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Existence of a Solution for a Variational Inequality Associated with the Maxwell-Stokes Type Problem and the Continuous Dependence of the Solution on the Data

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Abstract

In this paper, we prove the existence of a solution for a variational inequality associated with the Maxwell-Stokes type equation in a bounded multiply connected domain with holes. Our equation is nonlinear and contains, the so called, p-curlcurl equation. Furthermore, we obtain the continuous dependence of the solution on the data.

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1. INTRODUCTION

In this paper, we consider a stationary nonlinear electromagnetic field in a multiply connected domain in $\mathbb{R}^3$ with holes. The electric and magnetic fields $\boldsymbol{e}$ and $\boldsymbol{h}$ satisfy the following Maxwell equations

$$\begin{cases} \boldsymbol{j} = \operatorname{curl} \boldsymbol{h} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{h} = 0 & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{e} = q & \text{in } \Omega, \\ \operatorname{curl} \boldsymbol{e} = \boldsymbol{f} & \text{in } \Omega, \end{cases}$$

where $\boldsymbol{j}$ denotes the total current density, q is the electric charge and $\boldsymbol{f}$ is here a given external field. We use the following nonlinear extension of Ohm's law

$$|\boldsymbol{j}|^{p-2}\boldsymbol{j} = \sigma \boldsymbol{e},$$

where σ is the electric conductivity. Then the magnetic field $\boldsymbol{h}$ satisfies

$$\begin{cases} \operatorname{curl}\left[\frac{1}{\sigma}|\operatorname{curl}\boldsymbol{h}|^{p-2}\operatorname{curl}\boldsymbol{h}\right] = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{h} = 0 & \text{in } \Omega. \end{cases} \tag{1.1}$$

The left-hand side of the first equation in (1.1) is called the p-curlcurl operator. For a weak solution to such a system under certain boundary condition, see Yin et al. [17], Miranda et al. [10], [11], Pan [13], and Aramaki [4]. A necessary condition for the existence of a solution of the problem (1.1) is that the external field $\boldsymbol{f}$ must satisfy $\operatorname{div}\boldsymbol{f} = 0$ in Ω. However, if this condition is not satisfied, then it is expected to demand an unknown potential function π such that

$$\begin{cases} \operatorname{curl}\left[\frac{1}{\sigma}|\operatorname{curl}\boldsymbol{h}|^{p-2}\operatorname{curl}\boldsymbol{h}\right] + \boldsymbol{\nabla}\pi = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{h} = 0 & \text{in } \Omega. \end{cases} \tag{1.2}$$

Whether a solution to (1.2) exists or not depends heavily on the boundary conditions and the geometry of the domain Ω.

We also consider another constitutive law that arises in type-II superconductors, which is known as an extension of the Bean critical-state model in Prigozhin [14]. In this case the current density $\boldsymbol{j} = \operatorname{curl}\boldsymbol{h}$ cannot exceed the critical value $\Psi = \Psi(x) > 0$ and we have

$$\boldsymbol{e} = \begin{cases} \frac{1}{\sigma}|\operatorname{curl}\boldsymbol{h}|^{p-2}\operatorname{curl}\boldsymbol{h} & \text{if } |\operatorname{curl}\boldsymbol{h}| < \Psi(x), \\ \left(\frac{1}{\sigma}\Psi^{p-2} + \lambda\right)\operatorname{curl}\boldsymbol{h} & \text{if } |\operatorname{curl}\boldsymbol{h}| = \Psi(x), \end{cases}$$

where $\lambda = \lambda(x) \geq 0$ is regarded as a unknown Lagrange multiplier. This leads to the variational inequality

$$\int_\Omega \frac{1}{\sigma}|\operatorname{curl}\boldsymbol{h}|^{p-2}\operatorname{curl}\boldsymbol{h}\cdot\operatorname{curl}(\boldsymbol{v}-\boldsymbol{h})dx + \int_\Omega \boldsymbol{\nabla}\pi\cdot(\boldsymbol{v}-\boldsymbol{h})dx \geq \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v}-\boldsymbol{h})dx$$

for any test function $\boldsymbol{v}$ such that $|\operatorname{curl}\boldsymbol{v}| \leq \Psi(x)$ a.e. in Ω.

In this paper, we consider such a variational inequality. We use a nicely extended Carathéodory function $S(x,t)$ defined in $\Omega \times [0,\infty)$ by Aramaki [5], and we consider the following system

$$\begin{cases} \operatorname{curl}\left[S_t(x,|\operatorname{curl}\boldsymbol{u}|^2)\operatorname{curl}\boldsymbol{u}\right] + \boldsymbol{\nabla}\pi = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega, \end{cases} \tag{1.3}$$

where $S_t = \partial S/\partial t$. Since we allow that Ω is multiply connected and has holes, we assume that Ω satisfies (O1) and (O2) defined in section 2. In particular, the boundary Γ of Ω has finitely many connected components $\Gamma_0, \Gamma_1, \ldots, \Gamma_I$ with Γ_0 denoting the boundary of the infinite connected component of $\mathbb{R}^3 \setminus \overline{\Omega}$.

We impose boundary conditions to system (1.3),

$$\begin{cases} \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0 & \text{for } i = 1, \ldots, I, \end{cases} \tag{1.4}$$

where $\boldsymbol{n}$ is the outer unit normal vector to Γ and $\langle \cdot, \cdot \rangle_{\Gamma_i}$ denotes some duality bracket defined in section 2.

