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Oscillatory Behavior of Second-Order Half-Linear Neutral Differential Equations with Damping

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Abstract

This paper discusses the oscillatory behavior of solutions to a class of second-order half-linear neutral differential equations with a damping term. Some new sufficient conditions for all solutions to be oscillatory are given. Examples illustrating our results are also included.

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1 Introduction

This paper deals with the oscillatory behavior of all solutions of the second-order half-linear neutral differential equation with a damping term

$$\left(r(t)\left(z'(t)\right)^{\alpha}\right)' + p(t)\left(z'(t)\right)^{\alpha} + q(t)f(t, x(\sigma(t))) = 0, \quad t \geq t_0 > 0, \tag{1.1}$$

where $z(t) = x(t) + h(t)x(\tau(t))$, and $\alpha \geq 1$ is the ratio of two positive odd integers. Throughout this paper, we always assume that the following conditions are satisfied:

(i) $p, q, r : [t_0, \infty) \to \mathbb{R}$ are continuous functions with $p(t) \geq 0$, $r(t) > 0$, $q(t) > 0$, and

$$\int_{t_0}^{\infty} \left[\frac{1}{r(t)} \exp\left(-\int_{t_0}^{t} \frac{p(s)}{r(s)} ds\right)\right]^{1/\alpha} dt = \infty; \tag{1.2}$$

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(ii) $h : [t_0, \infty) \to \mathbb{R}$ is a continuous function with $h(t) \geq 1$, and $h(t) \not\equiv 1$ for large t;

(iii) $\tau, \sigma : [t_0, \infty) \to \mathbb{R}$ are continuous functions such that τ is strictly increasing, $\tau(t) < t$, and $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty$;

(iv) $f(t, u) : [t_0, \infty) \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $uf(t, u) > 0$ for all $u \neq 0$ and there exists a positive constant k such that

$$f(t, u)/u^\alpha \geq k \quad \text{for } u \neq 0.$$

The cases where

$$\tau(t) \geq \sigma(t) \tag{1.3}$$

and

$$\tau(t) \leq \sigma(t) \tag{1.4}$$

are considered.

By a *solution* of equation (1.1), we mean a function $x \in C([t_x, \infty), \mathbb{R})$ for some $t_x \geq t_0$ that has the properties $z \in C^1([t_x, \infty), \mathbb{R})$, $r(z')^\alpha \in C^1([t_x, \infty), \mathbb{R})$, and satisfies (1.1) on $[t_x, \infty)$. We only consider those solutions of (1.1) that exist on some half-line $[t_x, \infty)$ and satisfy the condition

$$\sup\{|x(t)| : T \leq t < \infty\} > 0 \text{ for any } T \geq t_x;$$

moreover, we tacitly assume that (1.1) possesses such solutions. Such a solution $x(t)$ of (1.1) is said to be *oscillatory* if it has arbitrarily large zeros on $[t_x, \infty)$, i.e., for any $t_1 \in [t_x, \infty)$ there exists $t_2 \geq t_1$ such that $x(t_2) = 0$; otherwise it is called *nonoscillatory*, i.e., if it is eventually positive or eventually negative. Equation (1.1) itself is termed oscillatory if all its solutions are oscillatory.

The oscillatory behavior of solutions to various classes of second order functional differential equations has been the object of research of a number of authors and many interesting results have been obtained. For some typical results, we refer the reader to [2–4,7,8,10–12,15–20,23] and the references cited therein as examples of recent results on this topic. However, results on the oscillatory behavior of solutions of second-order neutral differential equations with damping term are relatively scarce in the literature; some results can be found, for example, in [5,6,21,22]. It should be noted that although papers [5, 6, 21, 22] deal with second-order neutral differential equations with damping term, the results obtained in these papers except [22] cannot be applied to the case where $h(t) \to \infty$ as $t \to \infty$. Motivated by the above observations, here we wish to develop sufficient conditions for equation (1.1) to be oscillatory in the case where $h(t) > 1$ and/or $h(t) \to \infty$ as $t \to \infty$. The results of the present paper are obtained by using an integral averaging technique due to Philos [13] (see also [9, 14] for the refined integral averaging technique) and can easily be extended to more general second-order nonlinear neutral differential equations with damping term. It is therefore hoped that the present paper will contribute significantly to the study of oscillatory behavior of solutions of second-order neutral differential equations with damping term.

2 Main Results

In the following theorems, we establish new oscillation criteria for (1.1) by using the integral averaging technique due to Philos [13]. In order to present our theorems, following Philos [13], we first introduce the function class $\mathcal{P}$. Namely, let $D_0 = \{(t,s) : t > s \geq t_0\}$ and $D = \{(t,s) : t \geq s \geq t_0\}$. We say that the function $H \in C(D, \mathbb{R})$ belongs to the class $\mathcal{P}$, denoted by $H \in \mathcal{P}$, if

(i) $H(t,t) = 0$ for $t \geq t_0$, and $H(t,s) > 0$ on $(t,s) \in D_0$;

(ii) H has a continuous and nonpositive partial derivative on D_0 with respect to the second variable.

