction

This paper is part of a series of five papers on q,ω-calculus. In each paper we start with many similar definitions, since the subject is quite new.

Let $\omega \in \mathbb{R},\ 0 < \omega < 1$. Put $\omega_0 \equiv \frac{\omega}{1-q},\ 0 < q < 1$. Let I be an interval which contains ω_0. Throughout, we assume that the variable x belongs to I.

We introduce a new calculus, which will be very similar to the well-known q-calculus, where many functions and operators appear again, with a similar name. The reason is that the q,ω-Appell polynomials form a ring, which is proved in one of these papers [9]. The convergence region in ω will always be a small interval above 0, depending on q. The subtle properties of absolute maximum for the two q,ω-additions are exemplified in [8].

The paper is organized as follows: In Section 2 we present preliminary definitions and theorems for q,ω-calculus, like a q,ω-analogue of a function.

In Section 3 we define the four q,ω-additions, natural generalizations of the four q-additions, and point out that they obey identical laws. It is an intriguing fact that the q,ω-difference operator of the q,ω-addition is identical with the corresponding q-analogue, a formula that seems to be new. The q,ω-addition formulas and q,ω-differences for the q,ω-exponential functions are also identical. In Section 5 we study general q,ω-Appell polynomials in the spirit of [7]. In Section 6 we study three general q-Apostol polynomials, which are natural generalizations of polynomials in [7]. First of all, the important operators $\triangle_{\text{NWA},\mathcal{A},q}$ and $\nabla_{\text{NWA},\mathcal{A},q}$ are defined. In Section 7 we study q,ω-$\mathcal{H}$ polynomials, a special case of one of the three general polynomials above, and show its connection to the q,ω-Apostol–Euler polynomials, see below. In Section 8, with the help of the $\triangle_{\text{NWA},\mathcal{A},q,\omega}$ operator, the q,ω-Apostol–Bernoulli polynomials are investigated. In Section 9, with the help of the $\nabla_{\text{NWA},\mathcal{A},q,\omega}$ operator, the q,ω-Apostol–Euler polynomials are investigated.

2 Preliminary Definitions and Theorems

Definition 2.1. The automorphism ϵ on the vector space of polynomials is defined by

$$\epsilon f(x) \equiv f(qx+\omega). \tag{2.1}$$

This automorphism is a generalization of the operator with the same name in q-calculus [6]. In [1, p. 136] it is proved that

$$\epsilon^k f(x) = f(q^k x + \omega\{k\}_q). \tag{2.2}$$

Definition 2.2. A q,ω-analogue of the mathematical object G is a mathematical function $F(q,\omega)$, with the property $\lim\limits_{\omega\to 0} F(q,\omega) = G_q$, the q-analogue of G. Both F and G can depend on more, common variables. They can also be operators.

Definition 2.3. Let φ be a continuous real function of x. Then we define the q,ω-difference operator $\mathrm{D}_{q,\omega}$ as follows:

$$\mathrm{D}_{q,\omega}\left(\varphi\right)(x) \equiv \begin{cases} \dfrac{\varphi(qx+\omega)-\varphi(x)}{(q-1)x+\omega}, & \text{if } x \neq \omega_0; \\ \dfrac{d\varphi}{dx}(x) & \text{if } x = \omega_0. \end{cases} \tag{2.3}$$

We say that a function $f(x)$ is n times q,ω-differentiable if $\mathrm{D}^n_{q,\omega} f(x)$ exists. If we want to point out that this operator operates on the variable x, we write $\mathrm{D}_{q,\omega,x}$ for the operator. Furthermore, $\mathrm{D}_{q,\omega}(K) = 0$, like for the derivative.

Theorem 2.4. *The q,ω-difference operator is linear.*

$$\mathrm{D}_{q,\omega} \sum_{k=0}^{\infty} a_k f_k(x) = \sum_{k=0}^{\infty} a_k \mathrm{D}_{q,\omega} f_k(x). \tag{2.4}$$

Proof. This is obvious, since the definition of $\mathrm{D}_{q,\omega}$ is linear in the function. □

We now introduce two basic sequences, which generalize the Ciglerian polynomials in [6, 5.5].

Definition 2.5.

$$[21, (16)]\ (x)^k_{q,\omega} \equiv \prod_{m=0}^{k-1} (x - \omega\{m\}_q),\ (x)^0_{q,\omega} = 1. \tag{2.5}$$

$$[21, (15)]\ [x]^k_{q,\omega} \equiv \prod_{m=0}^{k-1} (q^m x + \omega\{m\}_q),\ [x]^0_{q,\omega} = 1. \tag{2.6}$$

The following names will be used for the ensuing q,ω-trigonometric and hyperbolic functions [8].

Definition 2.6. A function f of two variables x,ω is called x,ω-even if $f(-x,-\omega)=f(x,\omega)$. A function f of two variables x,ω is called x,ω-odd if $f(-x,-\omega)=-f(x,\omega)$.

Lemma 2.7. *Products and sums of any number of x,ω-even functions are x,ω-even. The product and quotient of an x,ω-even function and an x,ω-odd function are x,ω-odd.*

Lemma 2.8. *The two functions $(x)_{q,\omega}^{2k}$ and $[x]_{q,\omega}^{2k}$ are x,ω-even. The two functions $(x)_{q,\omega}^{2k+1}$ and $[x]_{q,\omega}^{2k+1}$ are x,ω-odd.*

The two following formulas correspond to the formula $\mathrm{D}x^n=nx^{n-1}$:

$$\text{[11, 2.5], [21, (17)] } \mathrm{D}_{q,\omega}(x)_{q,\omega}^n=\{n\}_q(x)_{q,\omega}^{n-1}. \tag{2.7}$$

$$\text{[21, (18)] } \mathrm{D}_{q,\omega}[x]_{q,\omega}^n=\{n\}_q[qx+\omega]_{q,\omega}^{n-1}. \tag{2.8}$$

We next introduce two q,ω-analogues of the exponential function:

Definition 2.9. The q,ω-exponential function $\mathrm{E}_{q,\omega}(z)$ [21, (21)] is defined by

$$\mathrm{E}_{q,\omega}(z)\equiv\sum_{k=0}^{\infty}\frac{(z)_{q,\omega}^k}{\{k\}_q!},\ |(1-q)z-\omega|<1. \tag{2.9}$$

The complementary q,ω-exponential function $\mathrm{E}_{\frac{1}{q},\omega}(z)$ [21, (26)] is defined by

$$\mathrm{E}_{\frac{1}{q},\omega}(z)\equiv\sum_{k=0}^{\infty}\frac{[z]_{q,\omega}^k}{\{k\}_q!},\ |\omega|<1. \tag{2.10}$$

We have changed the name to $\mathrm{E}_{\frac{1}{q},\omega}(z)$ since $\mathrm{E}_{\frac{1}{q},0}(z)=\mathrm{E}_{\frac{1}{q}}(z)$ [6]. Observe that, as before, $\mathrm{E}_{q,\omega}(0)=\mathrm{E}_{\frac{1}{q},\omega}(0)=1$.

Theorem 2.10. *[21, (19)] The q,ω-exponential function is the unique solution of the first order initial value problem*

$$\mathrm{D}_{q,\omega}f(z)=f(z),\ f(0)=1. \tag{2.11}$$

[21, (24)] The complementary q,ω-exponential function is the unique solution of the first order initial value problem

$$\mathrm{D}_{q,\omega}f(z)=f(qz+\omega),\ f(0)=1. \tag{2.12}$$

Theorem 2.11. *[21, (21)] The meromorphic continuation of the q, ω-exponential function $\mathrm{E}_{q,\omega}(z)$ is given by*

$$\mathrm{E}_{q,\omega}(z) = \frac{(-\omega; q)_\infty}{((1-q)z - \omega; q)_\infty}. \tag{2.13}$$

[21, (26)] The meromorphic continuation of the complementary q, ω-exponential function $\mathrm{E}_{\frac{1}{q},\omega}(z)$ is given by

$$\mathrm{E}_{\frac{1}{q},\omega}(z) = \frac{((q-1)z + \omega; q)_\infty}{(\omega; q)_\infty}. \tag{2.14}$$

3 On the q, ω-Addition with Applications to q, ω-Special Functions

In order to use these functions, we need to generalize the q-addition. The ordinary q-addition is the special case $\omega = 0$. Just like for the q-addition, we use letters in an alphabet for the q, ω-additions. Equality between letters is denoted by $\sim$. In the following, beware of the fact that whenever we multiply the function argument x in $(x)_{q,\omega}^\nu$ or in $[x]_{q,\omega}^\nu$ by the constant a, we must also multiply ω by a.

Definition 3.1. Let $\{f_\nu\}_{\nu=0}^\infty$ denote an arbitrary sequence of real numbers. The q, ω-addition for the sequences $(x)_{q,\omega}^k$ is defined by

$$(f \oplus_q (x)_{q,\omega})^\nu \equiv \sum_{k=0}^{\nu} \binom{\nu}{k}_q f_{\nu-k} (x)_{q,\omega}^k. \tag{3.1}$$

The NWA q, ω-addition is defined as follows:

$$(x \oplus_{q,\omega} y)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q (x)_{q,\omega}^{n-k} (y)_{q,\omega}^k. \tag{3.2}$$

The NWA q, ω-subtraction is defined as follows:

$$(x \ominus_{q,\omega} y)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q (x)_{q,\omega}^{n-k} (-y)_{q,-\omega}^k. \tag{3.3}$$

The JHC q, ω-addition is defined as follows:

$$(x \boxplus_{q,\omega} y)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q (x)_{q,\omega}^{n-k} [y]_{q,\omega}^k. \tag{3.4}$$

The JHC q, ω-subtraction is defined as follows:

$$(x \boxminus_{q,\omega} y)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q (x)_{q,\omega}^{n-k} [-y]_{q,-\omega}^k. \tag{3.5}$$

Theorem 3.2. *The NWA q,ω-addition is commutative and associative.*

Proof. Similar to the proof for NWA q-addition. □

Corollary 3.3. *Two extensions of the formula [6, 4.29]*

$$\mathrm{D}_{q,\omega,x}(x \oplus_{q,\omega} y)^n = \{n\}_q (x \oplus_{q,\omega} y)^{n-1},\ \oplus_{q,\omega} \equiv \oplus_{q,\omega} \vee \boxplus_{q,\omega}. \tag{3.6}$$

Proof.

$$\mathrm{D}_{q,\omega,x}(x \oplus_{q,\omega} y)^n \overset{\text{by}(2.7)}{=} \sum_{k=0}^{n-1} \binom{n}{k}_q \{n-k\}_q (x)_{q,\omega}^{n-k-1} (y)_{q,\omega}^k = \text{RHS}. \tag{3.7}$$

□

Corollary 3.4. *Four q,ω-additions for the q,ω-exponential function.*

$$\mathrm{E}_{q,\omega}(x \oplus_{q,\omega} y) \equiv \mathrm{E}_{q,\omega}(x)\mathrm{E}_{q,\omega}(y). \tag{3.8}$$

$$\mathrm{E}_{q,\omega}(x \ominus_{q,\omega} y) \equiv \mathrm{E}_{q,\omega}(x)\mathrm{E}_{q,-\omega}(-y). \tag{3.9}$$

$$\mathrm{E}_{q,\omega}(x \boxplus_{q,\omega} y) \equiv \mathrm{E}_{q,\omega}(x)\mathrm{E}_{\frac{1}{q},\omega}(y). \tag{3.10}$$

$$\mathrm{E}_{q,\omega}(x \boxminus_{q,\omega} y) \equiv \mathrm{E}_{q,\omega}(x)\mathrm{E}_{\frac{1}{q},-\omega}(-y). \tag{3.11}$$

Theorem 3.5. *The q,ω-differences for the q,ω-exponential functions are:*

$$\mathrm{D}_{q,\omega}\ \mathrm{E}_{q,a\omega}(ax) = a\ \mathrm{E}_{q,a\omega}(ax), \tag{3.12}$$

$$\mathrm{D}_{q,\omega}\ \mathrm{E}_{\frac{1}{q},a\omega}(ax) = a\ \mathrm{E}_{\frac{1}{q},a\omega}(aqx + a\omega), \tag{3.13}$$

Proof. This follows from the chain rule [9]. □

The following umbral numbers can only be function arguments in formal power series.

In our second book [7] we introduced several new q-deformed number systems, semiring, biring etc., each with an extra index q. By a miracle, we can extend these number systems by adding another index ω. The proofs will be very similar, and we just state the definitions.

Definition 3.6. The Ward-ω number $\overline{n}_{q,\omega}$ is defined by

$$\overline{n}_{q,\omega} \sim 1 \oplus_{q,\omega} 1 \oplus_{q,\omega} \ldots \oplus_{q,\omega} 1, \tag{3.14}$$

where the number of 1 on the RHS is n.