Thus we consider the following variational inequality: to find $(\boldsymbol{u}, \pi)$ in an appropriate space such that

$$\int_\Omega S_t(x, |\mathrm{curl}\, \boldsymbol{u}|^2) \mathrm{curl}\, \boldsymbol{u} \cdot \mathrm{curl}\, (\boldsymbol{v} - \boldsymbol{u}) dx - \int_\Omega \pi \mathrm{div}\, (\boldsymbol{v} - \boldsymbol{u}) dx \geq \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{v} - \boldsymbol{u}) dx \tag{1.5}$$

for all $\boldsymbol{v}$ such that $|\mathrm{curl}\, \boldsymbol{v}| \leq \Psi(x)$ a.e. in Ω.

The first purpose of this paper is to show the existence of a unique solution to (1.5) under boundary conditions (1.4) (Theorem 3 3). More precisely, let the constrained function Ψ be of the form $\Psi(x) = F(\varphi(x))$, where $F : \mathbb{R} \to [0, \infty)$ is a continuous function and $\varphi \in L^\infty(\Omega)$, To get a solution to (1.5), we use the standard minimization problem of some functional on a closed convex subset

$$\mathbb{K}_\varphi = \{\boldsymbol{v} \in \mathbb{X}_N^p(\Omega) : |\mathrm{curl}\, \boldsymbol{v}| \leq F(\varphi) \text{ a.e. in } \Omega\},$$

where $\mathbb{X}_N^p(\Omega)$ is a reflexive Banach space associated with the boundary condition (1.4) defined in section 2. We note that the functional

$$\int_\Omega S(x, |\mathrm{curl}\, \boldsymbol{v}|^2) dx$$

is not coercive on $\mathbb{K}_\varphi$. To overcome this, we use the penalty method introduced by Temam [16]. As a result, we can find a unique solution to (1.5) as $(\boldsymbol{u}, \pi) \in \widehat{\mathbb{K}}_\varphi \times L^{p'}(\Omega)$, where

$$\widehat{\mathbb{K}}_\varphi = \{\boldsymbol{v} \in \mathbb{V}_N^p(\Omega) : |\mathrm{curl}\, \boldsymbol{v}| \leq F(\varphi) \text{ a.e. in } \Omega\}.$$

Here $\mathbb{V}_N^p(\Omega)$ is a reflexive Banach space defined in section 2.

The second purpose of this paper is to derive the continuity of the solution to (1.5) on the data $\boldsymbol{f}$ and φ. Let $\boldsymbol{f}_n, \boldsymbol{f} \in \mathbb{X}_N^p(\Omega)'$ ($\mathbb{X}_N^p(\Omega)'$ denotes the dual space of $\mathbb{X}_N^p(\Omega)$), and let $(\boldsymbol{u}_n, \pi_n) \in \widehat{\mathbb{K}}_{\varphi_n} \times L^{p'}(\Omega)$ be the solution to (1.5) with $\boldsymbol{f} = \boldsymbol{f}_n$ and $\varphi = \varphi_n$. We show that if $\boldsymbol{f}_n \to \boldsymbol{f}$ in $\mathbb{X}_N^p(\Omega)'$ and $\varphi_n \to \varphi$ in $L^\infty(\Omega)$, then we can prove that $\boldsymbol{u}_n \to \boldsymbol{u}$ strongly in $\mathbb{V}_N^p(\Omega)$ and $\pi_n \to \pi$ weakly in $L^{p'}(\Omega)$. To show that $\{\boldsymbol{u}_n\}$ converges strongly in $\mathbb{V}_N^p(\Omega)$, we use the celebrated result of Mosco [12] (Theorem 4.2).

This paper is organized as follows. Section 2 covers preliminaries in which we give the geometry of the domain Ω, some spaces of functions and their properties. In section 3, we consider a variational inequality as in (1.5) and give the main theorem on the existence of a solution. In section 4, we consider the continuity of the solution obtained in section 3 on the data $\boldsymbol{f}$ and the constrained function. We apply the result of Mosco [12].

2. PRELIMINARIES

In this section, we introduce the geometry of the domain, a Carathéodory function $S(x, t)$ on $\Omega \times [0, +\infty)$ satisfying some structural conditions, and some spaces of functions.

Let Ω be a bounded domain in $\mathbb{R}^3$ with a $C^{1,1}$ boundary Γ. Since we allow Ω to be a multiply-connected domain with holes in $\mathbb{R}^3$, we assume that Ω satisfies the following conditions as in Amrouche and Seloula [2] (cf. Amrouche and Seloula [1], Dautray and Lions [7, vol. 3] and Girault and Raviart [9]). Ω is locally situated on one side of Γ and satisfies the following (O1) and (O2).

(O1) Γ has a finite number of connected components $\Gamma_0, \Gamma_1, \ldots, \Gamma_I$ with Γ_0 denoting the boundary of the infinite connected component of $\mathbb{R}^3 \setminus \overline{\Omega}$.

(O2) There exist J connected open surfaces Σ_j, $(j = 1, \ldots, J)$, called cuts, contained in Ω such that

- (a) each surface Σ_j is an open subset of a smooth manifold $\mathcal{M}_j$,
- (b) $\partial\Sigma_j \subset \Gamma$ $(j = 1, \ldots, J)$, where $\partial\Sigma_j$ denotes the boundary of Σ_j, and Σ_j is non-tangential to Γ,
- (c) $\overline{\Sigma_j} \cap \overline{\Sigma_k} = \emptyset$ $(j \neq k)$,
- (d) the open set $\Omega^\circ = \Omega \setminus (\cup_{j=1}^J \Sigma_j)$ is simply connected and has Lipschitz-continuous boundary.