For notational purposes, we let

$$A(t,t_*) := \int_{t_*}^{t} \frac{ds}{r^{1/\alpha}(s)}, \quad t_* \geq t_0,$$

for any positive function $\eta \in C^1([t_0,\infty), \mathbb{R})$,

$$\xi(t) = \frac{\eta'(t)r(t) - \eta(t)p(t)}{\eta(t)r(t)},$$

and

$$\psi(t,t_*) := \frac{1}{h(\tau^{-1}(t))}\left(1 - \frac{1}{h(\tau^{-1}(\tau^{-1}(t)))}\frac{A(\tau^{-1}(\tau^{-1}(t)),t_*)}{A(\tau^{-1}(t),t_*)}\right), \quad t_* \geq t_0,$$

where τ^{-1} is the inverse function of τ. Throughout this section we assume that $\psi(t,t_*) > 0$ for all sufficiently large t.

Our first main result is contained in the following theorem.

Theorem 2.1. *Let conditions (i)–(iv),* (1.2) *and* (1.3) *hold, and let $h, H : D \to \mathbb{R}$ be continuous functions such that H belongs to the class $\mathcal{P}$ and*

$$-\frac{\partial H}{\partial s}(t,s) = h(t,s)\sqrt{H(t,s)} \quad for\ all\ (t,s) \in D_0. \tag{2.1}$$

If there exists a positive function $\eta \in C^1([t_0,\infty), \mathbb{R})$ such that, for some $\gamma \geq 1$,

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\Psi(s) - \frac{\gamma}{4\alpha}\frac{\eta(s)r^{1/\alpha}(s)\Phi(t,s)}{A^{\alpha-1}(s,t_2)}\right] ds = \infty, \tag{2.2}$$

for all sufficiently large $t_2 \in [t_1,\infty) \subseteq [t_0,\infty)$, and all $T > t_2$ with $\sigma(t) > t_2$ for all $t \geq T$, where

$$\Psi(t) = k\eta(t)q(t)\psi^{\alpha}(\sigma(t),t_2)\frac{A^{\alpha}(\tau^{-1}(\sigma(t)),t_2)}{A^{\alpha}(t,t_2)}, \tag{2.3}$$

and

$$\Phi(t,s) = \left(-h(t,s) + \xi(s)\sqrt{H(t,s)}\right)^2, \tag{2.4}$$

then every solution of (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \in [t_{0,}\infty)$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \geq t_1$. If $x(t)$ is eventually negative, the proof is similar, so we omit the details of that case here, as well as in the remaining proofs in this paper. Then, it follows from (1.1) that

$$\left(r(t)\left(z'(t)\right)^{\alpha}\right)' + p(t)\left(z'(t)\right)^{\alpha} + kq(t)x^{\alpha}(\sigma(t)) \leq 0, \tag{2.5}$$

and so

$$\left(r(t)\left(z'(t)\right)^{\alpha}\right)' + p(t)\left(z'(t)\right)^{\alpha} < 0 \;\text{ for } t \geq t_1. \tag{2.6}$$

Letting $v(t) = r(t)\left(z'(t)\right)^{\alpha}$, it follows from (2.6) that

$$v'(t) + \frac{p(t)}{r(t)}v(t) < 0 \;\text{ for } t \geq t_1,$$

which implies

$$\left(\exp\left(\int_{t_1}^{t}\frac{p(s)}{r(s)}ds\right)v(t)\right)' < 0 \quad \text{for } t \geq t_1,$$

and so, $v(t)\exp\left(\int_{t_1}^{t}\frac{p(s)}{r(s)}ds\right)$ is decreasing and eventually does not change its sign, say on $[t_2,\infty)$ for some $t_2 \geq t_1$. Therefore, $z'(t)$ eventually has a fixed sign on $[t_2,\infty)$, and so we have two cases to consider: (I) $z'(t) > 0$ for $t \geq t_2$ or (II) $z'(t) < 0$ for $t \geq t_2$.

We first assume that case (I) holds. It then follows from (2.5) and the definition of z that

$$z(t) > 0,\; z'(t) > 0, \;\text{ and }\; \left(r(t)\left(z'(t)\right)^{\alpha}\right)' < 0 \;\text{ for } t \geq t_2,$$

from which, we see that

$$z(t) = z(t_2) + \int_{t_2}^{t}\frac{1}{r^{1/\alpha}(s)}\left(r(s)\left(z'(s)\right)^{\alpha}\right)^{1/\alpha}ds \geq r^{1/\alpha}(t)z'(t)A(t,t_2). \tag{2.7}$$

In view of (2.7), we have for all $t \geq t_3$ for $t_3 \in (t_2,\infty)$ that

$$\left(\frac{z(t)}{A(t,t_2)}\right)' = \frac{r^{-1/\alpha}(t)[r^{1/\alpha}(t)z'(t)A(t,t_2) - z(t)]}{A^2(t,t_2)} \leq 0,$$

i.e., $z(t)/A(t,t_2)$ is nonincreasing for $t \geq t_3$.