Definition 3.7. An extension of [6, 4.70]:

$$(\overline{n}_{q,\omega})^k \equiv \sum_{m_1+\ldots+m_n=k} \binom{k}{m_1,\ldots,m_n}_q \prod_{i=1}^{n}(1)_{q,\omega}^{m_i}, \tag{3.15}$$

where each partition of k is multiplied with its number of permutations. We have the following special cases:

$$(\overline{0}_{q,\omega})^k = \delta_{k,0};\ (\overline{n}_{q,\omega})^0 = 1;\ (\overline{n}_{q,\omega})^1 = n. \tag{3.16}$$

Theorem 3.8. *Functional equations for Ward-ω numbers operating on the $q, omega$-exponential function. First assume that the letters $\overline{m}_{q,\omega}$ and $\overline{n}_{q,\omega}$ are independent, i.e. come from two different functions, when operating with the functional. Furthermore, $mnt < \dfrac{1+\omega}{1-q}$. Then we have*

$$\mathrm{E}_{q,\omega}(\overline{m}_{q,\omega}\overline{n}_{q,\omega}t) = \mathrm{E}_{q,\omega}(\overline{mn}_{q,\omega}t). \tag{3.17}$$

Furthermore,

$$\mathrm{E}_{q,\omega}(\overline{jm}_{q,\omega}) = \mathrm{E}_{q,\omega}(\overline{j}_{q,\omega})^m = \mathrm{E}_{q,\omega}(\overline{m}_{q,\omega})^j = \mathrm{E}_{q,\omega}(\overline{j}_{q,\omega} \odot_{q,\omega} \overline{m}_{q,\omega}). \tag{3.18}$$

Definition 3.9. Let the q,ω-rational numbers $\mathbb{Q}_{q,\omega}$ be defined as follows:

$$\mathbb{Q}_{q,\omega} \equiv \left\{\frac{\overline{m}_{q,\omega}}{\overline{n}_{q,\omega}},\ m \in \mathbb{N}\cup\{0\}, n \in \mathbb{N},\ m \neq n, \frac{\overline{0}_{q,\omega}}{\overline{n}_{q,\omega}} \sim \theta,\ \frac{\overline{n}_{q,\omega}}{\overline{n}_{q,\omega}} \sim 1\right\}, \tag{3.19}$$

together with a linear functional

$$v,\ \mathbb{R}[x] \times \mathbb{Q}_{\oplus_{q,\omega}} \to \mathbb{R}, \tag{3.20}$$

called the evaluation. If $v(x) = \sum_{k=0}^{n} a_k x^k$, then

$$v\left(\frac{\overline{m}_{q,\omega}}{\overline{n}_{q,\omega}}\right) \equiv \sum_{k=0}^{n} a_k \frac{(\overline{m}_{q,\omega})^k}{(\overline{n}_{q,\omega})^k}. \tag{3.21}$$

4 Multiple q,ω-Power Sums

Definition 4.1. A q,ω-analogue of [14, (20) p. 381], the multiple q,ω-power sum is defined by

$$s^{(l)}_{\mathrm{NWA},\lambda,m,q,\omega}(n) \equiv \sum_{|\vec{j}|=l} \binom{l}{\vec{j}} \lambda^k \left(\overline{k}_{q,\omega}\right)^m, \tag{4.1}$$

where $k \equiv j_1 + 2j_2 + \cdots + (n-1)j_{n-1},\ \forall j_i \geq 0$.

Definition 4.2. A q,ω-analogue of [14, (46) p. 386], the multiple alternating q,ω-power sum is defined by

$$\sigma^{(l)}_{\text{NWA},\lambda,m,q,\omega}(n) \equiv (-1)^l \sum_{|\vec{j}|=l} \binom{l}{\vec{j}} (-\lambda)^k \left(\overline{k}_{q,\omega}\right)^m, \tag{4.2}$$

where $k \equiv j_1 + 2j_2 + \cdots + (n-1)j_{n-1},\ \forall j_i \geq 0$.

For $l = 1$, formulas (4.1) and (4.2) reduce to single sums. In order to keep the same notation as in [6], we make a slight change from [22, p.309]. The following definitions are special cases of the q,ω-power sums in section 8.2.

Definition 4.3. Almost a q,ω-analogue of [22, p.309], the q,ω-power sum and the alternate q,ω-power sum (with respect to λ), are defined by

$$s_{\text{NWA},\lambda,m,q,\omega}(n) \equiv \sum_{k=0}^{n-1} \lambda^k (\overline{k}_{q,\omega})^m, \tag{4.3}$$

$$\sigma_{\text{NWA},\lambda,m,q,\omega}(n) \equiv \sum_{k=0}^{n-1} (-1)^k \lambda^k (\overline{k}_{q,\omega})^m. \tag{4.4}$$

Their respective generating functions are

$$\sum_{m=0}^{\infty} s_{\text{NWA},\lambda,m,q,\omega}(n) \frac{t^m}{\{m\}_q!} = \frac{\lambda^n \mathrm{E}_{q,\omega}(\overline{n}_{q,\omega} t) - 1}{\lambda \mathrm{E}_{q,\omega}(t) - 1} \tag{4.5}$$

and

$$\sum_{m=0}^{\infty} \sigma_{\text{NWA},\lambda,m,q,\omega}(n) \frac{t^m}{\{m\}_q!} = \frac{(-1)^{n+1} \lambda^n \mathrm{E}_{q,\omega}(\overline{n}_{q,\omega} t) + 1}{\lambda \mathrm{E}_{q,\omega}(t) + 1}. \tag{4.6}$$

Proof. Let us prove (4.5). We have

$$\sum_{m=0}^{\infty} s_{\text{NWA},\lambda,m,q,\omega}(n) \frac{t^m}{\{m\}_q!} = \sum_{m=0}^{\infty} \sum_{k=0}^{n-1} \lambda^k \frac{(\overline{k}_{q,\omega} t)^m}{\{m\}_q!} = \sum_{k=0}^{n-1} \lambda^k (\mathrm{E}_{q,\omega}(t))^k = \text{RHS}. \tag{4.7}$$

□

We have the following special cases:

$$s_{\text{NWA},\lambda,m,q,\omega}(1) = \sigma_{\text{NWA},\lambda,m,q,\omega}(1) = \delta_{0,m}, \tag{4.8}$$

$$s_{\text{NWA},\lambda,m,q,\omega}(2) = \delta_{0,m} + \lambda,\ \sigma_{\text{NWA},\lambda,m,q,\omega}(2) = \delta_{0,m} - \lambda. \tag{4.9}$$

5 q,ω-Appell Polynomials

The most general form of polynomial in this article is the Hahn–Appell polynomial, which we will now define.

Definition 5.1. Let $\mathcal{A}_{q,\omega}$ denote the set of real sequences $\{u_{\nu,q}\}_{\nu=0}^{\infty}$ such that

$$\sum_{\nu=0}^{\infty}|u_{\nu,q}|\frac{r^{\nu}}{\{\nu\}_q!}<\infty, \tag{5.1}$$

for some q,ω-dependent convergence radius $r=r(q)>0$, where $0<q<1$.

The q,ω-Appell number sequence is denoted by $\{\Phi_{\nu,q,\omega}^{(n)}\}_{\nu=0}^{\infty}$.

Definition 5.2. Assume that $h(t,q,\omega),\ h(t,q,\omega)^{-1}\in\mathbb{R}[[t]]$. For $f_n(t,q,\omega)=h(t,q,\omega)^n$, the multiplicative q,ω-Appell numbers of degree ν and order n $\Phi_{\nu,q,\omega}\in\mathcal{A}_{q,\omega}$ are given by the generating function

$$f_n(t,q,\omega)=\sum_{\nu=0}^{\infty}\frac{t^{\nu}}{\{\nu\}_q!}\Phi_{\nu,q,\omega}^{(n)},\ \Phi_{0,q,\omega}^{(n)}=1. \tag{5.2}$$

It will be convenient to fix the value for $n=0$ and $n=1$:

$$\Phi_{\nu,q,\omega}^{(0)}\equiv\delta_{0,\nu},\ \Phi_{\nu,q,\omega}^{(1)}\equiv\Phi_{\nu,q,\omega}. \tag{5.3}$$

Definition 5.3. Compare with [21, (31)]. For $f_n(t,q,\omega)\in\mathbb{R}[[t]]$ as above, the multiplicative q,ω-Appell polynomial sequence $\{\Phi_{\nu;q,\omega}^{(n)}(x)\}_{\nu=0}^{\infty}$ of degree ν and order n is defined by the generating function

$$f_n(t,q,\omega)\mathrm{E}_{q,\omega t}(xt)=\sum_{\nu=0}^{\infty}\frac{t^{\nu}}{\{\nu\}_q!}\Phi_{\nu;q,\omega}^{(n)}(x). \tag{5.4}$$

It will be convenient to fix the value for $n=0$ and $n=1$:

Theorem 5.4. *[9]*

$$\Phi_{\nu,q,\omega}^{(0)}(x)=(x)_{q,\omega}^{\nu},\ \Phi_{\nu,q,\omega}^{(1)}(x)\equiv\Phi_{\nu,q,\omega}(x). \tag{5.5}$$

Definition 5.5. For $f_n(t,q,\omega)\in\mathbb{R}[[t]]$ as above, the complementary, multiplicative q,ω-Appell polynomial sequence $\{\Phi_{\nu;\frac{1}{q},\omega}^{(n)}(x)\}_{\nu=0}^{\infty}$ of degree ν and order n is defined by the generating function

$$f_n(t,q,\omega)\mathrm{E}_{\frac{1}{q},\omega t}(xt)=\sum_{\nu=0}^{\infty}\frac{t^{\nu}}{\{\nu\}_q!}\Phi_{\nu;\frac{1}{q},\omega}^{(n)}(x). \tag{5.6}$$

It will be convenient to fix the value for $n = 0$ and $n = 1$:

Definition 5.6.
$$\Phi^{(0)}_{\nu,\frac{1}{q},\omega}(x) \equiv [x]^{\nu}_{q,\omega}, \ \Phi^{(1)}_{\nu,\frac{1}{q},\omega}(x) \equiv \Phi_{\nu,\frac{1}{q},\omega}(x). \tag{5.7}$$

We next present generalizations of the three formulas [6, 4.107, 4.108, 4.111].

Theorem 5.7.
$$\mathrm{D}_{q,\omega}\Phi_{\nu;q,\omega}(x) = \{\nu\}_q \Phi_{\nu-1;q,\omega}(x). \tag{5.8}$$

[21, (30)] in umbral form:

$$\Phi_{\nu;q,\omega}(x) \doteq (\Phi_{q,\omega} \oplus_{q,\omega} x)^{\nu}. \tag{5.9}$$

Theorem 5.8. *[9] Assume that M and K are the x-order and y-order, respectively. Then we have:*

$$\Phi^{(M+K)}_{\nu,q,\omega}(x \oplus_{q,\omega} y) = \sum_{k=0}^{\nu} \binom{\nu}{k}_q \Phi^{(M)}_{k,q,\omega}(x)\Phi^{(K)}_{\nu-k,q,\omega}(y). \tag{5.10}$$

6 Three General q,ω-Apostol Polynomials

The following operators are generalizations of the $\triangle_{\mathrm{NWA},\mathcal{A},q}$ and $\nabla_{\mathrm{NWA},\mathcal{A},q}$ operators [7].

Definition 6.1. Let I denote the identity operator. The Apostol NWA q,ω-difference operator is given by
$$\triangle_{\mathrm{NWA},\mathcal{A},q,\omega} \equiv \lambda\mathrm{E}(\oplus_{q,\omega}) - \mathrm{I}. \tag{6.1}$$
The Apostol NWA q,ω-mean value operator is given by
$$\nabla_{\mathrm{NWA},\mathcal{A},q,\omega} \equiv \frac{\lambda\mathrm{E}(\oplus_{q,\omega}) + \mathrm{I}}{2}. \tag{6.2}$$
The Apostol JHC q,ω-difference operator is given by
$$\triangle_{\mathrm{JHC},\mathcal{A},q,\omega} \equiv \lambda\mathrm{E}(\boxplus_{q,\omega}) - \mathrm{I}. \tag{6.3}$$
The Apostol JHC q,ω-mean value operator is given by
$$\nabla_{\mathrm{JHC},\mathcal{A},q,\omega} \equiv \frac{\lambda\mathrm{E}(\boxplus_{q,\omega}) + \mathrm{I}}{2}. \tag{6.4}$$

Theorem 6.2. *An extension of [6, 4.112]:*

$$(\lambda\mathrm{E}_{q,\omega}(t) - 1)f_n(t,q,\omega)\mathrm{E}_{q,\omega t}(xt) = \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\{\nu\}_q!} \triangle_{\mathrm{NWA},\mathcal{A},q,\omega}\Phi^{(n)}_{\nu,q,\omega}(x). \tag{6.5}$$

There are three similar formulas with the other previously defined operators.

We will now define three quite general q,ω-Apostol polynomials, which have some similarities with the Appell polynomials in Prabkakar and Reva [19]. Two of the names are chosen to resemble the Euler and Bernoulli polynomials.

Definition 6.3. A q,ω-analogue of Lu, Luo [13, p.4], [10, p.203]. The generating function for the generalized NWA q,ω-Apostol $\mathcal{E}$ polynomials of degree ν and order n, $\mathcal{E}^{(n)}_{\text{NWA},\lambda,\mu,\theta;\nu,q,\omega}(x)$, is given by

$$\left(\frac{2^\mu t^\theta}{\lambda \mathrm{E}_{q,\omega}(t)+1}\right)^n \mathrm{E}_{q,\omega t}(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \mathcal{E}^{(n)}_{\text{NWA},\lambda,\mu,\theta;\nu,q,\omega}(x), \theta \in \mathbb{N}. \tag{6.6}$$

Several q,ω-Appell polynomials in this article are special cases of these polynomials, e.g. the q,ω-Euler polynomial is the case $\theta = 0,\ \mu = 1$.