The number J is called the first Betti number and I is the second Betti number. We say that Ω is simply connected if $J = 0$ and Ω has no holes if $I = 0$. If we define

$$\mathbb{K}_T^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega); \operatorname{curl} \boldsymbol{v} = \boldsymbol{0}, \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega, \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\}$$

and

$$\mathbb{K}_N^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega); \operatorname{curl} \boldsymbol{v} = \boldsymbol{0}, \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega, \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma\},$$

then it is well known that $\dim \mathbb{K}_T^p(\Omega) = J$ and $\dim \mathbb{K}_N^p(\Omega) = I$.

Throughout this paper, let $1 < p < \infty$ and we denote the conjugate exponent of p by p', i.e., $(1/p) + (1/p') = 1$. From now on we use $L^p(\Omega)$, $W_0^{1,p}(\Omega)$ and $W^{1,p}(\Omega)$ for the standard L^p and Sobolev spaces of functions. For any Banach space B, we denote $B \times B \times B$ by the boldface character $\boldsymbol{B}$. We use this character to denote vector and vector-valued functions, and we denote the standard Euclidean inner product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbb{R}^3$ by $\boldsymbol{a} \cdot \boldsymbol{b}$. For the dual space $\boldsymbol{B}'$ of $\boldsymbol{B}$, we write $\langle \cdot, \cdot \rangle_{\boldsymbol{B}', \boldsymbol{B}}$ for the duality bracket.

We assume that a Carathéodory function $S(x,t)$ in $\Omega \times [0,\infty)$ satisfies the following structural conditions. For a.e. $x \in \Omega$, $S(x,t) \in C^2((0,\infty)) \cap C^0([0,\infty))$, and positive constants $0 < \lambda \le \Lambda < \infty$ such that for a.e. $x \in \Omega$,

$$S(x,0) = 0 \text{ and } \quad \lambda t^{(p-2)/2} \le S_t(x,t) \le \Lambda t^{(p-2)/2} \text{ for } t > 0, \tag{2.1a}$$

$$\lambda t^{(p-2)/2} \le S_t(x,t) + 2tS_{tt}(x,t) \le \Lambda t^{(p-2)/2} \text{ for } t > 0, \tag{2.1b}$$

$$\text{If } 1 < p < 2, S_{tt}(x,t) < 0, \text{ and if } p \ge 2, S_{tt}(x,t) \ge 0 \text{ for } t > 0, \tag{2.1c}$$

where $S_t = \partial S/\partial t$ and $S_{tt} = \partial^2 S/\partial t^2$. We note that from (2.1a), it follows that

$$\frac{2}{p}\lambda t^{p/2} \le S(x,t) \le \frac{2}{p}\Lambda t^{p/2} \text{ for } t \ge 0. \tag{2.2}$$

Example 2.1. *If $S(x,t) = \nu(x)g(t)t^{p/2}$, where ν is a measurable function in Ω and satisfies $0 < \nu_* \le \nu(x) \le \nu^* < \infty$ for a.e. $x \in \Omega$ for some constants ν_* and ν^*, and $g \in C^\infty([0,\infty))$,*

When $g(t) \equiv 1$, it follows from elementary calculations that (2.1a)-(2.1c) *hold.*

As another example, we can take

$$g(t) = \begin{cases} a(e^{-1/t} + 1) & \text{if } t > 0, \\ a & \text{if } t = 0 \end{cases}$$

with a constant $a > 0$. Then $S(x,t) = \nu(x)g(t)t^{p/2}$ satisfies (2.1a)-(2.1c) *if $p \ge 2$. (cf. Aramaki [6, Example 3.2]).*

We remember the monotonic property of S_t.

Lemma 2.2. *There exists a constant $c > 0$ such that for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$,*

$$\big(S_t(x, |\boldsymbol{a}|^2)\boldsymbol{a} - S_t(x, |\boldsymbol{b}|^2)\boldsymbol{b}\big) \cdot (\boldsymbol{a} - \boldsymbol{b}) \geq \begin{cases} c|\boldsymbol{a} - \boldsymbol{b}|^p & \text{if } p \geq 2, \\ c(|\boldsymbol{a}| + |\boldsymbol{b}|)^{p-2}|\boldsymbol{a} - \boldsymbol{b}|^2 & \text{if } 1 < p < 2. \end{cases}$$

In particular, if $\boldsymbol{a} \neq \boldsymbol{b}$, we have

$$\big(S_t(x, |\boldsymbol{a}|^2)\boldsymbol{a} - S_t(x, |\boldsymbol{b}|^2)\boldsymbol{b}\big) \cdot (\boldsymbol{a} - \boldsymbol{b}) > 0.$$

For the proof, see Aramaki [5, Lemma 3.6].

Lemma 2.3. *There exists a constant $C_1 > 0$ depending only on Λ and p such that for any $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$,*

$$|S_t(x, |\boldsymbol{a}|^2)\boldsymbol{a} - S_t(x, |\boldsymbol{b}|^2)\boldsymbol{b}| \leq \begin{cases} C_1|\boldsymbol{a} - \boldsymbol{b}|^{p-1} & \text{if } 1 < p < 2, \\ C_1(|\boldsymbol{a}| + |\boldsymbol{b}|)^{p-2}|\boldsymbol{a} - \boldsymbol{b}| & \text{if } p \geq 2. \end{cases}$$

For the proof, see Aramaki [3].

