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A Criterion for the Unique Existence of the Limit Cycle of a Liénard-Type System with One Parameter

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Abstract

A criterion for the unique existence of limit cycles of a Liénard-type system with one parameter is investigated. Our idea is to estimate the solution orbit starting from the initial point on some invariant domain. In this light, our tool constructed by using four Lyapunov functions is applied for the unique existence. The several examples shall be discussed under the new parameter for the uniqueness of the limit cycle.

AMS Subject Classifications: 34C07, 34C25, 34C26, 34D20.
Keywords: Liénard system, Uniqueness, Limit cycles, Invariant domain.

1 Introduction

In this paper, we consider a Liénard-type system

$$\dot{x} = \frac{1}{a(x)}\left\{y - \lambda F(x)\right\}, \qquad \dot{y} = -a(x)g(x) \tag{1.1}$$

where $a(x) > 0$ for every $x \in \mathbb{R}$ and λ is a positive real number. This system is a generalization of the Liénard system and was discussed in [1]. They investigated some properties (X^+) or (Y^+) and so on for the system.

Our aim is to improve the conditions for the parameter λ in order that system (1.1) has a unique limit cycle under the assumptions

(C1) $F(x)$, $g(x)$ and $a(x)$ are locally Lipschitz continuous functions,

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(C2) $F(0) = g(0) = 0$ and $g(x)/x > 0$,

(C3) there exist α and β with $\alpha < 0 < \beta$ such that $x(x-\alpha)(x-\beta)F(x) > 0$.

In [3, 4, 6, 7, 11], system (1.1) with $a(x) \equiv 1$ was treated and a sufficient condition for λ in order that it has a unique limit cycle was given.

It is well known under the above assumptions that the uniqueness of solutions of system (1.1) for initial value problems is guaranteed and the only equilibrium point $(0,0)$ is unstable. Under these conditions, the qualitative property of the limit cycles of system (1.1), specially $a(x) \equiv 1$, has studied by many mathematicians, physicists, economists and engineers and so on. See [1,4,5,12]. Thus, our results play an important role to resolve the scientific phenomenon.

In [2, 11], the following system with one parameter λ has been introduced.

$$\dot{x} = y - \lambda\frac{x}{\pi}\Big(\frac{64}{35}x^6 - \frac{112}{45}x^4 + \frac{1}{2}x^3 + \frac{196}{243}x^2 + \frac{1}{200}x - \frac{4}{81}\Big), \qquad \dot{y} = -x \tag{1.2}$$

G. Villari and F. Zanolin in [11] introduced from the Maple software that system (1.2) has three limit cycles if λ is small (for instance $\lambda = 30$) and conjectured the unique existence of the limit cycle intersecting the lines $x = \alpha$ and $x = \beta$ for $\lambda \geq 141.515778$. By combining the recent tool in [10] with [8,9], we can approach the existence criterion of the unique limit cycle for λ until $\lambda \geq 200.690379$ as is shown in Section 4. For the polynomial system as system (1.2), it has been known by V. A. Gaiko ([4]) that it has at most four limit cycles.

In [6], the relation between the magnitude of $F(x)$ and the unique existence of a limit cycle of system (1.1) has been investigated by constructing some positive invariant domain. Also see [2]. In [7], the author gave the weak condition than [6] by using the existence of some invariant curve of system (1.1). For instance, the methods were applied to the Duff–Levinson system in [6,7]. We reuse these ideas to prove our results. Recently, in [10], it was shown that the estimation of the solution orbits starting from the initial point on some invariant domain for system (1.1) was useful to the unique existence of the limit cycles. Our goal is to give the uniqueness theorem constructed by [8,9] and this idea ([10]) for the system with one parameter as system (1.1).

In Section 2, the main results are given and the proofs are shown in Section 3. Further, the applications for system (1.2) shall be discussed in Section 4.

2 Main Results

Let $G(x) = \displaystyle\int_0^x a^2(\xi)g(\xi)d\xi$. First, we confirm that Theorem 2.1 in [6] also holds for this new function $G(x)$.

Theorem 2.1. *Let* $G(\alpha) = G(\beta)$. *If system* (1.1) *satisfies the conditions* (C1) – (C3) *and besides*

(C4) *$F(x)$ is monotone increasing for $x \leq \alpha$ and $x \geq \beta$,*

then it has at most one limit cycle for all $\lambda > 0$. It intersects both the lines $x = \alpha$ and $x = \beta$, is stable and hyperbolic.

We assume $G(\alpha) > G(\beta)$ without loss of generality.