Definition 6.4. The generalized NWA q,ω-$\mathcal{H}$ polynomials are defined by

$$\frac{(2t)^n}{(\lambda \mathrm{E}_{q,\omega}(t)+1)^n} \mathrm{E}_{q,\omega t}(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu \mathcal{H}^{(n)}_{\text{NWA},\lambda,\nu,q,\omega}(x)}{\{\nu\}_q!},\ |t+\log\lambda| < \pi. \tag{6.7}$$

Definition 6.5. The generalized JHC q,ω-$\mathcal{H}$ polynomials are defined by

$$\frac{(2t)^n}{(\lambda \mathrm{E}_{\frac{1}{q},\omega}(t)+1)^n} \mathrm{E}_{q,\omega t}(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu \mathcal{H}^{(n)}_{\text{JHC},\lambda,\nu,q,\omega}(x)}{\{\nu\}_q!},\ |t+\log\lambda| < \pi. \tag{6.8}$$

Theorem 6.6. *We have*

$$\nabla_{\text{NWA},\mathcal{A},q} \mathcal{H}^{(n)}_{\text{NWA},\lambda,\nu,q,\omega}(x) = \{\nu\}_q \mathcal{H}^{(n-1)}_{\text{NWA},\lambda,\nu-1,q,\omega}(x),\ \text{NWA} = \text{NWA} \vee \text{JHC}. \tag{6.9}$$

Theorem 6.7. *A symmetry relation for the generalized q,ω-$\mathcal{H}$ numbers.*

$$(-1)^\nu \mathcal{H}_{\text{JHC},\lambda^{-1},\nu,q,\omega} = \mathcal{H}_{\text{NWA},\lambda,\nu,q,\omega},\ \nu > 1. \tag{6.10}$$

Proof. A computation with generating functions.

$$\begin{aligned}\sum_{\nu=0}^{\infty} \frac{(-t)^\nu \mathcal{H}_{\text{JHC},\lambda^{-1},\nu,q,\omega}}{\{\nu\}_q!} &\overset{\text{by}(6.8)}{=} \frac{-2t}{\lambda^{-1}\mathrm{E}_{\frac{1}{q},-\omega}(-t)+1} = \frac{-2t\lambda \mathrm{E}_{q,\omega}(t)}{\lambda \mathrm{E}_{q,\omega}(t)+1} \\ &\overset{\text{by}(6.7)}{=} -\lambda \sum_{\nu=0}^{\infty} \frac{t^\nu \mathcal{H}_{\text{NWA},\lambda,\nu,q,\omega}(1)}{\{\nu\}_q!} \overset{\text{by}(6.9)}{=} \sum_{\nu=0}^{\infty} \frac{t^\nu \mathcal{H}_{\text{NWA},\lambda,\nu,q,\omega}}{\{\nu\}_q!}.\end{aligned} \tag{6.11}$$

Equating the coefficients of t^ν gives (6.10). □

Theorem 6.8. *A complementary argument theorem for the generalized q,ω-$\mathcal{H}$ polynomials.*

$$\mathcal{H}^{(n)}_{\text{JHC},\lambda^{-1},\nu,q,\omega}(x) = (-1)^{\nu+n}\lambda^n \mathcal{H}^{(n)}_{\text{NWA},\lambda,\nu,q,\omega}(\overline{n}_{q,\omega} \ominus_{q,\omega} x). \tag{6.12}$$

Theorem 6.9. *A q,ω-analogue of [13, (2.3) p. 5], first multiplication formula for q,ω-Apostol-$\mathcal{E}$ polynomials*

$$\begin{aligned}\mathcal{E}^{(n)}_{\text{NWA},\lambda,\mu,\theta;\nu,q,\omega}(\overline{m}_{q,\omega}x) = \frac{(\overline{m}_{q,\omega})^\nu}{((\overline{m}_{q,\omega})^\theta)^n}\sum_{|\vec{j}|=n}(-\lambda)^k\binom{n}{\vec{j}}\\ \mathcal{E}^{(n)}_{\text{NWA},\lambda^m,\mu,\theta;\nu,q,\omega}\left(x \oplus_{q,\omega} \frac{\overline{k}_{q,\omega}}{\overline{m}_{q,\omega}}\right),\end{aligned} \tag{6.13}$$

where $k = j_1 + 2j_2 + \cdots + (m-1)j_{m-1}$, m odd.

Proof.

$$\begin{aligned}&\sum_{\nu=0}^{\infty}\mathcal{E}^{(n)}_{\text{NWA},\lambda,\mu,\theta;\nu,q,\omega}(\overline{m}_{q,\omega}x)\frac{t^\nu}{\{\nu\}_q!} = \frac{(2^\mu t^\theta)^n}{(\lambda \mathrm{E}_{q,\omega}(t)+1)^n}\mathrm{E}_{q,\omega t}(\overline{m}_{q,\omega}xt)\\ &= \frac{(2^\mu t^\theta)^n}{(\lambda^m \mathrm{E}_{q,\omega}(\overline{m}_{q,\omega}t)+1)^n}\left(\sum_{i=0}^{m-1}(-\lambda)^i \mathrm{E}_{q,\omega}(\overline{i}_{q,\omega}t)\right)^n \mathrm{E}_{q,\omega t}(\overline{m}_{q,\omega}xt)\\ &\overset{\text{by}(3.18)}{=} \left(\frac{2^\mu t^\theta \overline{m}^\theta_{q,\omega}}{(\lambda^m \mathrm{E}_{q,\omega}(\overline{m}_{q,\omega}t)+1)}\right)^n \sum_{|\vec{j}|=n}\binom{n}{\vec{j}}(-\lambda)^k\\ &\mathrm{E}_{q,\omega}\left((x \oplus_{q,\omega} \frac{\overline{k}_{q,\omega}}{\overline{m}_{q,\omega}})\overline{m}_{q,\omega}t\right)\frac{1}{((\overline{m}_{q,\omega})^\theta)^n}\\ &= \sum_{\nu=0}^{\infty}\left(\frac{(\overline{m}_{q,\omega})^\nu}{((\overline{m}_{q,\omega})^\theta)^n}\sum_{|\vec{j}|=n}\binom{n}{\vec{j}}(-\lambda)^k\right.\\ &\left.\mathcal{E}^{(n)}_{\text{NWA},\lambda^m,\mu,\theta;\nu,q,\omega}\left(x \oplus_{q,\omega} \frac{\overline{k}_{q,\omega}}{\overline{m}_{q,\omega}}\right)\right)\frac{t^\nu}{\{\nu\}_q!}.\end{aligned} \tag{6.14}$$

The theorem follows by equating the coefficients of $\frac{t^\nu}{\{\nu\}_q!}$. ◻

The following formula only applies for special values of the integers.

Theorem 6.10. *A q,ω-analogue of [13, (2.4) p. 5], second multiplication formula for q,ω-Apostol-$\mathcal{E}$ polynomials.*

$$\begin{aligned}\mathcal{E}^{(n)}_{\text{NWA},\lambda,\mu,\theta;\nu,q,\omega}(\overline{m}_{q,\omega}x) = \frac{(-1)^n 2^{\mu n}(\overline{m}_{q,\omega})^{\nu+(1-\theta)n}}{\{\nu+1\}_{(1-\theta)n,q}(\overline{m}_{q,\omega})^n}\\ \times \sum_{|\vec{j}|=n}(-\lambda^k)\binom{n}{\vec{j}}\mathcal{B}^{(n)}_{\text{NWA},\lambda^m,\nu+(1-\theta)n,q,\omega}\left(x \oplus_{q,\omega} \frac{\overline{k}_{q,\omega}}{\overline{m}_{q,\omega}}\right),\end{aligned} \tag{6.15}$$

where $k = j_1 + 2j_2 + \cdots + (m-1)j_{m-1}$, m *even*, $\nu \geq (\theta-1)n$.

Proof.

$$\begin{aligned}
&\sum_{\nu=0}^{\infty} \mathcal{E}^{(n)}_{\mathrm{NWA},\lambda,\mu,\theta;\nu,q,\omega}(\overline{m}_{q,\omega}x)\frac{t^\nu}{\{\nu\}_q!} = \frac{(2^\mu t^\theta)^n}{(\lambda \mathrm{E}_{q,\omega}(t)+1)^n}\mathrm{E}_{q,\omega t}(\overline{m}_{q,\omega}xt) \\
&= \frac{(2^\mu t^\theta)^n}{(1-\lambda^m \mathrm{E}_{q,\omega}(\overline{m}_{q,\omega}t))^n}\left(\sum_{i=0}^{m-1}(-\lambda)^i \mathrm{E}_{q,\omega}(\overline{i}_{q,\omega}t)\right)^n \mathrm{E}_{q,\omega t}(\overline{m}_{q,\omega}xt) \\
&= \left(\frac{2^\mu t\overline{m}_{q,\omega}}{1-\lambda^m \mathrm{E}_{q,\omega}(\overline{m}_{q,\omega}t)}\right)^n \sum_{|\vec{j}|=n}\binom{n}{\vec{j}}(-\lambda)^k \qquad (6.16) \\
&\times \mathrm{E}_{q,\omega t}\left((x \oplus_{q,\omega} \frac{\overline{k}_{q,\omega}}{\overline{m}_{q,\omega}})\overline{m}_{q,\omega}t\right)\frac{t^{(\theta-1)n}}{(\overline{m}_{q,\omega})^n} = (-1)^n t^{(\theta-1)n}2^{\mu n} \\
&\sum_{\nu=0}^{\infty}\left(\frac{(\overline{m}_{q,\omega})^\nu}{(\overline{m}_{q,\omega})^n}\sum_{|\vec{j}|=n}\binom{n}{\vec{j}}(-\lambda)^k \mathcal{B}^{(n)}_{\mathrm{NWA},\lambda^m,\nu;q,\omega}\left(x \oplus_{q,\omega}\frac{\overline{k}_{q,\omega}}{\overline{m}_{q,\omega}}\right)\right)\frac{t^\nu}{\{\nu\}_q!}.
\end{aligned}$$

The theorem follows by equating the coefficients of $\dfrac{t^\nu}{\{\nu\}_q!}$. □

Corollary 6.11. *A q,ω-analogue of [17, (2.1) p. 49], [13, p.7], first multiplication formula for generalized q,ω-$\mathcal{H}$ polynomials.*

$$\begin{aligned}
&\mathcal{H}^{(n)}_{\mathrm{NWA},\lambda,\nu,q,\omega}(\overline{m}_{q,\omega}x) \\
&= \frac{(\overline{m}_{q,\omega})^\nu}{(\overline{m}_{q,\omega})^n}\sum_{|\vec{j}|=n}(-\lambda)^k\binom{n}{\vec{j}}\mathcal{H}^{(n)}_{\mathrm{NWA},\lambda^m,\nu,q,\omega}\left(x \oplus_{q,\omega}\frac{\overline{k}_{q,\omega}}{\overline{m}_{q,\omega}}\right),
\end{aligned} \qquad (6.17)$$

where $k = j_1 + 2j_2 + \cdots + (m-1)j_{m-1}$, m *odd.*

Corollary 6.12. *A q,ω-analogue of [17, (2.2) p. 49], [13, p.7], second multiplication formula for generalized q,ω-$\mathcal{H}$ polynomials.*

$$\begin{aligned}
&\mathcal{H}^{(n)}_{\mathrm{NWA},\lambda,\nu,q,\omega}(\overline{m}_{q,\omega}x) \\
&= \frac{(-2)^n(\overline{m}_{q,\omega})^\nu}{(\overline{m}_{q,\omega})^n}\sum_{|\vec{j}|=n}(-\lambda^k)\binom{n}{\vec{j}}\mathcal{B}^{(n)}_{\mathrm{NWA},\lambda^m,\nu,q,\omega}\left(x \oplus_{q,\omega}\frac{\overline{k}_{q,\omega}}{\overline{m}_{q,\omega}}\right),
\end{aligned} \qquad (6.18)$$

where $k = j_1 + 2j_2 + \cdots + (m-1)j_{m-1}$, m *even.*

Theorem 6.13. *A q,ω-analogue of [17, p.51], an explicit formula for the multiple al-*

ternating q,ω-power sum:

$$\sigma^{(l)}_{\mathrm{NWA},\lambda,\nu,q,\omega}(n)=2^{-l}\sum_{j=0}^{l}\binom{l}{j}\frac{(-1)^{jn}\lambda^{(n-1)j+l}}{\{\nu+1\}_{l,q}} \times\sum_{m=0}^{\nu+l}\binom{\nu+l}{m}_q \mathcal{H}^{(j)}_{\mathrm{NWA},\lambda,m,q,\omega}\left(\overline{(n-1)j+l}_{q,\omega}\right)\mathcal{H}^{(l-j)}_{\mathrm{NWA},\lambda,\nu+l-m,m,q,\omega}. \tag{6.19}$$

Proof. We use the generating function technique. Put $k=j_1+2j_2+\cdots+(n-1)j_{n-1}$. It is assumed that $j_i\geq 0, 1\leq i\leq n-1$. All zeros are neglected.