We can see the convexity of $S(x, t)$ in the following sense.

Lemma 2.4. *If $S(x, t)$ satisfies* (2.1a) *and* (2.1b)*, then for a.e. $x \in \Omega$, the function $\mathbb{R} \ni t \mapsto g[t] = S(x, t^2)$ is strictly convex.*

For the proof, see [6, Lemma 2.3].

The following inequality is used frequently (cf. [2]). If Ω is a bounded domain in $\mathbb{R}^3$ with a $C^{1,1}$ boundary Γ, and if $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ satisfies $\operatorname{curl} \boldsymbol{u} \in \boldsymbol{L}^p(\Omega), \operatorname{div} \boldsymbol{u} \in L^p(\Omega)$ and $\boldsymbol{u} \times \boldsymbol{n} \in \boldsymbol{W}^{1-1/p,p}(\Gamma)$, then $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ and there exists a constant $C > 0$ depending only on p and Ω such that

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C(\|\operatorname{curl} \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} + \|\operatorname{div} \boldsymbol{u}\|_{L^p(\Omega)} + \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{u} \times \boldsymbol{n}\|_{W^{1-1/p,p}(\Gamma)}). \quad (2.3)$$

Moreover, if $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ satisfies $\operatorname{curl} \boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$, then $\boldsymbol{u} \times \boldsymbol{n} \in \boldsymbol{W}^{-1/p,p}(\Gamma)$ is well defined, and if $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ satisfies $\operatorname{div} \boldsymbol{u} \in L^p(\Omega)$, then $\boldsymbol{u} \cdot \boldsymbol{n} \in W^{-1/p,p}(\Gamma)$ is well defined by the formulae

$$\langle \boldsymbol{u} \times \boldsymbol{n}, \boldsymbol{\phi}\rangle_{\boldsymbol{W}^{-1/p,p}(\Gamma), \boldsymbol{W}^{1-1/p',p'}(\Gamma)} = \int_\Omega \boldsymbol{u} \cdot \operatorname{curl} \boldsymbol{\phi} dx - \int_\Omega \operatorname{curl} \boldsymbol{u} \cdot \boldsymbol{\phi} dx$$

for all $\boldsymbol{\phi} \in \boldsymbol{W}^{1,p'}(\Omega)$ and

$$\langle \boldsymbol{u} \cdot \boldsymbol{n}, \phi \rangle_{W^{-1/p,p}(\Gamma), W^{1-1/p',p'}(\Gamma)} = \int_\Omega \boldsymbol{u} \cdot \boldsymbol{\nabla} \phi dx + \int_\Omega (\operatorname{div} \boldsymbol{u}) \phi dx$$

for all $\phi \in W^{1,p'}(\Omega)$. Furthermore, if $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ satisfies $\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0}$ on Γ, then there exists a constant $C > 0$ depending only on p and Ω such that

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq C(\|\operatorname{curl} \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} + \|\operatorname{div} \boldsymbol{u}\|_{L^p(\Omega)} + \sum_{i=1}^{I} |\langle \boldsymbol{u} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i}|$$

where $\langle \cdot, \cdot \rangle_{\Gamma_i} = \langle \cdot, \cdot \rangle_{W^{-1/p,p}(\Gamma_i), W^{1-1/p',p'}(\Gamma_i)}$.

Define a space

$$\begin{aligned} \mathbb{X}_N^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega); \operatorname{curl} \boldsymbol{u} \in \boldsymbol{L}^p(\Omega), \operatorname{div} \boldsymbol{v} \in L^p(\Omega), \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0 \text{ for } i = 1, \ldots, I\}. \end{aligned}$$

with the norm

$$\|\boldsymbol{v}\|_{\mathbb{X}_N^p(\Omega)} = (\|\operatorname{curl} \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}^p + \|\operatorname{div} \boldsymbol{u}\|_{L^p(\Omega)}^p)^{1/p}.$$

We note that $\|\boldsymbol{v}\|_{\mathbb{X}_N^p(\Omega)}$ is equivalent to $\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)}$ for $\boldsymbol{v} \in \mathbb{X}_N^p(\Omega)$ (cf. [2]). Since $\mathbb{X}_N^p(\Omega)$ is a closed subspace of $\boldsymbol{W}^{1,p}(\Omega)$, we can see that $\mathbb{X}_N^p(\Omega)$ is a reflexive Banach space and $\boldsymbol{W}_0^{1,p}(\Omega) \hookrightarrow \mathbb{X}_N^p(\Omega) \hookrightarrow \boldsymbol{W}^{1,p}(\Omega)$, where the symbol $\hookrightarrow$ means that the inclusion map is continuous. Furthermore, we define a closed subspace $\mathbb{V}_N^p(\Omega)$ of $\mathbb{X}_N^p(\Omega)$ by

$$\mathbb{V}_N^p(\Omega) = \{\boldsymbol{v} \in \mathbb{X}_N^p(\Omega); \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega\}$$

with the norm $\|\boldsymbol{v}\|_{\mathbb{V}_N^p(\Omega)} = \|\operatorname{curl} \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}$ which is also equivalent to $\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)}$. We note that $\mathbb{V}_N^p(\Omega)$ is also a reflexive Banach space.