Let $\beta^* \in (\alpha, 0)$ such that $G(\beta) = G(\beta^*)$ and $p = \min_{i \in \mathbb{N}} \{p_i \in (\alpha, \beta^*] \mid F^{'}(p_i) = 0, F^{''}(p_i) \neq 0\} < 0$.

From the powerful tools in [6, 8, 9] for system (1.1), we have the following.

Theorem 2.2. *Let $p \geq \beta^*$. Under the conditions* (C1) – (C3) *and besides*

(C5) *$F(x)$ is monotone increasing for $x \leq \beta^*$ and $x \geq \beta$,*

system (1.1) *has at most one limit cycle for all $\lambda > 0$. It intersects both the lines $x = \beta^*$ and $x = \beta$, is stable and hyperbolic.*

We define the supplement function

$$L(x; s) = \sqrt{\frac{1}{F(x)} \int_s^x \frac{a^2(\xi) g(\xi)}{F(\xi)} d\xi}$$

for some constant s.

Our main results are following

Theorem 2.3. *Let $p < \beta^*$. Assume the conditions* (C1) – (C3) *and besides*

(C6) *$F(x)$ is monotone increasing for $x \leq p$ and $x \geq \beta$.*

Then system (1.1) *has at most one limit cycle for all $\lambda > \lambda_1 = \max_{x \in [p, \beta^*]} L(x; \beta^*)$. It intersects both the lines $x = p$ and $x = \beta$, is stable and hyperbolic.*

Remark 2.4. If $G(\alpha) < G(\beta)$, (C6) or λ_1 in Theorem 2.3 is replaced by

$(\mathrm{C6})^*$ $F(x)$ is monotone increasing for $x \leq \alpha$ and $x \geq q$

or

$\lambda_2 = \max_{x \in [\alpha^*, q]} L(x; \alpha^*)$ where $\alpha^* \in (0, \beta)$ such that $G(\alpha) = G(\alpha^*)$, and $q = \max_{i \in \mathbb{N}} \{q_i \in [\alpha^*, \beta) \mid F^{'}(q_i) = 0, F^{''}(p_i) \neq 0\} > 0$, respectively.

From the uniformly boundedness of the solutions given in [2, 5] and Poincaré–Bendixson's theorem, we have the following.

Theorem 2.5. *Assume that system* (1.1) *satisfies the conditions* (C1) – (C3), [(C4) or (C5) or (C6)] *and besides*

(C7) $$\limsup_{x \to \pm\infty} \Big\{ G(x) \pm F(x) \Big\} = +\infty.$$

If $p \geq \beta^*$, then it has a unique limit cycle for all $\lambda > 0$.It intersects both the lines $x = \beta^*$ and $x = \beta$, is stable and hyperbolic.

If $p < \beta^*$, it has a unique limit cycle for all $\lambda > \lambda_1$. It intersects both the lines $x = p$ and $x = \beta$, is stable and hyperbolic.

As the special case for λ_1 or λ_2, the following criterions are used. For instance it shall be applied to the system in Example 4.3.

Corollary 2.6. *In* Theorem 2.3*, if $L^{'}(x;\beta^*) < 0$ for all $x \in [p, \beta^*]$, then $\lambda_1 = L(p;\beta^*)$.*

Corollary 2.7. *In* Remark 2.4*, if $L^{'}(x;\alpha^*) > 0$ for all $x \in [\alpha^*, q]$, then $\lambda_2 = L(q;\alpha^*)$.*

Remark 2.8. $L^{'}(x;s) > 0 (< 0)$ is equivalent to

$$M(x;s) = a^2(x)g(x) - F^{'}(x)\int_s^x \frac{a^2(\xi)g(\xi)}{F(\xi)}d\xi > 0(< 0),$$

respectively.

3 Proofs of Theorems

We assume $G(\alpha) \geq G(\beta)$ without loss of generality. Let Ω be the region surrounded by a closed curve $V(x,y) = (1/2)y^2 + G(x) = G(\beta)$. It surrounds the only one equilibrium point $(0,0)$. We prepare two lemmas.

Lemma 3.1. *Under the conditions* (C1) – (C3)*, every limit cycle of system* (1.1) *must intersect the lines $x = \beta^*$ and $x = \beta$ if it exists.*

Proof. First, we prove that no non-trivial closed orbit of system (1.1) exists in the domain Ω. We suppose that a non-trivial closed orbit C of system (1.1) exists in the domain Ω. Let $(x(t), y(t))$ be the periodic solution corresponding to C and let T be its period. We have

$$\oint_C dV = \int_0^T \frac{dV(x(t),y(t))}{dt}dt = [V(x(t),y(t))]_0^T = 0. \tag{3.1}$$

On the other hand, since

$$\frac{dV(x(t),y(t))}{dt} = -\lambda a(x(t))g(x(t))F(x(t)) > 0 \tag{3.2}$$

for $\beta^* < x(t) < \beta$, we have $\oint_C dV > 0$. This is in contradiction to the equality (3.1). Thus, a non-trivial closed orbit of the system cannot stay in the domain Ω.