$$\begin{aligned}
&\sum_{\nu=0}^{\infty}\sigma^{(l)}_{\mathrm{NWA},\lambda,\nu,q,\omega}(n)\frac{t^\nu}{\{\nu\}_q!}\overset{\text{by}(4.1)}{=}(-1)^l\sum_{\nu=0}^{\infty}\left(\sum_{|\vec{j}|=l}\binom{l}{\vec{j}}(-\lambda)^k\left(\overline{k}_{q,\omega}\right)^\nu\right)\frac{t^\nu}{\{\nu\}_q!}\\
&=\left(\lambda\mathrm{E}_{q,\omega}(t)-\lambda^2\mathrm{E}_{q,\omega}(\overline{2}_{q,\omega}t)+\cdots+(-1)^n\lambda^{n-1}\mathrm{E}_{q,\omega}(\overline{n-1}_{q,\omega}t)\right)^l\\
&=\left(\frac{(-\lambda)^n\mathrm{E}_{q,\omega}(\overline{n}_{q,\omega}t)}{\lambda\mathrm{E}_{q,\omega}(t)+1}+\frac{\lambda\mathrm{E}_{q,\omega}(t)}{\lambda\mathrm{E}_{q,\omega}(t)+1}\right)^l\\
&=\sum_{j=0}^{l}\binom{l}{j}\left(\frac{(-\lambda)^n\mathrm{E}_{q,\omega}(\overline{n}_{q,\omega}t)}{\lambda\mathrm{E}_{q,\omega}(t)+1}\right)^j\left(\frac{\lambda\mathrm{E}_{q,\omega}(t)}{\lambda\mathrm{E}_{q,\omega}(t)+1}\right)^{l-j}\\
&\overset{\text{by}(3.18)}{=}(2t)^{-l}\sum_{j=0}^{l}\binom{l}{j}(-1)^{jn}\lambda^{(n-1)j+l}\sum_{m=0}^{\infty}\mathcal{H}^{(j)}_{\mathrm{NWA},\lambda,m,q,\omega}\left(\overline{(n-1)j+l}_{q,\omega}\right)\\
&\times\frac{t^m}{\{m\}_q!}\sum_{i=0}^{\infty}\mathcal{H}^{(l-j)}_{\mathrm{NWA},\lambda,i,q,\omega}\frac{t^i}{\{i\}_q!}=\sum_{\nu=0}^{\infty}\left[2^{-l}\sum_{j=0}^{l}\binom{l}{j}\frac{(-1)^{jn}\lambda^{(n-1)j+l}}{\{\nu+1\}_{l,q}}\right.\\
&\left.\sum_{m=0}^{\nu+l}\binom{\nu+l}{m}_q\mathcal{H}^{(j)}_{\mathrm{NWA},\lambda,m,q,\omega}\left(\overline{(n-1)j+l}_{q,\omega}\right)\mathcal{H}^{(l-j)}_{\mathrm{NWA},\lambda,\nu+l-m,q,\omega}\right]\frac{t^\nu}{\{\nu\}_q!}.
\end{aligned}\tag{6.20}$$

The theorem follows by equating the coefficients of $\frac{t^\nu}{\{\nu\}_q!}$. □

Theorem 6.14. *For m odd, we have the following recurrence relation for q,ω-Apostol-$\mathcal{E}$-numbers.*

$$\mathcal{E}^{(l)}_{\mathrm{NWA},\lambda,\mu,\theta;n,q,\omega}=(-1)^l\sum_{j=0}^{n}\binom{n}{j}_q\frac{(\overline{m}_{q,\omega})^n}{((\overline{m}_{q,\omega})^\theta)^l(\overline{m}_{q,\omega})^{n-j}} \mathcal{E}^{(l)}_{\mathrm{NWA},\lambda^m,\mu,\theta;j,q,\omega}\sigma^{(l)}_{\mathrm{NWA},\lambda,n-j,q,\omega}(m), \tag{6.21}$$

where $k=j_1+2j_2+\cdots+(m-1)j_{m-1}$ in $\sigma^{(l)}_{\mathrm{NWA},\lambda,n-j,q,\omega}(m)$.

Proof.

$$\begin{aligned}
&\mathcal{E}^{(l)}_{\mathrm{NWA},\lambda,\mu,\theta;n,q,\omega} \overset{\mathrm{by}(6.13)}{=} \frac{(\overline{m}_{q,\omega})^n}{((\overline{m}_{q,\omega})^\theta)^l} \sum_{|\vec{\nu}|=l} (-\lambda)^k \binom{l}{\vec{\nu}} \mathcal{E}^{(l)}_{\mathrm{NWA},\lambda^m,\mu,\theta;n,q,\omega}\left(\frac{\overline{k}_{q,\omega}}{\overline{m}_{q,\omega}}\right)\\
&= \frac{(\overline{m}_{q,\omega})^n}{((\overline{m}_{q,\omega})^\theta)^l} \sum_{|\vec{\nu}|=l} (-\lambda)^k \binom{l}{\vec{\nu}} \sum_{j=0}^{n} \binom{n}{j}_q \mathcal{E}^{(l)}_{\mathrm{NWA},\lambda^m,\mu,\theta;j,q,\omega}\left(\frac{\overline{k}_{q,\omega}}{\overline{m}_{q,\omega}}\right)^{n-j}\\
&= \sum_{j=0}^{n} \binom{n}{j}_q \frac{(\overline{m}_{q,\omega})^n}{(\overline{m}_{q,\omega})^{n-j}((\overline{m}_{q,\omega})^\theta)^l}\\
&\mathcal{E}^{(l)}_{\mathrm{NWA},\lambda^m,\mu,\theta;j,q,\omega} \sum_{|\vec{\nu}|=l} (-\lambda)^k \binom{l}{\vec{\nu}} (\overline{k}_{q,\omega})^{n-j} \overset{\mathrm{by}(4.1)}{=} \mathrm{LHS}.
\end{aligned} \tag{6.22}$$

□

Definition 6.15. The generating function for the generalized NWA q, ω-Apostol $\mathcal{C}$ polynomials of degree ν and order n, $\mathcal{C}^{(n)}_{\mathrm{NWA},\lambda,\theta;\nu,q,\omega}(x)$, is given by

$$\left(\frac{t^\theta}{\lambda \mathrm{E}_{q,\omega}(t) - 1}\right)^n \mathrm{E}_{q,\omega t}(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \mathcal{C}^{(n)}_{\mathrm{NWA},\lambda,\theta;\nu,q,\omega}(x), \theta \in \mathbb{N}. \tag{6.23}$$

Theorem 6.16. *Multiplication formula for q, ω-Apostol-$\mathcal{C}$ polynomials*

$$\begin{aligned}
&\mathcal{C}^{(n)}_{\mathrm{NWA},\lambda,\theta;\nu,q,\omega}(\overline{m}_{q,\omega} x)\\
&= \frac{(\overline{m}_{q,\omega})^\nu}{((\overline{m}_{q,\omega})^\theta)^n} \sum_{|\vec{j}|=n} \lambda^k \binom{n}{\vec{j}} \mathcal{C}^{(n)}_{\mathrm{NWA},\lambda^m,\theta;\nu,q,\omega}\left(x \oplus_{q,\omega} \frac{\overline{k}_{q,\omega}}{\overline{m}_{q,\omega}}\right),
\end{aligned} \tag{6.24}$$

where $k = j_1 + 2j_2 + \cdots + (m-1)j_{m-1}$.

Proof.

$$\begin{aligned}
&\sum_{\nu=0}^{\infty} \mathcal{C}^{(n)}_{\mathrm{NWA},\lambda,\theta;\nu,q,\omega}(\overline{m}_{q,\omega} x) \frac{t^\nu}{\{\nu\}_q!} \overset{\mathrm{by}(6.23)}{=} \frac{t^{\theta n}}{(\lambda \mathrm{E}_{q,\omega}(t) - 1)^n} \mathrm{E}_{q,\omega t}(\overline{m}_{q,\omega} xt)\\
&= \frac{t^{\theta n}}{(\lambda^m \mathrm{E}_{q,\omega}(\overline{m}_{q,\omega} t) - 1)^n} \left(\sum_{i=0}^{m-1} \lambda^i \mathrm{E}_{q,\omega}(\overline{i}_{q,\omega} t)\right)^n \mathrm{E}_{q,\omega t}(\overline{m}_{q,\omega} xt)\\
&\overset{\mathrm{by}(3.18)}{=} \left(\frac{t^\theta \overline{m}^\theta_{q,\omega}}{(\lambda^m \mathrm{E}_{q,\omega}(\overline{m}_{q,\omega} t) - 1)}\right)^n \sum_{|\vec{j}|=n} \binom{n}{\vec{j}} \lambda^k\\
&\mathrm{E}_{q,\omega t}\left((x \oplus_{q,\omega} \frac{\overline{k}_{q,\omega}}{\overline{m}_{q,\omega}}) \overline{m}_{q,\omega} t\right) \frac{1}{((\overline{m}_{q,\omega})^\theta)^n}\\
&\overset{\mathrm{by}(6.23)}{=} \sum_{\nu=0}^{\infty} \left(\frac{(\overline{m}_{q,\omega})^\nu}{((\overline{m}_{q,\omega})^\theta)^n} \sum_{|\vec{j}|=n} \binom{n}{\vec{j}} \lambda^k \mathcal{C}^{(n)}_{\mathrm{NWA},\lambda^m,\theta;\nu,q,\omega}\left(x \oplus_{q,\omega} \frac{\overline{k}_{q,\omega}}{\overline{m}_{q,\omega}}\right)\right) \frac{t^\nu}{\{\nu\}_q!}.
\end{aligned} \tag{6.25}$$

The theorem follows by equating the coefficients of $\frac{t^\nu}{\{\nu\}_q!}$. □

7 The q,ω-H Polynomials

Definition 7.1. The generating function for $\mathrm{H}^{(n)}_{\mathrm{NWA},\nu,q,\omega}(x)$ is given by

$$\frac{(2t)^n}{(\mathrm{E}_{q,\omega}(t)+1)^n}\mathrm{E}_{q,\omega t}(xt)=\sum_{\nu=0}^{\infty}\frac{t^\nu \mathrm{H}^{(n)}_{\mathrm{NWA},\nu,q,\omega}(x)}{\{\nu\}_q!},\ |t|<2\pi. \tag{7.1}$$

Definition 7.2. The generating function for $\mathrm{H}^{(n)}_{\mathrm{JHC},\nu,q,\omega}(x)$ is given by

$$\frac{(2t)^n}{(\mathrm{E}_{\frac{1}{q},\omega}(t)+1)^n}\mathrm{E}_{q,\omega t}(xt)=\sum_{\nu=0}^{\infty}\frac{t^\nu \mathrm{H}^{(n)}_{\mathrm{JHC},\nu,q,\omega}(x)}{\{\nu\}_q!},\ |t|<2\pi. \tag{7.2}$$

The polynomials in (7.1) and (7.2) are q,ω-analogues of the generalized H polynomials.

Corollary 7.3. *A relation between generalized H-q,ω polynomials and generalized q,ω-Euler polynomials.*

$$\mathrm{F}^{(n)}_{\mathrm{NWA},\nu,q,\omega}(x)=\frac{\{\nu\}_q!}{\{\nu+n\}_q!}\mathrm{H}^{(n)}_{\mathrm{NWA},\nu+n,q,\omega}(x),\ \mathrm{NWA}=\mathrm{NWA}\vee\mathrm{JHC}. \tag{7.3}$$

We prove this later in formula (9.19).

Theorem 7.4. *We have*

$$\nabla_{\mathrm{NWA},q,\omega}\mathrm{H}^{(n)}_{\mathrm{NWA},\nu,q,\omega}(x)=\{\nu\}_q\mathrm{H}^{(n-1)}_{\mathrm{NWA},\nu-1,q,\omega}(x),\ \mathrm{NWA}=\mathrm{NWA}\vee\mathrm{JHC}. \tag{7.4}$$

The following recurrences hold:

$$\mathrm{H}_{\mathrm{NWA},0,q,\omega}=0,\ \mathrm{H}_{\mathrm{NWA},1,q,\omega}=1,\ (\mathrm{H}_{\mathrm{NWA},q,\omega}\oplus_{q,\omega}\ 1)^k+\mathrm{H}_{\mathrm{NWA},k,q,\omega}\doteq 0, k>1. \tag{7.5}$$

$$\mathrm{H}_{\mathrm{JHC},0,q,\omega}=0,\ \mathrm{H}_{\mathrm{JHC},1,q,\omega}=1,\ (\mathrm{H}_{\mathrm{JHC},q,\omega}\boxplus_{q,\omega}\ 1)^k+\mathrm{H}_{\mathrm{JHC},k,q,\omega}\doteq 0, k>1. \tag{7.6}$$

We need not calculate the $\mathrm{H}_{\mathrm{JHC},\nu,q,\omega}$ numbers, since we have the following symmetry relations:

Theorem 7.5.

$$\begin{aligned}&\text{For } \nu \text{ even}, \mathrm{H}_{\mathrm{NWA},\nu,q,\omega}=\mathrm{H}_{\mathrm{JHC},\nu,q,\omega}.\\ &\text{For } \nu \text{ odd},\ \mathrm{H}_{\mathrm{NWA},\nu,q,\omega}=-\mathrm{H}_{\mathrm{JHC},\nu,q,\omega},\ \nu>1.\end{aligned} \tag{7.7}$$