From the definition of z (see also inequality (8.6) in [1]), it follows that

$$\begin{aligned} x(t) &= \frac{1}{h(\tau^{-1}(t))}\left[z(\tau^{-1}(t)) - x(\tau^{-1}(t))\right] \\ &= \frac{z(\tau^{-1}(t))}{h(\tau^{-1}(t))} - \frac{[z(\tau^{-1}(\tau^{-1}(t))) - x(\tau^{-1}(\tau^{-1}(t)))]}{h(\tau^{-1}(t))h(\tau^{-1}(\tau^{-1}(t)))} \end{aligned}$$

$$\geq \frac{z(\tau^{-1}(t))}{h(\tau^{-1}(t))} - \frac{1}{h(\tau^{-1}(t))h(\tau^{-1}(\tau^{-1}(t)))} z(\tau^{-1}(\tau^{-1}(t))). \tag{2.8}$$

Now $\tau(t) < t$ and τ is strictly increasing, so τ^{-1} is increasing and $\tau^{-1}(t) > t$. Thus,

$$\tau^{-1}(\tau^{-1}(t)) > \tau^{-1}(t),$$

and since $z(t)/A(t,t_2)$ is nonincreasing for $t \geq t_3$, we have

$$\frac{A(\tau^{-1}(\tau^{-1}(t)),t_2)z(\tau^{-1}(t))}{A(\tau^{-1}(t),t_2)} \geq z(\tau^{-1}(\tau^{-1}(t))).$$

Substituting the last inequality into (2.8) yields

$$x(t) \geq \psi(t,t_2)z\left(\tau^{-1}(t)\right) \quad \text{for } t \geq t_3. \tag{2.9}$$

Since $\lim_{t\to\infty} \sigma(t) = \infty$, we can choose $t_4 \geq t_3$ such that $\sigma(t) \geq t_3$ for all $t \geq t_4$. Thus, it follows from (2.9) that

$$x(\sigma(t)) \geq \psi(\sigma(t),t_2)z\left(\tau^{-1}(\sigma(t))\right) \quad \text{for } t \geq t_4. \tag{2.10}$$

Using (2.10) in (2.5) gives

$$\left(r(t)\left(z'(t)\right)^{\alpha}\right)' + p(t)\left(z'(t)\right)^{\alpha} + kq(t)\psi^{\alpha}(\sigma(t),t_2)z^{\alpha}\left(\tau^{-1}(\sigma(t))\right) \leq 0 \tag{2.11}$$

for $t \geq t_4$. Define the function w by the Riccati type substitution

$$w(t) = \eta(t)\frac{r(t)\left(z'(t)\right)^{\alpha}}{z^{\alpha}(t)} \qquad \text{for } t \geq t_4. \tag{2.12}$$

Clearly, $w(t) > 0$, and from (2.11)–(2.12), we see that

$$w'(t) \leq \xi(t)w(t) - k\eta(t)q(t)\psi^{\alpha}(\sigma(t),t_2)\frac{z^{\alpha}\left(\tau^{-1}(\sigma(t))\right)}{z^{\alpha}(t)} - \alpha\frac{w^{(1+\alpha)/\alpha}(t)}{(\eta(t)r(t))^{1/\alpha}} \tag{2.13}$$

for $t \geq t_4$. From (1.3) and the fact that τ is strictly increasing, we have

$$\tau^{-1}(\sigma(t)) \leq t,$$

and since $z(t)/A(t,t_2)$ is nonincreasing on $[t_4,\infty) \subseteq [t_3,\infty)$, we get

$$\frac{z\left(\tau^{-1}(\sigma(t))\right)}{z(t)} \geq \frac{A\left(\tau^{-1}(\sigma(t)),t_2\right)}{A(t,t_2)}. \tag{2.14}$$

Using (2.14) in (2.13), we obtain

$$w'(t) \leq \xi(t)w(t) - k\eta(t)q(t)\psi^{\alpha}(\sigma(t),t_2)\frac{A^{\alpha}\left(\tau^{-1}(\sigma(t)),t_2\right)}{A^{\alpha}(t,t_2)} - \alpha\frac{w^{(1+\alpha)/\alpha}(t)}{(\eta(t)r(t))^{1/\alpha}},$$

which can be written as, for $t \geq t_4$,

$$w'(t) \leq \xi(t)w(t) - \Psi(t) - \frac{\alpha w^{1/\alpha-1}(t)}{(\eta(t)r(t))^{1/\alpha}} w^2(t). \tag{2.15}$$