Next, we investigate the direction of the field of velocity vectors defined by system (1.1) on the boundary Γ of Ω. From the inequality (3.2), the velocity vector on Γ points outwards. Thus, we conclude that a solution orbit starting from outside of Ω cannot get into Ω through Γ, namely the outside of Ω is an invariant set of system (1.1). These discussions imply that if system (1.1) has a non-trivial closed orbit, then it must exist outside Ω and intersect the lines $x = \beta^*$ and $x = \beta$. This concludes the proof. □

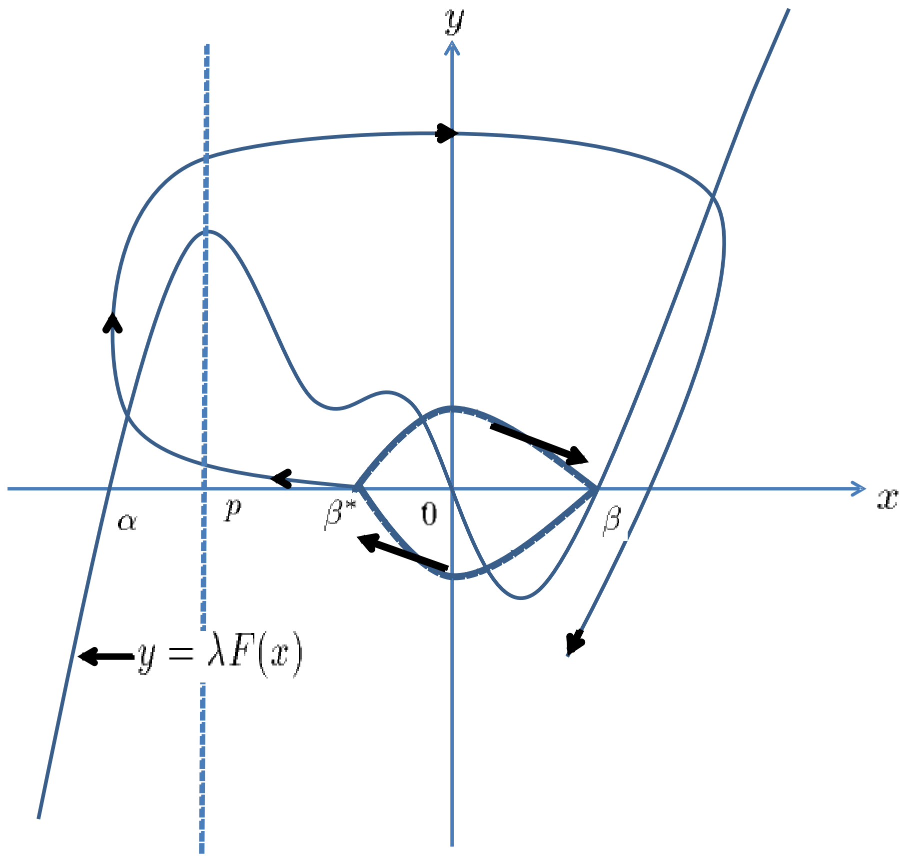

Figure 3.1 (The illustration for the proof of Theorem 2.3)

From [8, 9], we have the following.

Lemma 3.2. *Under the conditions* (C1) – (C3) *and* (C5)*, system* (1.1) *has at most one limit cycle intersecting the lines* $x = \beta^*$ *and* $x = \beta$ *if it exists.*

Proof. By the virtue of Lemma 3.1 and Lemma 3.2, if $p \geq \beta^*$, we see that the limit cycle is at most one and intersects the lines $x = \beta^*$ and $x = \beta$. The proofs of Theorem 2.1 and Theorem 2.2 are completed now. □

Proof. We shall prove Theorem 2.3. Let $p < \beta^*$ and the function $y = y(x)$ be the solution of system (1.1) with $y(\beta^*) = 0$. If the inequality $\lambda F(x) - y(x) > 0$ for all $x \in [p, \beta^*]$, then the solution orbit $y = y(x)$ starting from $(\beta^*, 0)$ must first intersect the line $x = p$.