Lemma 7.6. *Two q,ω-analogues of [18, (18)].*

$$(x)_{q,\omega}^{\nu}=\frac{1}{2\{\nu+1\}_q}\left[\mathrm{H}_{\mathrm{NWA},\nu+1,q,\omega}(x)+\sum_{k=0}^{\nu+1}\binom{\nu+1}{k}_q\mathrm{H}_{\mathrm{NWA},k,q,\omega}(x)(1)_{q,\omega}^{\nu+1-k}\right].\tag{7.8}$$

$$(x)_{q,\omega}^{\nu}=\frac{1}{2\{\nu+1\}_q}\left[\mathrm{H}_{\mathrm{JHC},\nu+1,q,\omega}(x)+\sum_{k=0}^{\nu+1}\binom{\nu+1}{k}_q\mathrm{H}_{\mathrm{JHC},k,q,\omega}(x)[1]_{q,\omega}^{\nu+1-k}\right].\tag{7.9}$$

Proof. Use formula (7.4). □

Theorem 7.7. *A q,ω-analogue of [18, p.489]. Let $\Phi_{\nu,q,\omega}^{(n)}(x)$ be a q,ω-Appell polynomial with x-order n and y-order 0. Then the following three q,ω-addition formulas hold:*

$$\Phi_{\nu,q,\omega}^{(n)}(x\oplus_{q,\omega}y)=\sum_{k=0}^{\nu}\binom{\nu}{k}_q\Phi_{\nu-k,q,\omega}^{(n)}(x)\frac{1}{\{k+1\}_q}\left[\sum_{m=0}^{k+1}\binom{k+1}{m}_q\mathrm{B}_{\mathrm{NWA},m,q,\omega}(y)(1)_{q,\omega}^{k+1-m}-\mathrm{B}_{\mathrm{NWA},k+1,q,\omega}(y)\right].\tag{7.10}$$

$$\Phi_{\nu,q,\omega}^{(n)}(x\oplus_{q,\omega}y)=\frac{1}{2}\sum_{k=0}^{\nu}\binom{\nu}{k}_q\Phi_{\nu-k,q,\omega}^{(n)}(x)\times\left[\mathrm{F}_{\mathrm{NWA},k,q,\omega}(y)+\sum_{m=0}^{k}\binom{k}{m}_q\mathrm{F}_{\mathrm{NWA},m,q,\omega}(y)(1)_{q,\omega}^{k-m}\right].\tag{7.11}$$

$$\Phi_{\nu,q,\omega}^{(n)}(x\oplus_{q,\omega}y)=\sum_{k=0}^{\nu}\binom{\nu}{k}_q\Phi_{\nu-k,q,\omega}^{(n)}(x)\frac{1}{2\{k+1\}_q}\left[\mathrm{H}_{\mathrm{NWA},k+1,q,\omega}(y)+\sum_{m=0}^{k+1}\binom{k+1}{m}_q\mathrm{H}_{\mathrm{NWA},m,q,\omega}(y)(1)_{q,\omega}^{k+1-m}\right].\tag{7.12}$$

Proof. Use formulas [9]

$$(x)_{q,\omega}^{n}=\frac{1}{\{n+1\}_q}\left[\sum_{k=0}^{n+1}\binom{n+1}{k}_q\mathrm{B}_{\mathrm{NWA},k,q,\omega}(x)(1)_{q,\omega}^{n+1-k}-\mathrm{B}_{\mathrm{NWA},n+1,q,\omega}(x)\right],\tag{7.13}$$

(7.9) and (5.10). □

Theorem 7.8. *A complementary argument theorem, a q, ω-analogue of [20, p.532]:*

$$\mathrm{H}^{(n)}_{\mathrm{JHC},\nu,q,\omega}(x) = (-1)^{\nu+n}\mathrm{H}^{(n)}_{\mathrm{NWA},\nu,q,\omega}(\overline{n}_{q,\omega} \ominus_{q,\omega} x). \tag{7.14}$$

Proof. Use the generating function. □

Corollary 7.9. *A q, ω-analogue of a generalization of [20, p.532].*

$$\mathrm{H}^{(n)}_{\mathrm{NWA},\nu,q,\omega}(x) + (-1)^{\nu+n}\mathrm{H}^{(n)}_{\mathrm{JHC},\nu,q,\omega}(-x) = 2\{\nu\}_q\mathrm{H}^{(n-1)}_{\mathrm{NWA},\nu-1,q,\omega}(x). \tag{7.15}$$

Proof. Use the generating function and formulas (7.4), (7.14). □

8 q, ω-Apostol–Bernoulli Polynomials

8.1 NWA q, ω-Apostol–Bernoulli Polynomials

Throughout, we assume that $\lambda \ni \{0, 1\}$.

Definition 8.1. The generalized NWA q, ω-Apostol–Bernoulli polynomials $\mathcal{B}^{(n)}_{\mathrm{NWA},\lambda,\nu,q,\omega}(x)$ are defined by

$$\frac{t^n}{(\lambda\mathrm{E}_{q,\omega}(t) - 1)^n}\mathrm{E}_{q,\omega t}(xt) = \sum_{\nu=0}^{\infty}\frac{t^\nu\mathcal{B}^{(n)}_{\mathrm{NWA},\lambda,\nu,q,\omega}(x)}{\{\nu\}_q!}, \ |t + \log\lambda| < 2\pi. \tag{8.1}$$

The poles in the denominator of (8.1) are the roots of $\mathrm{E}_{q,\omega}(t) = \lambda^{-1}$.
We have

$$\triangle_{\mathrm{NWA},\mathcal{A},q,\omega}\mathcal{B}^{(n)}_{\mathrm{NWA},\lambda,\nu,q,\omega}(x) = \{\nu\}_q\mathcal{B}^{(n-1)}_{\mathrm{NWA},\lambda,\nu-1,q,\omega}(x) = \mathrm{D}_{q,\omega}\mathcal{B}^{(n-1)}_{\mathrm{NWA},\lambda,\nu,q,\omega}(x). \tag{8.2}$$

This leads to the following recurrence for the NWA q, ω-Apostol–Bernoulli numbers:

$$\mathcal{B}_{\mathrm{NWA},\lambda,0,q,\omega} = 0, \ \lambda(\mathcal{B}_{\mathrm{NWA},\lambda,q,\omega} \oplus_{q,\omega} 1)^k - \mathcal{B}_{\mathrm{NWA},\lambda,k,q,\omega} \doteq \delta_{1,k}, \ k > 0. \tag{8.3}$$

These numbers are related to the Apostol–Bernoulli numbers [2], rather than to the q-Bernoulli numbers. For a table of the NWA q-Apostol–Bernoulli numbers, see [7].

Corollary 8.2. *A q, ω-analogue of [15, p.634, (28), (29)]:*

$$\mathcal{B}^{(n-1)}_{\mathrm{NWA},\lambda,\nu,q,\omega}(x) = \frac{1}{\{\nu+1\}_q} \left[\lambda\sum_{k=0}^{\nu+1}\binom{\nu+1}{k}_q\mathcal{B}^{(n)}_{\mathrm{NWA},\lambda,k,q,\omega}(x)(1)^{\nu+1-k}_{q,\omega} - \mathcal{B}^{(n)}_{\mathrm{NWA},\lambda,\nu+1,q,\omega}(x)\right]. \tag{8.4}$$

A generalization of [6, 4.149].

$$(x)_{q,\omega}^{\nu} = \frac{1}{\{\nu+1\}_q} \left[\lambda \sum_{k=0}^{\nu+1} \binom{\nu+1}{k}_q \mathcal{B}_{\text{NWA},\lambda,k,q,\omega}(x)(1)_{q,\omega}^{\nu+1-k} - \mathcal{B}_{\text{NWA},\lambda,\nu+1,q,\omega}(x)\right]. \tag{8.5}$$

We note that the limits $q \to 1$ and $\omega \to 0$ can be taken anywhere in the following theorems; see the subsequent corollaries.

8.2 Multiplication Formulas for q,ω-Apostol–Bernoulli Polynomials

Theorem 8.3. *A q,ω-analogue of [14, p.380], multiplication formula for q,ω-Apostol–Bernoulli polynomials.*

$$\mathcal{B}_{\text{NWA},\lambda,\nu,q,\omega}^{(n)}(\overline{m}_{q,\omega}x) = \frac{(\overline{m}_{q,\omega})^{\nu}}{(\overline{m}_{q,\omega})^{n}} \sum_{|\vec{j}|=n} \lambda^k \binom{n}{\vec{j}} \mathcal{B}_{\text{NWA},\lambda^m,\nu,q,\omega}^{(n)}\left(x \oplus_{q,\omega} \frac{\overline{k}_{q,\omega}}{\overline{m}_{q,\omega}}\right), \tag{8.6}$$

where $k = j_1 + 2j_2 + \cdots + (m-1)j_{m-1}$, *and* $\frac{\overline{k}_{q,\omega}}{\overline{m}_{q,\omega}} \in \mathbb{Q}_{\oplus_{q,\omega}}$.

Proof. This follows from (6.24). □

Corollary 8.4. *A q,ω-analogue of [14, p.381]:*

$$\mathcal{B}_{\text{NWA},\lambda,\nu,q,\omega}(\overline{m}_{q,\omega}x) = \frac{(\overline{m}_{q,\omega})^{\nu}}{m} \sum_{j=0}^{m-1} \lambda^j \mathcal{B}_{\text{NWA},\lambda^m,\nu,q,\omega}\left(x \oplus_{q,\omega} \frac{\overline{j}_{q,\omega}}{\overline{m}_{q,\omega}}\right). \tag{8.7}$$

Proof. Put $n = 1$ in (8.6). □

Corollary 8.5. *A q,ω-analogue of Carlitz formula [3], [14, p.381]*

$$\mathcal{B}_{\text{NWA},\nu,q,\omega}^{(n)}(\overline{m}_{q,\omega}x) = \frac{(\overline{m}_{q,\omega})^{\nu}}{(\overline{m}_{q,\omega})^{n}} \sum_{|\vec{j}|=n} \binom{n}{\vec{j}} \mathcal{B}_{\text{NWA},\nu,q,\omega}^{(n)}\left(x \oplus_{q,\omega} \frac{\overline{k}_{q,\omega}}{\overline{m}_{q,\omega}}\right), \tag{8.8}$$

where $k = j_1 + 2j_2 + \cdots + (m-1)j_{m-1}$, *and* $\frac{\overline{k}_{q,\omega}}{\overline{m}_{q,\omega}} \in \mathbb{Q}_{\oplus_{q,\omega}}$.

Proof. Put $\lambda = 1$ in (8.6). □

Theorem 8.6. *A formula for a multiple q,ω-power sum, a q,ω-analogue of [14, (25) p. 382]:*

$$s^{(l)}_{\mathrm{NWA},\lambda,m,q,\omega}(n) = \sum_{j=0}^{l} \binom{l}{j} \frac{(-1)^{l-j}\lambda^{(n-1)j+l}}{\{m+1\}_{l,q}} \left(\sum_{k=0}^{m+l} \binom{m+l}{k}_q \mathcal{B}^{(j)}_{\mathrm{NWA},\lambda,k,q,\omega}\left(\overline{(n-1)j+l}_{q,\omega}\right) \mathcal{B}^{(l-j)}_{\mathrm{NWA},\lambda,m+l-k,q,\omega} \right). \tag{8.9}$$

Proof. We use the generating function technique. Put $k = j_1 + 2j_2 + \cdots + (n-1)j_{n-1}$. It is assumed that $j_i \geq 0, 1 \leq i \leq n-1$, zeros are neglected.

$$\begin{aligned}
&\sum_{\nu=0}^{\infty} s^{(l)}_{\mathrm{NWA},\lambda,\nu,q,\omega}(n) \frac{t^\nu}{\{\nu\}_q!} \overset{\text{by}(4.1)}{=} \sum_{\nu=0}^{\infty} \left(\sum_{|\vec{j}|=l} \binom{l}{\vec{j}} \lambda^k \left(\overline{k}_{q,\omega}\right)^\nu \right) \frac{t^\nu}{\{\nu\}_q!} \\
&\overset{\text{by}(4.1)}{=} \left(\lambda \mathrm{E}_{q,\omega}(t) + \lambda^2 \mathrm{E}_{q,\omega}(\overline{2}_{q,\omega} t) + \cdots + \lambda^{n-1} \mathrm{E}_{q,\omega}(\overline{n-1}_{q,\omega} t)\right)^l \\
&= \left(\frac{\lambda^n \mathrm{E}_{q,\omega}(\overline{n}_{q,\omega} t)}{\lambda \mathrm{E}_{q,\omega}(t) - 1} - \frac{\lambda \mathrm{E}_{q,\omega}(t)}{\lambda \mathrm{E}_{q,\omega}(t) - 1} \right)^l \\
&= \sum_{j=0}^{l} \binom{l}{j} (-1)^{l-j} \left(\frac{\lambda^n \mathrm{E}_{q,\omega}(\overline{n}_{q,\omega} t)}{\lambda \mathrm{E}_{q,\omega}(t) - 1} \right)^j \left(\frac{\lambda \mathrm{E}_{q,\omega}(t)}{\lambda \mathrm{E}_{q,\omega}(t) - 1} \right)^{l-j} \\
&\overset{\text{by}(3.18)}{=} t^{-l} \sum_{j=0}^{l} \binom{l}{j} (-1)^{l-j} \lambda^{(n-1)j+l} \sum_{k=0}^{\infty} \mathcal{B}^{(j)}_{\mathrm{NWA},\lambda,k,q,\omega}\left(\overline{(n-1)j+l}_{q,\omega}\right) \frac{t^k}{\{k\}_q!} \\
&\sum_{i=0}^{\infty} \mathcal{B}^{(l-j)}_{\mathrm{NWA},\lambda,i,q,\omega} \frac{t^i}{\{i\}_q!} = \sum_{\nu=0}^{\infty} \left[\sum_{j=0}^{l} \binom{l}{j} \frac{(-1)^{l-j}\lambda^{(n-1)j+l}}{\{m+1\}_{l,q}} \right. \\
&\left. \sum_{k=0}^{m+l} \binom{m+l}{k}_q \mathcal{B}^{(j)}_{\mathrm{NWA},\lambda,k,q,\omega}\left(\overline{(n-1)j+l}_{q,\omega}\right) \mathcal{B}^{(l-j)}_{\mathrm{NWA},\lambda,m+l-k,q,\omega} \right] \frac{t^\nu}{\{\nu\}_q!}.
\end{aligned} \tag{8.10}$$