In view of (2.7) and (2.12), for $t \geq t_4$ we have

$$\begin{aligned} w^{\frac{1}{\alpha}-1}(t) = (\eta(t)r(t)^{\frac{1}{\alpha}-1}\left(\left(\frac{z'(t)}{z(t)}\right)^{\alpha}\right)^{\frac{1}{\alpha}-1} &= (\eta(t)r(t)^{\frac{1}{\alpha}-1}\left(\frac{z(t)}{z'(t)}\right)^{\alpha-1} \\ &\geq \eta^{\frac{1}{\alpha}-1}(t)A^{\alpha-1}(t,t_2). \end{aligned} \tag{2.16}$$

Using (2.16) in (2.15), we arrive at

$$w'(t) \leq \xi(t)w(t) - \Psi(t) - \frac{\alpha A^{\alpha-1}(t,t_2)}{\eta(t)r^{1/\alpha}(t)} w^2(t). \tag{2.17}$$

Multiplying (2.17) by $H(t,s)$ and integrating from T to t, we have, for some $\gamma \geq 1$ and for all $t \geq T \geq t_4$,

$$\begin{aligned} \int_T^t H(t,s)\Psi(s)ds \leq & - \int_T^t H(t,s)w'(s)ds + \int_T^t H(t,s)\xi(s)w(s)ds \\ & - \frac{\alpha}{\gamma}\int_T^t H(t,s)\frac{A^{\alpha-1}(s,t_2)}{\eta(s)r^{1/\alpha}(s)}w^2(s)ds \\ & - \frac{\alpha(\gamma-1)}{\gamma}\int_T^t H(t,s)\frac{A^{\alpha-1}(s,t_2)}{\eta(s)r^{1/\alpha}(s)}w^2(s)ds. \end{aligned} \tag{2.18}$$

An integrating by parts yields

$$\begin{aligned} \int_T^t H(t,s)w'(s)ds &= H(t,s)w(s)\mid_T^t - \int_T^t \frac{\partial H}{\partial s}(t,s)w(s)ds \\ &= -H(t,T)w(T) - \int_T^t \frac{\partial H}{\partial s}(t,s)w(s)ds. \end{aligned} \tag{2.19}$$

Substituting (2.19) into (2.18) and taking (2.1) into account yields

$$\begin{aligned} \int_T^t H(t,s)\Psi(s)ds \leq & \ H(t,T)w(T) \\ & + \int_T^t \left[-h(t,s)\sqrt{H(t,s)} + H(t,s)\xi(s)\right]w(s)ds \\ & - \frac{\alpha}{\gamma}\int_T^t H(t,s)\frac{A^{\alpha-1}(s,t_2)}{\eta(s)r^{1/\alpha}(s)}w^2(s)ds \\ & - \frac{\alpha(\gamma-1)}{\gamma}\int_T^t H(t,s)\frac{A^{\alpha-1}(s,t_2)}{\eta(s)r^{1/\alpha}(s)}w^2(s)ds. \end{aligned} \tag{2.20}$$

Completing the square with respect to w, it follows from (2.20) that

$$\int_T^t \left[H(t,s)\Psi(s) - \frac{\gamma}{4\alpha} \frac{\eta(s) r^{1/\alpha}(s) \Phi(t,s)}{A^{\alpha-1}(s,t_2)} \right] ds \le H(t,T) w(T) \\ - \frac{\alpha(\gamma-1)}{\gamma} \int_T^t H(t,s) \frac{A^{\alpha-1}(s,t_2)}{\eta(s) r^{1/\alpha}(s)} w^2(s) ds. \tag{2.21}$$

So, for every $t \ge t_4$, we obtain

$$\int_{t_4}^t \left[H(t,s)\Psi(s) - \frac{\gamma}{4\alpha} \frac{\eta(s) r^{1/\alpha}(s) \Phi(t,s)}{A^{\alpha-1}(s,t_2)} \right] ds \le H(t,t_4) w(t_4),$$

which contradicts (2.2).

Next, we consider case (II). Letting $u(t) = r(t)\left(-z'(t)\right)^{\alpha} > 0$ for $t \ge t_2$, it follows from (1.1) that

$$u'(t) + \frac{p(t)}{r(t)} u(t) \ge 0 \quad \text{for } t \ge t_2.$$

Integrating this relation from t_2 to t, we obtain

$$u(t) \ge u(t_2) \exp\left(-\int_{t_2}^t \frac{p(s)}{r(s)} ds \right),$$

from which we have

$$z'(t) \le r^{1/\alpha}(t_2) z'(t_2) \left[\frac{1}{r(t)} \exp\left(-\int_{t_2}^t \frac{p(s)}{r(s)} ds \right) \right]^{1/\alpha}. \tag{2.22}$$

Integrating (2.22) from t_2 to t and taking (1.2) into account, we see that

$$z(t) \le z(t_2) + r^{1/\alpha}(t_2) z'(t_2) \int_{t_2}^t \left[\frac{1}{r(s)} \exp\left(-\int_{t_2}^s \frac{p(u)}{r(u)} du \right) \right]^{1/\alpha} ds \to -\infty$$

as $t \to \infty$, which contradicts the positivity of $z(t)$ and completes the proof. □

The following oscillation criterion follows immediately from Theorem 2.1.