Thus, we have

$$\lambda F(x) - y(x) = F(x)\Big(\lambda - \frac{y(x)}{F(x)}\Big) = F(x)\Big(\lambda - \frac{1}{F(x)}\int_{\beta^*}^{x} \frac{-a^2(\xi)g(\xi)}{y(\xi) - \lambda F(\xi)} d\xi\Big) > 0$$

for all $x \in [p, \beta^*]$.

Since $y(x) \geq 0$ and $F(x) > 0$ on $x \in [p, \beta^*]$, we get

$$0 < \lambda - \frac{1}{F(x)}\int_{\beta^*}^{x} \frac{-a^2(\xi)g(\xi)}{y(\xi) - \lambda F(\xi)} d\xi < \lambda - \frac{1}{F(x)}\int_{\beta^*}^{x} \frac{a^2(\xi)g(\xi)}{\lambda F(\xi)} d\xi = \lambda - \frac{1}{\lambda}L^2(x;\beta^*)$$

for all $x \in [p, \beta^*]$.

This implies that if $\lambda > \lambda_1 = \max_{x\in[p,\beta^*]} L(x;\beta^*)$, then $\lambda F(x) - y(x) > 0$ for all $x \in [p, \beta^*]$. Also see Figure 3.1. Thus,the solution orbit starting from $(\beta^*, 0)$ must first intersect the line $x = p$ and next the curve $y = \lambda F(x)$ in $\alpha < x \leq p$. Therefore, if $\lambda > \lambda_1$,then all limit cycles of system (1.1) must exist outside the strip domain $D = \{(x, y) \mid p \leq x \leq \beta\}$. By the same reason as Lemma 3.2, we see under the condition (C6) that the limit cycle is at most one. Moreover, assuming the condition (C7), we conclude from the uniformly boundedness and Poincaré–Bendixson's theorem that the limit cycle of system (1.1) exists outside D and is exactly one. Further, it is stable and hyperbolic.

Similarly, we can prove the case $G(\alpha) < G(\beta)$ under the condition of Remark 2.4. This concludes the proof. □

4 Examples

We shall apply our methods (specially, Theorem 2.3) to several systems concretely.

Example 4.1. Consider system (1.2). This is system (1.1) with $a(x) \equiv 1$ and was introduced in [2, 11] as an interesting example for the unique existence of the limit cycle. Villari and Zanolin conjectured the uniqueness for $\lambda \geq 141.515778$ by the numerical analysis for the Maple software. In virtue of Theorem 2.5, the conjecture is now replaced by following statement.

Proposition 4.2. *System* (1.2) *has a unique limit cycle for all* $\lambda \geq 200.690379$.

In fact, it is trivial that system (1.2) satisfies the conditions (C1) – (C3) and (C7). Solving the equation $F(x) = 0$, we have $\alpha \risingdotseq -1.12959$ and $\beta \risingdotseq 0.247712 = -\beta^*$. Further, we can take $p \risingdotseq -0.968071$ as the minimum number in the interval $(\alpha, 0)$ satisfying the condition (C6). Then we get from the numerical calculation

$$\lambda_1 = \max_{x\in[p,\beta^*]} L(x) \risingdotseq 200.690379.$$

Figure 4.1 is the case of $\lambda = 201$. It is shown that the solution orbit moves fast on the horizontal direction and slow on the vertical direction along the characteristic curve $y = 201F(x)$. We can confirm that the system has a unique limit cycle.