Lemma 2.5. *If* $\boldsymbol{v} \in \boldsymbol{L}^{p'}(\Omega)$, *then* $\operatorname{curl} \boldsymbol{v} \in \mathbb{X}_N^p(\Omega)'$ *and*

$$\langle \operatorname{curl} \boldsymbol{v}, \boldsymbol{\varphi} \rangle_{\mathbb{X}_N^p(\Omega)', \mathbb{X}_N^p(\Omega)} = \int_\Omega \boldsymbol{v} \cdot \operatorname{curl} \boldsymbol{\varphi} dx \textit{ for all } \boldsymbol{\varphi} \in \mathbb{X}_N^p(\Omega). \tag{2.4}$$

Moreover, there exists a constant $C > 0$ *depending only on* p *and* Ω *such that*

$$\|\operatorname{curl} \boldsymbol{v}\|_{\mathbb{X}_N^p(\Omega)'} \leq C \|\boldsymbol{v}\|_{\boldsymbol{L}^{p'}(\Omega)} \textit{ for all } \boldsymbol{v} \in \boldsymbol{L}^{p'}(\Omega).$$

Proof. Let $\boldsymbol{v} \in \boldsymbol{L}^{p'}(\Omega)$. Then the distribution $\operatorname{curl} \boldsymbol{v} \in \boldsymbol{\mathcal{D}}'(\Omega)$ is defined by

$$\langle \operatorname{curl} \boldsymbol{v}, \boldsymbol{\varphi} \rangle = \int_\Omega \boldsymbol{v} \cdot \operatorname{curl} \boldsymbol{\varphi} dx \text{ for all } \boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega) = \boldsymbol{C}_0^\infty(\Omega).$$

Define temporarily a Banach space

$$H_0(\mathrm{curl}\,,\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega); \mathrm{curl}\,\boldsymbol{v} \in \boldsymbol{L}^p(\Omega), \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma\}$$

with the norm $\|\boldsymbol{v}\|_{H_0(\mathrm{curl}\,,\Omega)} = (\|\boldsymbol{v}\|^2_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{curl}\,\boldsymbol{v}\|^p_{\boldsymbol{L}^p(\Omega)})^{1/p}$. Then by Temam [16] or [9], $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $H_0(\mathrm{curl}\,,\Omega)$. Hence for any $\boldsymbol{\varphi} \in H_0(\mathrm{curl}\,,\Omega)$, there exists a sequence $\{\boldsymbol{\varphi}_j\} \subset \boldsymbol{\mathcal{D}}(\Omega)$ such that $\boldsymbol{\varphi}_j \to \boldsymbol{\varphi}$ in $H_0(\mathrm{curl}\,,\Omega)$. Define

$$\langle \mathrm{curl}\,\boldsymbol{v}, \boldsymbol{\varphi}\rangle = \lim_{j\to\infty} \int_\Omega \boldsymbol{v} \cdot \mathrm{curl}\,\boldsymbol{\varphi}_j dx.$$

Clearly, the definition is well defined (independent of the choice of a sequence $\{\boldsymbol{\varphi}_j\}$ such that $\boldsymbol{\varphi}_j \to \boldsymbol{\varphi}$ in $H_0(\mathrm{curl}\,,\Omega)$), and

$$\begin{aligned}|\langle \mathrm{curl}\,\boldsymbol{v}, \boldsymbol{\varphi}\rangle| = \lim_{j\to\infty} \left|\int_\Omega \boldsymbol{v} \cdot \mathrm{curl}\,\boldsymbol{\varphi}_j dx\right| &\le \lim_{j\to\infty} \|\boldsymbol{v}\|_{\boldsymbol{L}^{p'}(\Omega)} \|\mathrm{curl}\,\boldsymbol{\varphi}_j\|_{\boldsymbol{L}^p(\Omega)} \\ &\le \|\boldsymbol{v}\|_{\boldsymbol{L}^{p'}(\Omega)} \|\mathrm{curl}\,\boldsymbol{\varphi}\|_{\boldsymbol{L}^p(\Omega)}.\end{aligned}$$

Therefore, we have

$$\langle \mathrm{curl}\,\boldsymbol{v}, \boldsymbol{\varphi}\rangle = \lim_{j\to\infty} \int_\Omega \boldsymbol{v} \cdot \mathrm{curl}\,\boldsymbol{\varphi}_j dx = \int_\Omega \boldsymbol{v} \cdot \mathrm{curl}\,\boldsymbol{\varphi} dx \text{ for all } \boldsymbol{\varphi} \in H_0(\mathrm{curl}\,,\Omega).$$

Moreover, we have $|\langle \mathrm{curl}\,\boldsymbol{v}, \boldsymbol{\varphi}\rangle| \le \|\boldsymbol{v}\|_{\boldsymbol{L}^{p'}(\Omega)} \|\boldsymbol{\varphi}\|_{H_0(\mathrm{curl}\,,\Omega)}$. Thus we can see that $\mathrm{curl}\,\boldsymbol{v} \in H_0(\mathrm{curl}\,,\Omega)'$. On the other hand, since $\mathbb{X}^p_N(\Omega) \hookrightarrow H_0(\mathrm{curl}\,,\Omega)$, we have $H_0(\mathrm{curl}\,,\Omega)' \hookrightarrow \mathbb{X}^p_N(\Omega)'$, and there exists a constant $C > 0$ depending only on p and Ω such that

$$|\langle \mathrm{curl}\,\boldsymbol{v}, \boldsymbol{\varphi}\rangle| \le C\|\boldsymbol{v}\|_{\boldsymbol{L}^{p'}(\Omega)} \|\boldsymbol{\varphi}\|_{\mathbb{X}^p_N(\Omega)} \text{ for all } \boldsymbol{\varphi} \in \mathbb{X}^p_N(\Omega).$$