Corollary 2.2. *Let the assumptions of Theorem 2.1 be satisfied except that condition* (2.2) *is replaced by*

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t k^{-1} H(t,s) \Psi(s) ds = \infty \tag{2.23}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \frac{\eta(s) r^{1/\alpha}(s) \Phi(t,s)}{A^{\alpha-1}(s,t_2)} ds < \infty. \tag{2.24}$$

Then equation (1.1) *is oscillatory.*

Theorem 2.3. *Suppose that conditions (i)–(iv),* (1.2) *and* (1.3) *are satisfied. Let* H *and* h *be as in Theorem 2.1 such that* (2.1) *holds, and*

$$0 < \inf_{s \geq t_0} \left\{ \liminf_{t \to \infty} \frac{H(t,s)}{H(t,t_0)} \right\} \leq \infty. \tag{2.25}$$

If there exist functions $\phi \in C([t_0,\infty),\mathbb{R})$ *and* $\eta \in C^1([t_0,\infty),(0,\infty))$ *such that, for some* $\gamma > 1$*,*

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\Psi(s) - \frac{\gamma}{4\alpha} \frac{\eta(s) r^{1/\alpha}(s)\Phi(t,s)}{A^{\alpha-1}(s,t_2)} \right] ds \geq \phi(T) \tag{2.26}$$

and

$$\int_T^\infty \frac{A^{\alpha-1}(s,t_2)}{\eta(s) r^{1/\alpha}(s)} \phi_+^2(s) ds = \infty, \tag{2.27}$$

for all sufficiently large $t_2 \in [t_1,\infty) \subseteq [t_0,\infty)$*, and all* $T > t_2$ *with* $\sigma(t) > t_2$ *for all* $t \geq T$*, where* $\Psi(s)$ *and* $\Phi(t,s)$ *are as in Theorem 2.1, and* $\phi_+(t) = \max\{\phi(t),0\}$*, then every solution of* (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \geq t_1$ for some $t_1 \in [t_{0,}\infty)$. Proceeding as in the proof of Theorem 2.1, we again have the two cases to consider: (I) $z'(t) > 0$ for $t \geq t_2$ or (II) $z'(t) < 0$ for $t \geq t_2$. If case (II) holds, proceeding exactly as in the proof of Theorem 2.1, we obtain a contradiction to the positivity of z.

Next, assume that case (I) holds. Proceeding as in the proof of Theorem 2.1, we again arrive at (2.21), which can be written as, for $t > T \geq t_4$,

$$\begin{aligned} &\frac{1}{H(t,T)} \int_T^t \left[H(t,s)\Psi(s) - \frac{\gamma}{4\alpha} \frac{\eta(s) r^{1/\alpha}(s)\Phi(t,s)}{A^{\alpha-1}(s,t_2)} \right] ds \\ &\leq w(T) - \frac{1}{H(t,T)} \int_T^t \frac{\alpha(\gamma-1)}{\gamma} \frac{H(t,s) A^{\alpha-1}(s,t_2)}{\eta(s) r^{1/\alpha}(s)} w^2(s) ds. \end{aligned} \tag{2.28}$$

From (2.28), we see that

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\Psi(s) - \frac{\gamma}{4\alpha} \frac{\eta(s) r^{1/\alpha}(s)\Phi(t,s)}{A^{\alpha-1}(s,t_2)} \right] ds \\ &\leq w(T) - \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \frac{\alpha(\gamma-1)}{\gamma} \frac{H(t,s) A^{\alpha-1}(s,t_2)}{\eta(s) r^{1/\alpha}(s)} w^2(s). \end{aligned} \tag{2.29}$$

In view of (2.26), it follows from (2.29) that

$$w(T) \geq \phi(T) + \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \frac{\alpha(\gamma-1)}{\gamma} \frac{H(t,s) A^{\alpha-1}(s,t_2)}{\eta(s) r^{1/\alpha}(s)} w^2(s) ds \tag{2.30}$$

for all $t > T \geq t_4$ and for any $\gamma > 1$. Thus, for all $T \geq t_4$,

$$w(T) \geq \phi(T) \tag{2.31}$$

and

$$\liminf_{t\to\infty} \frac{1}{H(t,t_4)} \int_{t_4}^{t} \frac{H(t,s)A^{\alpha-1}(s,t_2)}{\eta(s)r^{1/\alpha}(s)} w^2(s)ds \leq \frac{\gamma(w(t_4)-\phi(t_4))}{\alpha(\gamma-1)} < \infty. \tag{2.32}$$