The theorem follows by equating the coefficients of $\frac{t^\nu}{\{\nu\}_q!}$. □

Corollary 8.7. *A q,ω-analogue of [14, (26) p. 382]: The generating function for $s^{(l)}_{\mathrm{NWA},\lambda,\nu,q,\omega}(n)$ is*

$$\begin{aligned}
&\sum_{\nu=0}^{\infty} s^{(l)}_{\mathrm{NWA},\lambda,\nu,q,\omega}(n) \frac{t^\nu}{\{\nu\}_q!} = \left(\frac{\lambda^n \mathrm{E}_{q,\omega}(\overline{n}_{q,\omega} t)}{\lambda \mathrm{E}_{q,\omega}(t) - 1} - \frac{\lambda \mathrm{E}_{q,\omega}(t)}{\lambda \mathrm{E}_{q,\omega}(t) - 1} \right)^l \\
&= \left(\lambda \mathrm{E}_{q,\omega}(t) + \lambda^2 \mathrm{E}_{q,\omega}(\overline{2}_{q,\omega} t) + \cdots + \lambda^{n-1} \mathrm{E}_{q,\omega}(\overline{n-1}_{q,\omega} t)\right)^l.
\end{aligned} \tag{8.11}$$

Theorem 8.8. *A recurrence relation for q,ω-Apostol–Bernoulli numbers, a q,ω-analogue of [14, (32) p. 384].*

$$(\overline{m}_{q,\omega})^l \mathcal{B}^{(l)}_{\text{NWA},\lambda,n,q,\omega} = \sum_{j=0}^{n} \binom{n}{j}_q \frac{(\overline{m}_{q,\omega})^n}{(\overline{m}_{q,\omega})^{n-j}} \mathcal{B}^{(l)}_{\text{NWA},\lambda^m,j,q,\omega} s^{(l)}_{\text{NWA},\lambda,n-j,q,\omega}(m), \tag{8.12}$$

where $k = j_1 + 2j_2 + \cdots + (m-1)j_{m-1}$ in $s^{(l)}_{\text{NWA},\lambda,n-j,q,\omega}(m)$.

Proof. We use the definition of q,ω-Appell numbers as q,ω-Appell polynomial at $x = 0$.

$$\begin{aligned}
&(\overline{m}_{q,\omega})^l \mathcal{B}^{(l)}_{\text{NWA},\lambda,n,q,\omega} \overset{\text{by}(8.6)}{=} (\overline{m}_{q,\omega})^n \sum_{|\vec{\nu}|=l} \lambda^k \binom{l}{\vec{\nu}} \mathcal{B}^{(l)}_{\text{NWA},\lambda^m,n,q,\omega}\left(\frac{\overline{k}_{q,\omega}}{\overline{m}_{q,\omega}}\right) \\
&= (\overline{m}_{q,\omega})^n \sum_{|\vec{\nu}|=l} \lambda^k \binom{l}{\vec{\nu}} \sum_{j=0}^{n} \binom{n}{j}_q \mathcal{B}^{(l)}_{\text{NWA},\lambda^m,j,q,\omega}\left(\frac{\overline{k}_{q,\omega}}{\overline{m}_{q,\omega}}\right)^{n-j} \\
&= \sum_{j=0}^{n} \binom{n}{j}_q \frac{(\overline{m}_{q,\omega})^n}{(\overline{m}_{q,\omega})^{n-j}} \mathcal{B}^{(l)}_{\text{NWA},\lambda^m,j,q,\omega} \sum_{|\vec{\nu}|=l} \lambda^k \binom{l}{\vec{\nu}} (\overline{k}_{q,\omega})^{n-j} \overset{\text{by}(4.1)}{=} \text{LHS}.
\end{aligned} \tag{8.13}$$

□

8.3 JHC q,ω-Apostol–Bernoulli Polynomials

Definition 8.9. The generalized JHC q,ω-Apostol–Bernoulli polynomials $\mathcal{B}^{(n)}_{\text{JHC},\lambda,\nu,q,\omega}(x)$ are defined by

$$\frac{t^n}{(\lambda \text{E}_{\frac{1}{q},\omega}(t) - 1)^n} \text{E}_{q,\omega t}(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu \mathcal{B}^{(n)}_{\text{JHC},\lambda,\nu,q,\omega}(x)}{\{\nu\}_q!}, \; |t + \log\lambda| < 2\pi. \tag{8.14}$$

We have

$$\triangle_{\text{JHC},\mathcal{A},q,\omega} \mathcal{B}^{(n)}_{\text{JHC},\lambda,\nu,q,\omega}(x) = \{\nu\}_q \mathcal{B}^{(n-1)}_{\text{JHC},\lambda,\nu-1,\omega}(x) = \text{D}_{q,\omega} \mathcal{B}^{(n-1)}_{\text{JHC},\lambda,\nu,\omega}(x). \tag{8.15}$$

This leads to the following recurrence:

Theorem 8.10.

$$\mathcal{B}_{\text{JHC},\lambda,0,q,\omega} = 0, \; \lambda(\mathcal{B}_{\text{JHC},\lambda,q,\omega} \boxplus_{q,\omega} 1)^k - \mathcal{B}_{\text{JHC},\lambda,k,q,\omega} \doteqdot \delta_{1,k}, \; k > 0. \tag{8.16}$$

We need not calculate the $\mathcal{B}_{\text{JHC},\lambda,\nu,q,\omega}$ numbers, since we have the following symmetry relation:

Theorem 8.11.

$$(-1)^{\nu}\mathcal{B}_{\mathrm{JHC},\lambda^{-1},\nu,q,\omega} = \mathcal{B}_{\mathrm{NWA},\lambda,\nu,q,\omega},\ \nu > 0. \tag{8.17}$$

Proof. We use the definition of the generating functions.

$$\begin{aligned}&\sum_{\nu=0}^{\infty}\frac{(-t)^{\nu}\mathcal{B}_{\mathrm{JHC},\lambda^{-1},\nu,q,\omega}}{\{\nu\}_q!} = \frac{t}{-\lambda^{-1}\mathrm{E}_{\frac{1}{q}}(-t)+1} = \frac{t\lambda\mathrm{E}_q(t)}{\lambda\mathrm{E}_q(t)-1} \overset{\text{by}(8.1),\ n=1}{=} \\ &\lambda\sum_{\nu=0}^{\infty}\frac{t^{\nu}\mathcal{B}_{\mathrm{NWA},\lambda,\nu,q,\omega}(1)}{\{\nu\}_q!} \overset{\text{by}(8.3)}{=} \sum_{\nu=1}^{\infty}\frac{t^{\nu}\mathcal{B}_{\mathrm{NWA},\lambda,\nu,q,\omega}}{\{\nu\}_q!}.\end{aligned} \tag{8.18}$$

Equating the coefficients of $\frac{t^{\nu}}{\{\nu\}_q!}$ gives (8.17). □

Theorem 8.12. *A q,ω-analogue of the complementary argument theorem [15, p.633, (19)].*

$$\mathcal{B}^{(n)}_{\mathrm{JHC},\lambda^{-1},\nu,q,\omega}(x) = (-1)^{\nu}\lambda^{n}\mathcal{B}^{(n)}_{\mathrm{NWA},\lambda,\nu,q,\omega}(\overline{n}_{q,\omega}\ominus_{q,\omega} x). \tag{8.19}$$

Corollary 8.13. *Another q,ω-analogue of [15, p.634, (28), (29)]:*

$$\begin{aligned}&\mathcal{B}^{(n-1)}_{\mathrm{JHC},\lambda,\nu,q,\omega}(x) = \frac{1}{\{\nu+1\}_q} \\ &\left[\lambda\sum_{k=0}^{\nu+1}\binom{\nu+1}{k}_q\mathcal{B}^{(n)}_{\mathrm{JHC},\lambda,\nu+1-k,q,\omega}(x)[1]^{k}_{q,\omega} - \mathcal{B}^{(n)}_{\mathrm{JHC},\lambda,\nu+1,q,\omega}(x)\right].\end{aligned} \tag{8.20}$$

Theorem 8.14. *A q,ω-analogue of [12, p.456]*

$$\begin{aligned}&\sum_{k=0}^{\nu}\binom{\nu}{k}_q\mathcal{B}^{(2n)}_{\mathrm{NWA},\lambda,k,q,\omega}(x\ominus_{q,\omega} y)(\overline{n}_{q,\omega})^{\nu-k} \\ &= \frac{1}{\lambda^{n}}\sum_{k=0}^{\nu}(-1)^{\nu-k}\binom{\nu}{k}_q\mathcal{B}^{(n)}_{\mathrm{NWA},\lambda,k,q,\omega}(x)\mathcal{B}^{(n)}_{\mathrm{JHC},\lambda^{-1},\nu-k,q,\omega}(y).\end{aligned} \tag{8.21}$$

Proof. We have that

$$\begin{aligned}&\frac{t^{n}}{(\lambda\mathrm{E}_{q,\omega}(t)-1)^{n}}\mathrm{E}_{q,\omega t}(xt)\frac{(-t)^{n}}{(\lambda^{-1}\mathrm{E}_{\frac{1}{q}}(-t)-1)^{n}}\mathrm{E}_{q,-\omega t}(-yt) \\ &= \mathrm{E}_{q,\omega t}((x\ominus_{q,\omega} y)t)\frac{t^{n}}{(\lambda\mathrm{E}_{q,\omega}(t)-1)^{n}}\left(\frac{t\lambda\mathrm{E}_{q,\omega}(t)}{\lambda\mathrm{E}_{q,\omega}(t)-1}\right)^{n},\end{aligned} \tag{8.22}$$

which implies that

$$\begin{aligned}&\sum_{\nu=0}^{\infty}\frac{t^{\nu}\mathcal{B}^{(2n)}_{\mathrm{NWA},\lambda,\nu,q,\omega}(x\ominus_{q,\omega} y)}{\{\nu\}_q!}\sum_{l=0}^{\infty}\frac{(\overline{n}_q t)^{l}}{\{l\}_q!} \\ &= \frac{1}{\lambda^{n}}\sum_{\nu=0}^{\infty}\frac{t^{\nu}\mathcal{B}^{(n)}_{\mathrm{NWA},\lambda,\nu,q,\omega}(x)}{\{\nu\}_q!}\sum_{m=0}^{\infty}\frac{(-t)^{m}\mathcal{B}^{(n)}_{\mathrm{JHC},\lambda^{-1},m,q,\omega}(y)}{\{m\}_q!}.\end{aligned} \tag{8.23}$$

Formula (8.21) now follows on equating the coefficients of t^{ν}. □

9 q, ω-Apostol–Euler Polynomials

This section has a similar structure as the section on q, ω-Apostol–Bernoulli polynomials, with similar function names. For completeness, we give all definitions and formulas.

9.1 NWA q, ω-Apostol–Euler Polynomials

Definition 9.1. The generalized NWA q, ω-Apostol–Euler polynomials $\mathcal{F}^{(n)}_{\text{NWA},\lambda,\nu,q,\omega}(x)$ are defined by

$$\frac{2^n}{(\lambda \mathrm{E}_{q,\omega}(t)+1)^n}\mathrm{E}_{q,\omega t}(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu \mathcal{F}^{(n)}_{\text{NWA},\lambda,\nu,q,\omega}(x)}{\{\nu\}_q!}, \ |t+\log\lambda| < \pi. \tag{9.1}$$

Assume that $\lambda \neq -1$. The poles in the denominator of (9.1) are the roots of $\mathrm{E}_{q,\omega}(t) = -\lambda^{-1}$. We have

$$\nabla_{\text{NWA},\mathcal{A},q,\omega}\mathcal{F}^{(n)}_{\text{NWA},\lambda,\nu,q,\omega}(x) = \mathcal{F}^{(n-1)}_{\text{NWA},\lambda,\nu,q,\omega}(x). \tag{9.2}$$

This leads to the following recurrence:

Theorem 9.2.

$$\mathcal{F}_{\text{NWA},\lambda,0,q,\omega} = \frac{2}{1+\lambda}, \ \lambda(\mathcal{F}_{\text{NWA},\lambda,q,\omega} \oplus_{q,\omega} \ 1)^k + \mathcal{F}_{\text{NWA},\lambda,k,q,\omega} \doteq 0, k > 1. \tag{9.3}$$

Observe that the limit $\lambda \to 1$ is the first q, ω-Euler numbers. A table of the first NWA q-Apostol–Euler numbers is given in [7].

The following two formulas are generalizations of [6, 4.202] and [6, 4.206].