Thus $\mathrm{curl}\,\boldsymbol{v} \in \mathbb{X}^p_N(\Omega)'$, (2.4) holds and $\|\mathrm{curl}\,\boldsymbol{v}\|_{\mathbb{X}^p_N(\Omega)'} \le C\|\boldsymbol{v}\|_{\boldsymbol{L}^{p'}(\Omega)}$ for any $\boldsymbol{v} \in \boldsymbol{L}^{p'}(\Omega)$. □

Corollary 2.6. *If* $\boldsymbol{v} \in \mathbb{X}^p_N(\Omega)$*, then* $\mathrm{curl}\,[S_t(x, |\mathrm{curl}\,\boldsymbol{v}|^2)\mathrm{curl}\,\boldsymbol{v}] \in \mathbb{X}^p_N(\Omega)'$*, and there exists a constant* $C > 0$ *depending only on* p, Λ *and* Ω *such that*

$$\|\mathrm{curl}\,[S_t(x|\mathrm{curl}\,\boldsymbol{v}|^2)\mathrm{curl}\,\boldsymbol{v}]\|_{\mathbb{X}^p_N(\Omega)'} \le C\|\boldsymbol{v}\|^{p-1}_{\mathbb{X}^p_N(\Omega)}.$$

Proof. If $\boldsymbol{v} \in \mathbb{X}^p_N(\Omega)$, then from (2.1b), $|S_t(x, |\mathrm{curl}\,\boldsymbol{v}|^2)\mathrm{curl}\,\boldsymbol{v}| \le \Lambda|\mathrm{curl}\,\boldsymbol{v}|^{p-1}$. Hence $S_t(x, |\mathrm{curl}\,\boldsymbol{v}|^2)\mathrm{curl}\,\boldsymbol{v} \in \boldsymbol{L}^{p'}(\Omega)$, and

$$\|S_t(x, |\mathrm{curl}\,\boldsymbol{v}|^2)\mathrm{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^{p'}(\Omega)} \le \Lambda \left(\int_\Omega |\mathrm{curl}\,\boldsymbol{v}|^p dx\right)^{1/p'} \le \Lambda\|\boldsymbol{v}\|^{p-1}_{\mathbb{X}^p_N(\Omega)}.$$

It suffices to apply Lemma 2.5. □

3. A VARIATIONAL INEQUALITY FOR THE MAXWELL-STOKES PROBLEM

In this section, we consider a variational inequality. Let $F : \mathbb{R} \to [0, \infty)$ be a continuous function, and let $\varphi \in L^\infty(\Omega)$. Define a closed convex subset $\mathbb{K}_\varphi$ of $\mathbb{X}^p_N(\Omega)$ and a closed convex subset $\widehat{\mathbb{K}}_\varphi$ of $\mathbb{V}^p_N(\Omega)$ by

$$\mathbb{K}_\varphi = \{\boldsymbol{v} \in \mathbb{X}^p_N(\Omega); |\mathrm{curl}\,\boldsymbol{v}| \le F(\varphi) \text{ a.e. in } \Omega\}$$

and

$$\widehat{\mathbb{K}}_\varphi = \{\boldsymbol{v} \in \mathbb{V}^p_N(\Omega); |\mathrm{curl}\,\boldsymbol{v}| \le F(\varphi) \text{ a.e. in } \Omega\},$$

respectively. For a given function $\boldsymbol{f} \in \mathbb{X}^p_N(\Omega)'$, we consider the following variational inequality: to find $(\boldsymbol{u}, \pi) \in \widehat{\mathbb{K}}_\varphi \times L^{p'}(\Omega)$ such that

$$\int_\Omega S_t(x, |\mathrm{curl}\,\boldsymbol{u}|^2)\mathrm{curl}\,\boldsymbol{u} \cdot \mathrm{curl}\,(\boldsymbol{v} - \boldsymbol{u})dx - \int_\Omega \pi \mathrm{div}\,\boldsymbol{v} dx \\ \ge \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u} \rangle_{\mathbb{X}^p_N(\Omega)', \mathbb{X}^p_N(\Omega)} \text{ for all } \boldsymbol{v} \in \mathbb{K}_\varphi. \quad (3.1)$$

We solve problem (3.1) by the penalty method introduced by Temam [16]. To do so, we consider the following functional E_ε on $\mathbb{K}_\varphi$ depending on a parameter $\varepsilon \in (0, 1]$ defined by

$$E_\varepsilon[\boldsymbol{v}] = \frac{1}{2}\left\{\int_\Omega S(x, |\mathrm{curl}\,\boldsymbol{v}|^2)dx + \frac{1}{\varepsilon}\int_\Omega S(x, (\mathrm{div}\,\boldsymbol{v})^2)dx\right\} \\ - \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\mathbb{X}^p_N(\Omega)', \mathbb{X}^p_N(\Omega)} \text{ for } \boldsymbol{v} \in \mathbb{K}_\varphi. \quad (3.2)$$

We derive the following minimization problem: to find $\boldsymbol{u}_\varepsilon \in \mathbb{K}_\varphi$ such that

$$E_\varepsilon[\boldsymbol{u}_\varepsilon] = \inf_{\boldsymbol{v} \in \mathbb{K}_\varphi} E_\varepsilon[\boldsymbol{v}]. \quad (3.3)$$

We call such a $\boldsymbol{u}_\varepsilon \in \mathbb{K}_\varphi$ a minimizer of E_ε.