Now, we claim that

$$\int_{t_4}^{\infty} \frac{A^{\alpha-1}(s,t_2)}{\eta(s)r^{1/\alpha}(s)} w^2(s)ds < \infty. \tag{2.33}$$

Suppose the contrary, that is,

$$\int_{t_4}^{\infty} \frac{A^{\alpha-1}(s,t_2)}{\eta(s)r^{1/\alpha}(s)} w^2(s)ds = \infty. \tag{2.34}$$

By (2.25), there exists a constant $\varepsilon > 0$ such that

$$\inf_{s\geq t_0} \left\{ \liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} \right\} > \varepsilon. \tag{2.35}$$

On the other hand, by virtue of (2.34), for any positive number δ, there exists a $t_5 > t_4$ such that

$$\int_{t_4}^{t} \frac{A^{\alpha-1}(s,t_2)}{\eta(s)r^{1/\alpha}(s)} w^2(s)ds \geq \frac{\delta}{\varepsilon} \quad \text{for all } t \geq t_5. \tag{2.36}$$

Using integration by parts and taking (2.36) into account, we conclude that, for all $t \geq t_5$,

$$\begin{aligned}
&\frac{1}{H(t,t_4)} \int_{t_4}^{t} H(t,s) \frac{A^{\alpha-1}(s,t_2)}{\eta(s)r^{1/\alpha}(s)} w^2(s)ds \\
&= \frac{1}{H(t,t_4)} \int_{t_4}^{t} H(t,s) d\left[\int_{t_4}^{s} \frac{A^{\alpha-1}(\xi,t_2)}{\eta(\xi)r^{1/\alpha}(\xi)} w^2(\xi)d\xi \right] \\
&= \frac{1}{H(t,t_4)} \int_{t_4}^{t} \left[\int_{t_4}^{s} \frac{A^{\alpha-1}(\xi,t_2)}{\eta(\xi)r^{1/\alpha}(\xi)} w^2(\xi)d\xi \right] \left[-\frac{\partial H(t,s)}{\partial s} \right] ds \\
&\geq \frac{\delta}{\varepsilon} \frac{1}{H(t,t_4)} \int_{t_5}^{t} \left[-\frac{\partial H(t,s)}{\partial s} \right] ds \\
&= \frac{\delta}{\varepsilon} \frac{H(t,t_5)}{H(t,t_4)} \geq \frac{\delta}{\varepsilon} \frac{H(t,t_5)}{H(t,t_0)}.
\end{aligned} \tag{2.37}$$

It follows from (2.35) that

$$\liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} > \varepsilon > 0, \tag{2.38}$$

and hence there exists a $t_6 \geq t_5$ such that

$$\frac{H(t,t_5)}{H(t,t_0)} \geq \varepsilon \quad \text{for all } t \geq t_6.$$

From the latter inequality and (2.37), we see that

$$\frac{1}{H(t,t_4)} \int_{t_4}^{t} H(t,s) \frac{A^{\alpha-1}(s,t_2)}{\eta(s) r^{1/\alpha}(s)} w^2(s) ds \geq \delta \quad \text{for } t \geq t_6. \tag{2.39}$$

Since δ is an arbitrary positive constant, we have

$$\liminf_{t\to\infty} \frac{1}{H(t,t_4)} \int_{t_4}^{t} H(t,s) \frac{A^{\alpha-1}(s,t_2)}{\eta(s) r^{1/\alpha}(s)} w^2(s) ds = \infty, \tag{2.40}$$

which contradicts (2.32). Thus, (2.33) should hold, and so, by (2.31) we have

$$\int_{t_4}^{\infty} \frac{A^{\alpha-1}(s,t_2)}{\eta(s) r^{1/\alpha}(s)} \phi_+^2(s) ds \leq \int_{t_4}^{\infty} \frac{A^{\alpha-1}(s,t_2)}{\eta(s) r^{1/\alpha}(s)} w^2(s) ds < \infty, \tag{2.41}$$

which contradicts (2.27). This proves the theorem. □

Theorem 2.4. *Let all conditions of Theorem 2.3 be satisfied except that condition* (2.26) *be replaced with*

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} \left[H(t,s)\Psi(s) - \frac{\gamma}{4\alpha} \frac{\eta(s) r^{1/\alpha}(s) \Phi(t,s)}{A^{\alpha-1}(s,t_2)} \right] ds \geq \phi(T). \tag{2.42}$$

Then, every solution of (1.1) *is oscillatory.*

Proof. The proof follows from the fact that

$$\begin{aligned} \phi(T) &\leq \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} \left[H(t,s)\Psi(s) - \frac{\gamma}{4\alpha} \frac{\eta(s) r^{1/\alpha}(s) \Phi(t,s)}{A^{\alpha-1}(s,t_2)} \right] ds \\ &\leq \limsup_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} \left[H(t,s)\Psi(s) - \frac{\gamma}{4\alpha} \frac{\eta(s) r^{1/\alpha}(s) \Phi(t,s)}{A^{\alpha-1}(s,t_2)} \right] ds, \end{aligned}$$

and so we omit the details. □

Next, we give oscillation results in the case when (1.4) holds.