Theorem 9.3. *A q, ω-analogue of [15, p.635, (31), (32)]:*

$$\mathcal{F}^{(n-1)}_{\text{NWA},\lambda,\nu,q,\omega}(x) = \frac{1}{2}\left[\lambda \sum_{k=0}^{\nu} \binom{\nu}{k}_q \mathcal{F}^{(n)}_{\text{NWA},\lambda,k,q,\omega}(x)(1)^{\nu-k}_{q,\omega} + \mathcal{F}^{(n)}_{\text{NWA},\lambda,\nu,q,\omega}(x)\right]. \tag{9.4}$$

$$(x)^{\nu}_{q,\omega} = \frac{1}{2}\left[\lambda \sum_{k=0}^{\nu} \binom{\nu}{k}_q \mathcal{F}_{\text{NWA},\lambda,k,q,\omega}(x)(1)^{\nu-k}_{q,\omega} + \mathcal{F}_{\text{NWA},\lambda,\nu,q,\omega}(x)\right]. \tag{9.5}$$

Theorem 9.4. *A q, ω-analogue of [15, p.636, (43)] and a generalization of [6, p.152]. Assume that y has order n and x has order 0. Then*

$$\begin{aligned}\mathcal{B}^{(n)}_{\text{NWA},\lambda,\nu,q,\omega}(x \oplus_{q,\omega} y) = \sum_{k=0}^{\nu} \binom{\nu}{k}_q \\ \left[\mathcal{B}^{(n)}_{\text{NWA},\lambda,k,q,\omega}(y) + \frac{\{k\}_q}{2}\mathcal{B}^{(n-1)}_{\text{NWA},\lambda,k-1,q,\omega}(y)\right]\mathcal{F}_{\text{NWA},\lambda,\nu-k,q,\omega}(x).\end{aligned} \tag{9.6}$$

Proof. We will use the q,ω-Taylor formula (5.10) twice, and then like in [15], the factor λ disappears.

$$
\begin{aligned}
&\mathcal{B}^{(n)}_{\mathrm{NWA},\lambda,\nu,q,\omega}(x \oplus_{q,\omega} y) \overset{\text{by}(5.10,9.5)}{=} \frac{1}{2}\sum_{k=0}^{\nu}\binom{\nu}{k}_q \mathcal{B}^{(n)}_{\mathrm{NWA},\lambda,k,q,\omega}(y)\Bigg[\mathcal{F}_{\mathrm{NWA},\lambda,\nu-k,q,\omega}(x)+ \\
&\lambda\sum_{j=0}^{\nu-k}\binom{\nu-k}{j}_q \mathcal{F}_{\mathrm{NWA},\lambda,j,q,\omega}(x)(1)^{\nu-k-j}_{q,\omega}\Bigg] \\
&= \frac{1}{2}\sum_{k=0}^{\nu}\binom{\nu}{k}_q \mathcal{B}^{(n)}_{\mathrm{NWA},\lambda,k,q,\omega}(y)\mathcal{F}_{\mathrm{NWA},\lambda,\nu-k,q,\omega}(x) \\
&+ \frac{\lambda}{2}\sum_{j=0}^{\nu}\binom{\nu}{j}_q \mathcal{F}_{\mathrm{NWA},\lambda,j,q,\omega}(x)\sum_{k=0}^{\nu-j}\binom{\nu-j}{k}_q \mathcal{B}^{(n)}_{\mathrm{NWA},\lambda,k,q,\omega}(y)(1)^{\nu-k-j}_{q,\omega} \\
&\overset{\text{by}(5.10)}{=} \frac{1}{2}\sum_{k=0}^{\nu}\binom{\nu}{k}_q \mathcal{B}^{(n)}_{\mathrm{NWA},\lambda,k,q,\omega}(y)\mathcal{F}_{\mathrm{NWA},\lambda,\nu-k,q,\omega}(x) \\
&+ \frac{\lambda}{2}\sum_{j=0}^{\nu}\binom{\nu}{j}_q \mathcal{B}^{(n)}_{\mathrm{NWA},\lambda,\nu-j,q,\omega}(y \oplus_{q,\omega} 1)\mathcal{F}_{\mathrm{NWA},\lambda,j,q,\omega}(x) \overset{\text{by}(8.2)}{=} \text{RHS}.
\end{aligned}
\tag{9.7}
$$

□

Theorem 9.5. *A q,ω-analogue of the addition theorem [12, p.458].*

$$
\begin{aligned}
&\mathcal{B}^{(n)}_{\mathrm{NWA},\lambda^2,\nu,q,\omega}(x \oplus_{q,\omega} y) \\
&= \frac{(\overline{2}_{q,\omega})^n}{2^n(\overline{2}_{q,\omega})^\nu}\sum_{k=0}^{\nu}\binom{\nu}{k}_q \mathcal{B}^{(n)}_{\mathrm{NWA},\lambda,k,q,\omega}(\overline{2}_{q,\omega}x)\mathcal{F}^{(n)}_{\mathrm{NWA},\lambda,\nu-k,q,\omega}(\overline{2}_{q,\omega}y).
\end{aligned}
\tag{9.8}
$$

Proof. We find that

$$
\begin{aligned}
&\frac{2^n}{(\overline{2}_{q,\omega})^n}\left(\frac{\overline{2}_{q,\omega}t}{\lambda^2\mathrm{E}_{q,\omega}(\overline{2}_{q,\omega}t)-1}\right)^n \mathrm{E}_{q,\omega t}(\overline{2}_{q,\omega}x \oplus_{q,\omega} \overline{2}_{q,\omega}y)t) \\
&= \left(\frac{t}{(\lambda\mathrm{E}_{q,\omega}(t)-1}\right)^n \mathrm{E}_{q,\omega t}(\overline{2}_{q,\omega}xt)\frac{2^n}{(\lambda\mathrm{E}_{q,\omega}(t)+1)^n}\mathrm{E}_{q,\omega t}(\overline{2}_{q,\omega}yt),
\end{aligned}
\tag{9.9}
$$

which implies that

$$
\begin{aligned}
&\frac{2^n}{(\overline{2}_{q,\omega})^n}\sum_{\nu=0}^{\infty}\frac{(\overline{2}_{q,\omega}t)^\nu\mathcal{B}^{(n)}_{\mathrm{NWA},\lambda^2,\nu,q,\omega}(x \oplus_{q,\omega} y)}{\{\nu\}_q!} \\
&= \sum_{\nu=0}^{\infty}\frac{t^\nu\mathcal{B}^{(n)}_{\mathrm{NWA},\lambda,\nu,q,\omega}(\overline{2}_{q,\omega}x)}{\{\nu\}_q!}\sum_{m=0}^{\infty}\frac{t^m\mathcal{F}^{(n)}_{\mathrm{NWA},\lambda,m,q,\omega}(\overline{2}_{q,\omega}y)}{\{m\}_q!}.
\end{aligned}
\tag{9.10}
$$

Formula (9.8) now follows on equating the coefficients of t^ν. □

9.2 Multiplication Formulas for q,ω-Apostol–Euler Polynomials

Theorem 9.6. *A q,ω-analogue of [14, (37) p. 385], first multiplication formula for q,ω-Apostol–Euler polynomials.*

$$\mathcal{F}^{(n)}_{\text{NWA},\lambda,\nu,q,\omega}(\overline{m}_{q,\omega}x) = (\overline{m}_{q,\omega})^{\nu} \sum_{|\vec{j}|=n} (-\lambda)^k \binom{n}{\vec{j}} \mathcal{F}^{(n)}_{\text{NWA},\lambda^m,\nu,q,\omega}\left(x \oplus_{q,\omega} \frac{\overline{k}_{q,\omega}}{\overline{m}_{q,\omega}}\right), \quad (9.11)$$

where $k = j_1 + 2j_2 + \cdots + (m-1)j_{m-1}$, m odd.

Proof. This follows from (6.13). □

Theorem 9.7. *A q,ω-analogue of [14, (38) p. 385], second multiplication formula for q,ω-Apostol–Euler polynomials.*

$$\begin{aligned}\mathcal{F}^{(n)}_{\text{NWA},\lambda,\nu,q,\omega}(\overline{m}_{q,\omega}x) &= \frac{(-2)^n(\overline{m}_{q,\omega})^{\nu+n}}{\{\nu+1\}_{n,q}(\overline{m}_{q,\omega})^n} \\ \sum_{|\vec{j}|=n} (-\lambda)^k \binom{n}{\vec{j}} &\mathcal{B}^{(n)}_{\text{NWA},\lambda^m,\nu+n,q,\omega}\left(x \oplus_{q,\omega} \frac{\overline{k}_{q,\omega}}{\overline{m}_{q,\omega}}\right),\end{aligned} \quad (9.12)$$

where $k = j_1 + 2j_2 + \cdots + (m-1)j_{m-1}$, m even.

Proof. This follows from (6.15). □

Corollary 9.8. *A q,ω-analogue of [14, (43) p. 386].*

$$\begin{aligned}&\mathcal{F}_{\text{NWA},\lambda,\nu,q,\omega}(\overline{m}_{q,\omega}x) \\ &= \begin{cases} (\overline{m}_{q,\omega})^{\nu} \sum_{j=0}^{m-1} (-\lambda)^j \mathcal{F}_{\text{NWA},\lambda^m,\nu,q,\omega}\left(x \oplus_{q,\omega} \frac{\overline{j}_{q,\omega}}{\overline{m}_{q,\omega}}\right), \ m \text{ odd}, \\ \frac{-2(\overline{m}_{q,\omega})^{\nu+1}}{m\{\nu+1\}_{q,\omega}} \sum_{j=0}^{m-1} (-\lambda)^j \mathcal{B}_{\text{NWA},\lambda^m,\nu+1,q,\omega}\left(x \oplus_{q,\omega} \frac{\overline{j}_{q,\omega}}{\overline{m}_{q,\omega}}\right), \ m \text{ even}, \end{cases}\end{aligned} \quad (9.13)$$

where $\dfrac{\overline{j}_{q,\omega}}{\overline{m}_{q,\omega}} \in \mathbb{Q}_{\oplus_{q,\omega}}$.

Theorem 9.9. *A formula for a multiple alternating q,ω-power sum, a q,ω-analogue of [14, (51) p. 387]:*

$$\begin{aligned}\sigma^{(l)}_{\text{NWA},\lambda,m,q,\omega}(n) &= 2^{-l} \sum_{j=0}^{l} \binom{l}{j} \frac{(-1)^{jn}\lambda^{(n-1)j+l}}{\{m+1\}_{l,q}} \\ &\times \left(\sum_{k=0}^{m+l} \binom{m+l}{k}_q \mathcal{F}^{(j)}_{\text{NWA},\lambda,k,q,\omega}\left(\overline{(n-1)j+l}_{q,\omega}\right) \mathcal{F}^{(l-j)}_{\text{NWA},\lambda,n+l-k,k,q,\omega} \right).\end{aligned} \quad (9.14)$$

Proof. We use the generating function technique. Put $k = j_1 + 2j_2 + \cdots + (n-1)j_{n-1}$. It is assumed that $j_i \geq 0, 1 \leq i \leq n-1$.

$$\sum_{\nu=0}^{\infty} \sigma^{(l)}_{\mathrm{NWA},\lambda,\nu,q,\omega}(n)\frac{t^\nu}{\{\nu\}_q!} \overset{\mathrm{by}(4.2)}{=} \sum_{\nu=0}^{\infty}\left(\sum_{|\vec{j}|=l}\binom{l}{\vec{j}}(-1)^l(-\lambda)^k\left(\overline{k_{q,\omega}}\right)^\nu\right)\frac{t^\nu}{\{\nu\}_q!}$$
$$\overset{\mathrm{by}(4.2)}{=}(-1)^l\sum_{|\vec{j}|=l}\binom{l}{\vec{j}}\left(-\lambda \mathrm{E}_{q,\omega}(t)\right)^k$$
$$=\left(\lambda\mathrm{E}_{q,\omega}(t)-\lambda^2\mathrm{E}_{q,\omega}(\overline{2}_{q,\omega}t)+\cdots+(-1)^n\lambda^{n-1}\mathrm{E}_{q,\omega}(\overline{n-1}_{q,\omega}t)\right)^l$$
$$=\left(\frac{(-\lambda)^n\mathrm{E}_{q,\omega}(\overline{n}_{q,\omega}t)}{\lambda\mathrm{E}_{q,\omega}(t)+1}+\frac{\lambda\mathrm{E}_{q,\omega}(t)}{\lambda\mathrm{E}_{q,\omega}(t)+1}\right)^l$$
$$=\sum_{j=0}^{l}\binom{l}{j}(-1)^{l-j}\left(\frac{(-\lambda)^n\mathrm{E}_{q,\omega}(\overline{n}_{q,\omega}t)}{\lambda\mathrm{E}_{q,\omega}(t)+1}\right)^j\left(\frac{\lambda\mathrm{E}_{q,\omega}(t)}{\lambda\mathrm{E}_{q,\omega}(t)+1}\right)^{l-j} \tag{9.15}$$
$$\overset{\mathrm{by}(3.18)}{=}2^{-l}\sum_{j=0}^{l}\binom{l}{j}(-1)^{jn}\lambda^{(n-1)j+l}\sum_{k=0}^{\infty}\mathcal{F}^{(j)}_{\mathrm{NWA},\lambda,k,q,\omega}\left(\overline{(n-1)j+l}_{q,\omega}\right)\frac{t^k}{\{k\}_q!}$$
$$\sum_{i=0}^{\infty}\mathcal{F}^{(l-j)}_{\mathrm{NWA},\lambda,i,q,\omega}\frac{t^i}{\{i\}_q!}=\sum_{\nu=0}^{\infty}\left[2^{-l}\sum_{j=0}^{l}\binom{l}{j}\frac{(-1)^{jn}\lambda^{(n-1)j+l}}{\{m+1\}_{l,q}}\right.$$
$$\left.\sum_{k=0}^{m+l}\binom{m+l}{k}_q\mathcal{F}^{(j)}_{\mathrm{NWA},\lambda,k,q,\omega}\left(\overline{(n-1)j+l}_{q,\omega}\right)\mathcal{F}^{(l-j)}_{\mathrm{NWA},\lambda,n+l-k,k,q,\omega}\right]\frac{t^\nu}{\{\nu\}_q!}.$$