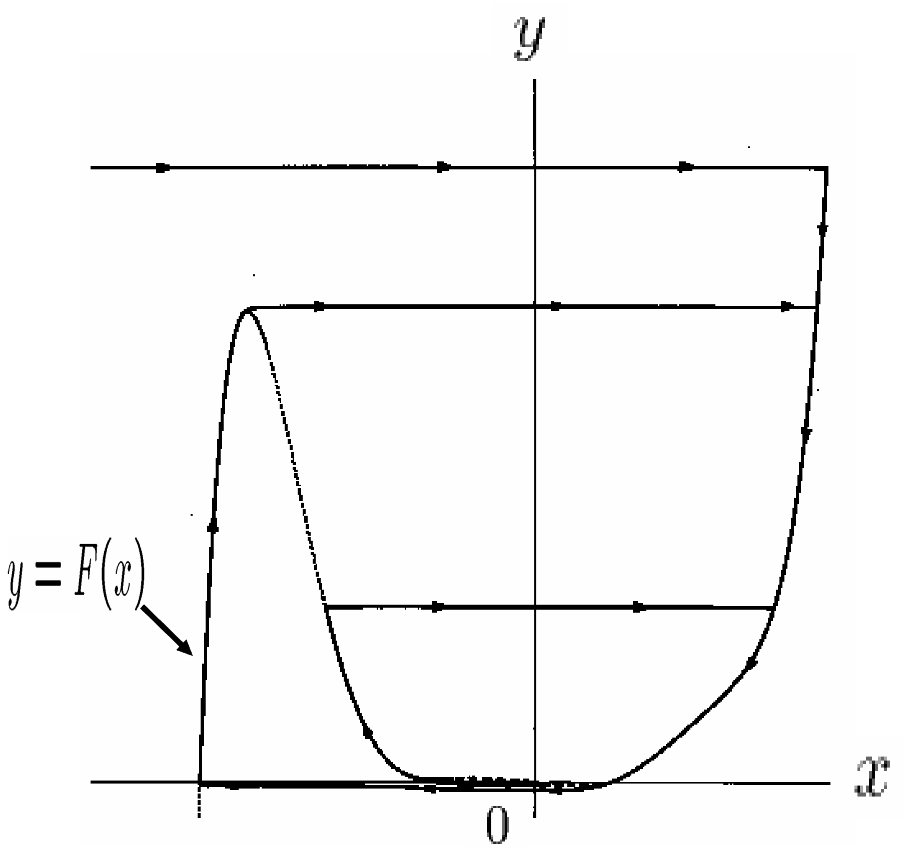

Figure 4.1 (System (1.2) for $\lambda = 201$)

Example 4.3. Consider the following system with a parameter λ

$$\dot{x} = y - \lambda x(x-3)(x+1), \qquad \dot{y} = -x. \tag{4.1}$$

System (4.1) with $\lambda = \sqrt{3}$ was introduced as an example having the unique limit cycle in [10]. We note that the method in [10] cannot be applied to the system with $\lambda < \sqrt{3}$. In virtue of Theorem 2.3 and Remark 2.4, the improved result for λ is stated as the following.

Proposition 4.4. *System* (4.1) *has a unique limit cycle for all* $\lambda \geq 0.195827$.

It is trivial that system (4.1) satisfies the conditions (C1) – (C3) and (C7). Since $F(x) = \lambda x(x-3)(x+1)$, $g(x) = x$ and $a(x) \equiv 1$, we have $\beta = 3$, $\alpha^* = -\alpha = 1$ and $\alpha^* < q = (2+\sqrt{13})/3 < \beta = 3$ for system (4.1). Then, we can prove $L^{'}(x;\alpha^*) > 0$ for all $x \in [1, q]$. Namely, from Remark 2.8, we have $M(x;1) > 0$ for all $x \in [1, q]$. In facts, we get for $x \in [1, q]$

$$M(x;1) = g(x) - F^{'}(x)\int_1^x \frac{g(\xi)}{F(\xi)}d\xi$$

$$= x - (3x^2 - 4x - 3)\int_1^x \frac{1}{(\xi-3)(\xi+1)}d\xi$$
$$= x - \frac{3x^2-4x-3}{4}\log\Big(\frac{3-x}{x+1}\Big)$$

and

$$M^{'}(x;1) = 1 - \frac{3x^2-4x-3}{x^2-2x-3} - \frac{3x-2}{2}\log\Big(\frac{3-x}{x+1}\Big)$$
$$= -\frac{2x(x-1)}{x^2-2x-3} - \frac{3x-2}{2}\log\Big(\frac{3-x}{x+1}\Big).$$

Since $\dfrac{x^2-x}{x^2-2x-3} < 0$ and $\log\dfrac{3-x}{x+1} < 0$ for $x \in [1,q]$, we get $M^{'}(x;1) > 0$ for $x \in [1,q]$ and $M(1;1) = 1 > 0$. This implies from Remark 2.8 that $L(x;\alpha^*)$ or $M(x;1)$ is monotone increasing for $x \in [1,q]$. Thus, by Remark 2.4 and Corollary 2.7, we conclude that system (4.1) has a unique limit cycle for

$$\lambda_2 = \max_{x\in[1,q]} L(x;\alpha^*) = L(q;\alpha^*) = L(q;1)$$
$$= \sqrt{\frac{1}{F(q)}\int_1^q \frac{1}{(x-3)(x+1)}dx} = \frac{3}{2}\sqrt{\frac{3\log\Big(\frac{5+\sqrt{13}}{7-\sqrt{13}}\Big)}{70+26\sqrt{13}}} \fallingdotseq 0.195827.$$

Remark 4.5. By [4], we see that system (4.1) has at most three limit cycles.

Remark 4.6. Our results can be applied to system (1.1) except polynomial systems.

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