Theorem 2.5. *Let conditions (i)–(iv),* (1.2) *and* (1.4) *be fulfilled, and let H and h be as in Theorem 2.1 such that* (2.1) *holds. If there exists a positive function $\eta \in C^1([t_0,\infty),\mathbb{R})$ such that, for some $\gamma \geq 1$,*

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} \left[H(t,s)\Omega(s) - \frac{\gamma}{4\alpha} \frac{\eta(s) r^{1/\alpha}(s) \Phi(t,s)}{A^{\alpha-1}(s,t_2)} \right] ds = \infty, \tag{2.43}$$

for all sufficiently large $t_2 \in [t_1, \infty) \subseteq [t_0, \infty)$*, and all* $T > t_2$ *with* $\sigma(t) > t_2$ *for all* $t \geq T$*, where*

$$\Omega(t) = k\eta(t)q(t)\psi^\alpha(\sigma(t), t_2), \tag{2.44}$$

and $\Phi(t, s)$ *is as in* (2.4)*, then every solution of (1.1) is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1) with $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \geq t_1$ for some $t_1 \in [t_{0,}\infty)$. Proceeding as in the proof of Theorem 2.1, we again have two cases to consider: (I) $z'(t) > 0$ for $t \geq t_2$ or (II) $z'(t) < 0$ for $t \geq t_2$. If case (II) holds, as in the proof of Theorem 2.1, we contradict the positivity of $z(t)$.

If case (I) holds, then, as in the proof of Theorem 2.1, we again arrive at (2.13) for $t \geq t_4$. From (1.4) and the fact that τ is strictly increasing, we have

$$\tau^{-1}(\sigma(t)) \geq t,$$

and since z is increasing, we obtain

$$\frac{z\left(\tau^{-1}(\sigma(t))\right)}{z(t)} \geq 1. \tag{2.45}$$

Using (2.45) in (2.13) yields

$$w'(t) \leq \xi(t)w(t) - k\eta(t)q(t)\psi^\alpha(\sigma(t), t_2) - \alpha\frac{w^{(1+\alpha)/\alpha}(t)}{(\eta(t)r(t))^{1/\alpha}}. \tag{2.46}$$

The remainder of the proof is similar to the first part of the proof of Theorem 2.1 and hence is omitted. □

Corollary 2.6. *The conclusion of Theorem 2.5 remains intact if assumption* (2.43) *is replaced by the two conditions*

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t k^{-1}H(t,s)\Omega(s)ds = \infty, \tag{2.47}$$

and (2.24).

Theorem 2.7. *Suppose that conditions (i)–(iv),* (1.2) *and* (1.4) *are satisfied. Let* H *and* h *be as in Theorem 2.1 such that* (2.1) *and* (2.25) *hold. If there exist functions* $\phi \in C([t_0, \infty), \mathbb{R})$ *and* $\eta \in C^1([t_0, \infty), (0, \infty))$ *such that* (2.27) *holds, and for some* $\gamma > 1$,

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\Omega(s) - \frac{\gamma}{4\alpha}\frac{\eta(s)r^{1/\alpha}(s)\Phi(t,s)}{A^{\alpha-1}(s,t_2)}\right] ds \geq \phi(T), \tag{2.48}$$

for all sufficiently large $t_2 \in [t_1, \infty) \subseteq [t_0, \infty)$*, and all* $T > t_2$ *with* $\sigma(t) > t_2$ *for all* $t \geq T$*, where* $\Omega(t)$ *is as in* (2.44)*, then every solution of* (1.1) *is oscillatory.*

Proof. The proof follows from (2.45), (2.46) and Theorem 2.3, so we omit the details. □

Theorem 2.8. *Let all conditions of Theorem 2.7 be satisfied except that condition* (2.48) *be replaced with*

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\Omega(s) - \frac{\gamma}{4\alpha} \frac{\eta(s) r^{1/\alpha}(s) \Phi(t,s)}{A^{\alpha-1}(s,t_2)} \right] ds \geq \phi(T). \tag{2.49}$$

Then, every solution of (1.1) *is oscillatory.*

Proof. The proof follows from (2.45), (2.46) and Theorem 2.4, so we omit the details. □

3 Examples

We conclude this paper with the following examples to illustrate the above results. The first example is concerned with the case where $h(t) \to \infty$ as $t \to \infty$, and the second example is concerned with the case where h is a constant function.