Proposition 3.1. *Assume that $\boldsymbol{f} \in \mathbb{X}^p_N(\Omega)'$. Then the minimization problem* (3.3) *has a unique minimizer $\boldsymbol{u}_\varepsilon \in \mathbb{K}_\varphi$, and there exists a constant $C > 0$ depending only on p, λ and Ω, but independent of $\varepsilon \in (0, 1]$ such that*

$$\|\boldsymbol{u}_\varepsilon\|^p_{\mathbb{X}^p_N(\Omega)} \le C\|\boldsymbol{f}\|^{p'}_{\mathbb{X}^p_N(\Omega)'} \quad (3.4)$$

and

$$\|\mathrm{div}\,\boldsymbol{u}_\varepsilon\|^p_{L^p(\Omega)} \le C\varepsilon\|\boldsymbol{f}\|^{p'}_{\mathbb{X}^p_N(\Omega)'}. \quad (3.5)$$

Proof. It is clear that E_ε is proper from (2.2), and that the functional E_ε is strictly convex from Lemma 2.4. Moreover, E_ε is lower semi-continuous on $\mathbb{K}_\varphi$ (cf. [5]). For any $\varepsilon \in (0,1]$ and for any $\boldsymbol{v} \in \mathbb{K}_\varphi$, it follows from (2.2) and the Young inequality that

$$\begin{aligned} E_\varepsilon[\boldsymbol{v}] &\geq \frac{\lambda}{p}\left\{\int_\Omega |\mathrm{curl}\,\boldsymbol{v}|^p dx + \frac{1}{\varepsilon}\int_\Omega |\mathrm{div}\,\boldsymbol{v}|^p dx\right\} - \|\boldsymbol{f}\|_{\mathbb{X}^p_N(\Omega)'}\|\boldsymbol{v}\|_{\mathbb{X}^p_N(\Omega)} \\ &\geq \frac{\lambda}{p}\|\boldsymbol{v}\|^p_{\mathbb{X}^p_N(\Omega)} - C(\delta)\|\boldsymbol{f}\|^{p'}_{\mathbb{X}^p_N(\Omega)'} - \delta\|\boldsymbol{v}\|^p_{\mathbb{X}^p_N(\Omega)} \end{aligned}$$

for any $\delta > 0$ and some $C(\delta) > 0$. If we choose $\delta = \lambda/(2p)$, then we have

$$E_\varepsilon[\boldsymbol{v}] \geq \frac{\lambda}{2p}\|\boldsymbol{v}\|^p_{\mathbb{X}^p_N(\Omega)} - C\big(\frac{\lambda}{2p}\big)\|\boldsymbol{f}\|^{p'}_{\mathbb{X}^p_N(\Omega)'}.$$

Hence E_ε is coercive on $\mathbb{K}_\varphi$. From Ekeland and Témam [8, Chapter II, Proposition 1.2], problem (3.3) has a unique minimizer $\boldsymbol{u}_\varepsilon \in \mathbb{K}_\varphi$.

For any $\boldsymbol{v} \in \mathbb{K}_\varphi$ and $0 \leq \mu \leq 1$, since $(1-\mu)\boldsymbol{u}_\varepsilon + \mu\boldsymbol{v} = \boldsymbol{u}_\varepsilon + \mu(\boldsymbol{v} - \boldsymbol{u}_\varepsilon) \in \mathbb{K}_\varphi$, we have

$$\frac{d}{d\mu}E_\varepsilon[\boldsymbol{u}_\varepsilon + \mu(\boldsymbol{v} - \boldsymbol{u}_\varepsilon)]\bigg|_{\mu=0} \geq 0.$$

That is to say, the minimizer $\boldsymbol{u}_\varepsilon$ satisfies the following inequality

$$\begin{aligned} &\int_\Omega S_t(x, |\mathrm{curl}\,\boldsymbol{u}_\varepsilon|^2)\mathrm{curl}\,\boldsymbol{u}_\varepsilon \cdot \mathrm{curl}\,(\boldsymbol{v} - \boldsymbol{u}_\varepsilon)dx \\ &\qquad + \frac{1}{\varepsilon}\int_\Omega S_t(x, (\mathrm{div}\,\boldsymbol{u}_\varepsilon)^2)(\mathrm{div}\,\boldsymbol{u}_\varepsilon)\mathrm{div}\,(\boldsymbol{v} - \boldsymbol{u}_\varepsilon)dx \\ &\qquad\qquad \geq \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}_\varepsilon\rangle_{\mathbb{X}^p_N(\Omega)', \mathbb{X}^p_N(\Omega)} \text{ for all } \boldsymbol{v} \in \mathbb{K}_\varphi. \end{aligned} \tag{3.6}$$

Taking $\boldsymbol{v} = \boldsymbol{0} \in \mathbb{K}_\varphi$ in (3.6) as a test function, we have

$$\begin{aligned} \lambda(\|\mathrm{curl}\,\boldsymbol{u}_\varepsilon\|^p_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\,\boldsymbol{u}_\varepsilon\|^p_{L^p(\Omega)}) &\leq \lambda\big(\|\mathrm{curl}\,\boldsymbol{u}_\varepsilon\|^p_{\boldsymbol{L}^p(\Omega)} + \frac{1}{\varepsilon}\|\mathrm{div}\,\boldsymbol{u}_\varepsilon\|^p_{L^p(\Omega)}\big) \\ &\leq \langle \boldsymbol{f}, \boldsymbol{u}_\varepsilon\rangle_{\mathbb{X}^p_N(\Omega)', \mathbb{X}^p_N(\Omega)} \leq C(\delta)\|\boldsymbol{f}\|^{p'}_{\mathbb{X}^p_N(\Omega)'} + \delta\|\boldsymbol{u}_\varepsilon\|^p_{\mathbb{X}^p_N(\Omega)} \end{aligned}$$

for any $\delta > 0$. If we choose $\delta > 0$ so that $\delta < \lambda$, we have estimate (3.4). Using (3.4), we also get estimate (3.5). □

Thus we showed that the variational problem (3.6) has a solution. We derive the uniqueness of solution to the problem (3.6).