The theorem follows by equating the coefficients of $\frac{t^\nu}{\{\nu\}_q!}$. □

Corollary 9.10. *A q,ω-analogue of [14, (52) p. 387]: The generating function for $\sigma^{(l)}_{\mathrm{NWA},\lambda,\nu,q,\omega}(n)$ is*

$$\sum_{\nu=0}^{\infty}\sigma^{(l)}_{\mathrm{NWA},\lambda,\nu,q,\omega}(n)\frac{t^\nu}{\{\nu\}_q!}=\left(\frac{(-\lambda)^n\mathrm{E}_{q,\omega}(\overline{n}_{q,\omega}t)}{\lambda\mathrm{E}_{q,\omega}(t)-1}+\frac{\lambda\mathrm{E}_{q,\omega}(t)}{\lambda\mathrm{E}_{q,\omega}(t)+1}\right)^l$$
$$=\left(\lambda\mathrm{E}_{q,\omega}(t)-\lambda^2\mathrm{E}_{q,\omega}(\overline{2}_{q,\omega}t)+\cdots+(-1)^n\lambda^{n-1}\mathrm{E}_{q,\omega}(\overline{n-1}_{q,\omega}t)\right)^l. \tag{9.16}$$

Theorem 9.11. *A q,ω-analogue of [14, p.389]. For m odd, we have the following recurrence relation for q,ω-Apostol–Euler numbers:*

$$\mathcal{F}^{(l)}_{\mathrm{NWA},\lambda,n,q,\omega}=(-1)^l\sum_{j=0}^{n}\binom{n}{j}_q\frac{(\overline{m}_{q,\omega})^n}{(\overline{m}_{q,\omega})^{n-j}}\mathcal{F}^{(l)}_{\mathrm{NWA},\lambda^m,j,q,\omega}\sigma^{(l)}_{\mathrm{NWA},\lambda,n-j,q,\omega}(m), \tag{9.17}$$

where $k = j_1 + 2j_2 + \cdots + (m-1)j_{m-1}$ in $\sigma^{(l)}_{\mathrm{NWA},\lambda,n-j,q,\omega}(m)$.

Proof. This follows from (6.21). □

9.3 JHC q,ω-Apostol–Euler Polynomials

Definition 9.12. The generalized JHC q,ω-Apostol–Euler polynomials $\mathcal{F}^{(n)}_{\text{JHC},\lambda,\nu,q,\omega}(x)$ are defined by

$$\frac{2^n}{(\lambda \mathrm{E}_{\frac{1}{q},\omega}(t)+1)^n}\mathrm{E}_{q,\omega t}(xt)=\sum_{\nu=0}^{\infty}\frac{t^\nu \mathcal{F}^{(n)}_{\text{JHC},\lambda,\nu,q,\omega}(x)}{\{\nu\}_q!},\ |t+\log\lambda|<\pi. \tag{9.18}$$

Theorem 9.13. *A relation between generalized q,ω-$\mathcal{H}$ polynomials and generalized q,ω-Euler polynomials. A q,ω-analogue of [16, (38), p.5707].*

$$\mathcal{F}^{(n)}_{\text{NWA},\lambda,\nu,q,\omega}(x)=\frac{\{\nu\}_q!}{\{\nu+n\}_q!}\mathcal{H}^{(n)}_{\text{NWA},\lambda,\nu+n,q,\omega}(x),\ \text{NWA}\equiv\text{NWA}\vee\text{JHC}. \tag{9.19}$$

Theorem 9.14. *We have*

$$\nabla_{\text{JHC},\mathcal{A},q,\omega}\mathcal{F}^{(n)}_{\text{NWA},\lambda,\nu,q,\omega}(x)=\mathcal{F}^{(n-1)}_{\text{NWA},\lambda,\nu,q,\omega}(x), \tag{9.20}$$

This leads to the following recurrence:

Theorem 9.15.

$$\mathcal{F}_{\text{JHC},\lambda,0,q,\omega}=\frac{2}{1+\lambda},\ \lambda(\mathcal{F}_{\text{JHC},\lambda,q,\omega}\boxplus_{q,\omega}\ 1)^k+\mathcal{F}_{\text{JHC},\lambda,k,q,\omega}\doteq 0,\ k>1. \tag{9.21}$$

Theorem 9.16.

$$(-1)^\nu\mathcal{F}_{\text{JHC},\lambda^{-1},\nu,q,\omega}=-\mathcal{F}_{\text{NWA},\lambda,\nu,q,\omega},\ \nu>0. \tag{9.22}$$

Proof. We use the generating function technique to prove (9.22).

$$\begin{aligned}&\sum_{\nu=0}^{\infty}\frac{(-t)^\nu\mathcal{F}_{\text{JHC},\lambda^{-1},\nu,q,\omega}}{\{\nu\}_q!}=\frac{2}{\lambda^{-1}\mathrm{E}_{\frac{1}{q}}(-t)+1}=\frac{2\lambda\mathrm{E}_{q,\omega}(t)}{\lambda\mathrm{E}_{q,\omega}(t)+1}\\&\overset{\text{by}(9.1),\ n=1}{=}\lambda\sum_{\nu=0}^{\infty}\frac{t^\nu\mathcal{F}_{\text{NWA},\lambda,\nu,q,\omega}(1)}{\{\nu\}_q!}\overset{\text{by}(9.3)}{=}-\sum_{\nu=0}^{\infty}\frac{t^\nu\mathcal{F}_{\text{NWA},\lambda,\nu,q,\omega}}{\{\nu\}_q!}.\end{aligned} \tag{9.23}$$

The result follows by equating the coefficients of $\frac{t^\nu}{\{\nu\}_q!}$. □

Theorem 9.17. *A q,ω-analogue of the complementary argument theorem [15, p.634, (20)].*

$$\mathcal{F}^{(n)}_{\text{JHC},\lambda^{-1},\nu,q,\omega}(x)=(-1)^\nu\lambda^n\mathcal{F}^{(n)}_{\text{NWA},\lambda,\nu,q,\omega}(\overline{n}_{q,\omega}\ominus_{q,\omega}x). \tag{9.24}$$

Theorem 9.18. *A generalization of [6, 4.224] and another q,ω-analogue of [15, p.635, (31)]:*

$$\mathcal{F}^{(n-1)}_{\text{JHC},\lambda,\nu,q,\omega}(x)=\frac{1}{2}\left[\lambda\sum_{k=0}^{\nu}\binom{\nu}{k}_q\mathcal{F}^{(n)}_{\text{JHC},\lambda,\nu-k,q,\omega}(x)[1]^k_{q,\omega}+\mathcal{F}^{(n)}_{\text{JHC},\lambda,\nu,q,\omega}(x)\right]. \tag{9.25}$$

Theorem 9.19. *A q,ω-analogue of [12, p.456]:*

$$\begin{aligned}&\sum_{k=0}^{\nu}\binom{\nu}{k}_q \mathcal{F}^{(2n)}_{\mathrm{NWA},\lambda,k,q,\omega}(x\ominus_{q,\omega} y)(\overline{n}_{q,\omega})^{\nu-k}\\&=\frac{1}{\lambda^n}\sum_{k=0}^{\nu}(-1)^{\nu-k}\binom{\nu}{k}_q \mathcal{F}^{(n)}_{\mathrm{NWA},\lambda,k,q,\omega}(x)\mathcal{F}^{(n)}_{\mathrm{JHC},\lambda^{-1},\nu-k,q,\omega}(y).\end{aligned} \tag{9.26}$$

Proof. We have that

$$\begin{aligned}&\frac{2^n}{(\lambda \mathrm{E}_{q,\omega}(t)+1)^n}\mathrm{E}_{q,\omega t}(xt)\frac{2^n}{(\lambda^{-1}\mathrm{E}_{\frac{1}{q},\omega}(-t)+1)^n}\mathrm{E}_{q,-\omega t}(-yt)\\&=\mathrm{E}_{q,\omega t}((x\ominus_{q,\omega} y)t)\frac{2^n}{(\lambda \mathrm{E}_{q,\omega}(t)+1)^n}\left(\frac{2\lambda \mathrm{E}_{q,\omega}(t)}{\lambda \mathrm{E}_{q,\omega}(t)+1}\right)^n,\end{aligned} \tag{9.27}$$

which implies that

$$\begin{aligned}&\sum_{\nu=0}^{\infty}\frac{t^\nu \mathcal{F}^{(2n)}_{\mathrm{NWA},\lambda,\nu,q,\omega}(x\ominus_{q,\omega} y)}{\{\nu\}_q!}\sum_{l=0}^{\infty}\frac{(\overline{n}_{q,\omega}t)^l}{\{l\}_q!}\\&=\frac{1}{\lambda^n}\sum_{\nu=0}^{\infty}\frac{t^\nu \mathcal{F}^{(n)}_{\mathrm{NWA},\lambda,\nu,q,\omega}(x)}{\{\nu\}_q!}\sum_{m=0}^{\infty}\frac{(-t)^m \mathcal{F}^{(n)}_{\mathrm{JHC},\lambda^{-1},m,q,\omega}(y)}{\{m\}_q!}.\end{aligned} \tag{9.28}$$

Formula (9.26) now follows on equating the coefficients of t^ν. □

We summarize four interconnections between q,ω-Appell polynomials and numbers in two tables. Everywhere the most general form is found on top, and special cases are placed lower in the same column. In the first table the transition to q,ω-Bernoulli polynomials applies only to the recurrences.

Multiple q,ω-Appell	$\Phi^{(n)}_{\mathcal{M},\nu,q,\omega}(x)$	$\Phi^{(n)}_{\mathcal{M},\nu,q,\omega}(x)$
General q,ω-Appell	$\mathrm{b}^{(n)}_{\lambda,\nu,q,\omega}(x)$	$\mathrm{c}^{(n)}_{\lambda,\nu,q,\omega}(x)$
General q,ω-Apostol	$\mathcal{B}^{(n)}_{\mathrm{NWA},\lambda,\nu,q,\omega}(x)$	$\mathcal{B}^{(n)}_{\mathrm{JHC},\lambda,\nu,q,\omega}(x)$
Order n polynomial	$\mathrm{B}^{(n)}_{\mathrm{NWA},\nu,q,\omega}(x)$	$\mathrm{B}^{(n)}_{\mathrm{JHC},\nu,q,\omega}(x)$
Apostol q,ω-B polynomial	$\mathrm{B}_{\mathrm{NWA},\lambda,\nu,q,\omega}(x)$	$\mathrm{B}_{\mathrm{JHC},\lambda,\nu,q,\omega}(x)$
q,ω-B polynomial	$\mathrm{B}_{\mathrm{NWA},\nu,q,\omega}(x)$	$\mathrm{B}_{\mathrm{JHC},\nu,q,\omega}(x)$
q,ω-B number	$\mathrm{B}_{\mathrm{NWA},\nu,q,\omega}$	$\mathrm{B}_{\mathrm{JHC},\nu,q,\omega}$
Bernoulli number	B_ν	B_ν

Multiple q,ω-Appell	$\Phi^{(n)}_{\mathcal{M},\nu,q,\omega}(x)$	$\Phi^{(n)}_{\mathcal{M},\nu,q,\omega}(x)$
General q,ω-Appell	$\mathrm{e}^{(n)}_{\lambda,\nu,q,\omega}(x)$	$\mathrm{f}^{(n)}_{\lambda,\nu,q,\omega}(x)$
General q,ω-Apostol	$\mathcal{F}^{(n)}_{\mathrm{NWA},\lambda,\nu,q,\omega}(x)$	$\mathcal{F}^{(n)}_{\mathrm{JHC},\lambda,\nu,q,\omega}(x)$
Order n polynomial	$\mathrm{F}^{(n)}_{\mathrm{NWA},\nu,q,\omega}(x)$	$\mathrm{F}^{(n)}_{\mathrm{JHC},\nu,q,\omega}(x)$
Apostol q,ω-F-pol.	$\mathrm{F}_{\mathrm{NWA},\lambda,\nu,q,\omega}(x)$	$\mathrm{F}_{\mathrm{JHC},\lambda,\nu,q,\omega}(x)$
q,ω Euler-pol.	$\mathrm{F}_{\mathrm{NWA},\nu,q,\omega}(x)$	$\mathrm{F}_{\mathrm{JHC},\nu,q,\omega}(x)$
q,ω-Euler number	$\mathrm{F}_{\mathrm{NWA},\nu,q,\omega}$	$\mathrm{F}_{\mathrm{JHC},\nu,q,\omega}$
Euler number	F_{ν}	F_{ν}

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