Example 3.1. Consider the half-linear neutral differential equation with damping

$$\left((z'(t))^5 \right)' + \frac{1}{t} (z'(t))^5 + (t+1)^4 x^5(t-1) = 0, \; t \geq 2, \tag{3.1}$$

with $z(t) = x(t) + tx(t-2)$. Here we have $\alpha = 5$, $\tau(t) = t-2$, $\sigma(t) = t-1$, $r(t) = 1$, $p(t) = 1/t$, $q(t) = (t+1)^4$, $h(t) = t$, and $f(t, x(\sigma(t))) = x^5(t-1)$. It is easy to see that conditions (i)–(iv), (1.2) and (1.4) hold. Choosing $t_2 = t_1 = t_0 = 2$, we have

$$A(t,t_2) = A(t,2) = t-2,$$

$$A(\tau^{-1}(t), t_2) = A(t+2, 2) = t,$$

$$A(\tau^{-1}(\tau^{-1}(t)), t_2) = A(t+4, 2) = t+2,$$

$$\psi(t,t_2) = \frac{1}{t+2} \left(1 - \frac{t+2}{t(t+4)} \right) > 0 \quad \text{for } t \geq t_0 = 2.$$

Letting $H(t,s) = (t-s)^2$, we see that $H \in \mathcal{P}$ and $h(t,s) = 2$. With $\eta(t) = t$, we see that $\xi(t) = 0$, and conditions (2.47) and (2.24) become, for all $T \in (3, \infty)$,

$$\begin{aligned} \limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t k^{-1} H(t,s)\Omega(s) ds &\geq \limsup_{t\to\infty} \frac{(2/5)^5}{(t-T)^2} \int_T^t (t-s)^2 \frac{s}{s+1} ds \\ &\geq \limsup_{t\to\infty} \frac{(2/5)^5 T}{T+1} \frac{1}{(t-T)^2} \int_T^t (t-s)^2 ds \end{aligned}$$

$$= \limsup_{t\to\infty} \frac{(2/5)^5 T(t^3 - 3t^2T + 3tT^2 - T^3)}{3(T+1)(t-T)^2} = \infty$$

and

$$\begin{aligned}\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \frac{\eta(s) r^{1/\alpha}(s) \Phi(t,s)}{A^{\alpha-1}(s,t_2)} ds &= \limsup_{t\to\infty} \frac{1}{(t-T)^2} \int_T^t \frac{4s}{(s-2)^4} ds \\ &\le \limsup_{t\to\infty} \frac{1}{(t-T)^2} \frac{4}{(T-2)^4} \int_T^t s ds \\ &= \limsup_{t\to\infty} \frac{2(t^2-T^2)}{(T-2)^4 (t-T)^2} = \frac{2}{(T-2)^4} < \infty,\end{aligned}$$

i.e., conditions (2.47) and (2.24) hold. Thus, all conditions of Corollary 2.6 are satisfied, so equation (3.1) is oscillatory.

Example 3.2. Consider the half-linear neutral differential equation with damping

$$\left(\frac{1}{t^3} (z'(t))^3\right)' + \frac{1}{t^4} (z'(t))^3 + t^2 x^3(t/2) = 0, \ t \ge 2, \tag{3.2}$$

with $z(t) = x(t) + 10x(t-1)$. Here we have $\alpha = 3$, $\tau(t) = t-1$, $\sigma(t) = t/2$, $r(t) = 1/t^3$, $p(t) = 1/t^4$, $q(t) = t^2$, $h(t) = 10$, and $f(t, x(\sigma(t))) = x^3(t/2)$. It is easy to see that conditions (i)–(iv), (1.2) and (1.3) hold, and with $t_2 = t_1 = t_0 = 2$, we have

$$A(t,t_2) = A(t,2) = (t^2 - 4)/2,$$

$$\begin{aligned}A(\tau^{-1}(t), t_2) &= A(t+1, 2) = ((t+1)^2 - 4)/2, \\ A(\tau^{-1}(\tau^{-1}(t)), t_2) &= A(t+2, 2) = ((t+2)^2 - 4)/2, \\ \psi(t,t_2) &= \frac{1}{10}\left(1 - \frac{t(t+4)}{10(t-1)(t+3)}\right) > 0 \quad \text{for } t \ge t_0 = 2.\end{aligned}$$

Letting $H(t,s) = (t-s)^2$, we see that $H \in \mathcal{P}$ and $h(t,s) = 2$. With $\eta(t) = t$, we have $\xi(t) = 0$, and as in Example 3.1, it is easy to see that conditions (2.23) and (2.24) hold. Thus, all conditions of Corollary 2.2 are satisfied, so equation (3.2) is oscillatory.

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