Lemma 3.2. *The variational inequality* (3.6) *has a unique solution.*

Proof. It suffices to prove the uniqueness. Let $\boldsymbol{u}_\varepsilon^1, \boldsymbol{u}_\varepsilon^2 \in \mathbb{K}_\varphi$ be two solutions to (3.6). Then we have

$$\int_\Omega S_t(x, |\mathrm{curl}\, \boldsymbol{u}_\varepsilon^1|^2)\mathrm{curl}\, \boldsymbol{u}_\varepsilon^1 \cdot \mathrm{curl}\, (\boldsymbol{u}_\varepsilon^2 - \boldsymbol{u}_\varepsilon^1)dx + \frac{1}{\varepsilon}\int_\Omega S_t(x, (\mathrm{div}\, \boldsymbol{u}_\varepsilon^1)^2)(\mathrm{div}\, \boldsymbol{u}_\varepsilon^1)\mathrm{div}\, (\boldsymbol{u}_\varepsilon^2 - \boldsymbol{u}_\varepsilon^1)dx \geq \langle \boldsymbol{f}, \boldsymbol{u}_\varepsilon^2 - \boldsymbol{u}_\varepsilon^1\rangle_{\mathbb{X}_N^p(\Omega)', \mathbb{X}_N^p(\Omega)}$$

and

$$\int_\Omega S_t(x, |\mathrm{curl}\, \boldsymbol{u}_\varepsilon^2|^2)\mathrm{curl}\, \boldsymbol{u}_\varepsilon^2 \cdot \mathrm{curl}\, (\boldsymbol{u}_\varepsilon^1 - \boldsymbol{u}_\varepsilon^2)dx + \frac{1}{\varepsilon}\int_\Omega S_t(x, (\mathrm{div}\, \boldsymbol{u}_\varepsilon^2)^2)(\mathrm{div}\, \boldsymbol{u}_\varepsilon^2)\mathrm{div}\, (\boldsymbol{u}_\varepsilon^1 - \boldsymbol{u}_\varepsilon^2)dx \geq \langle \boldsymbol{f}, \boldsymbol{u}_\varepsilon^1 - \boldsymbol{u}_\varepsilon^2\rangle_{\mathbb{X}_N^p(\Omega)', \mathbb{X}_N^p(\Omega)}.$$

Hence we have

$$\int_\Omega \big(S_t(x, |\mathrm{curl}\, \boldsymbol{u}_\varepsilon^1|^2)\mathrm{curl}\, \boldsymbol{u}_\varepsilon^1 - S_t(x, |\mathrm{curl}\, \boldsymbol{u}_\varepsilon^2|^2)\mathrm{curl}\, \boldsymbol{u}_\varepsilon^2\big) \cdot \mathrm{curl}\, (\boldsymbol{u}_\varepsilon^1 - \boldsymbol{u}_\varepsilon^2)dx + \frac{1}{\varepsilon}\int_\Omega \big(S_t(x, (\mathrm{div}\, \boldsymbol{u}_\varepsilon^1)^2)(\mathrm{div}\, \boldsymbol{u}_\varepsilon^1) - S_t(x, (\mathrm{div}\, \boldsymbol{u}_\varepsilon^2))(\mathrm{div}\, \boldsymbol{u}_\varepsilon^2)\big)\mathrm{div}\, (\boldsymbol{u}_\varepsilon^1 - \boldsymbol{u}_\varepsilon^2)dx \leq 0.$$

By Lemma 2.2, we can see that $\mathrm{curl}\, (\boldsymbol{u}_\varepsilon^1 - \boldsymbol{u}_\varepsilon^2) = \boldsymbol{0}$ and $\mathrm{div}\, (\boldsymbol{u}_\varepsilon^1 - \boldsymbol{u}_\varepsilon^2) = 0$ in Ω, so $\boldsymbol{u}_\varepsilon^1 = \boldsymbol{u}_\varepsilon^2$. □

Here we prepare the following lemma.

Lemma 3.3. *For any* $\psi \in L^p(\Omega)$*, there exists* $\boldsymbol{v}_\psi \in \mathbb{X}_N^p(\Omega)$ *such that* $\mathrm{curl}\, \boldsymbol{v}_\psi = \boldsymbol{0}, \mathrm{div}\, \boldsymbol{v}_\psi = \psi$ *in* Ω*, and there exists a constant* $C > 0$ *such that*

$$\|\boldsymbol{v}_\psi\|_{\mathbb{X}_N^p(\Omega)} \leq C\|\psi\|_{L^p(\Omega)}.$$

Thus $\boldsymbol{v}_\psi \in \mathbb{K}_\varphi$.

Proof. For any $\psi \in L^p(\Omega)$, the following Dirichlet problem

$$\begin{cases} \Delta\phi = \psi & \text{in } \Omega, \\ \phi = 0 & \text{on } \Gamma \end{cases}$$

has a unique solution $\phi \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. If we define $\boldsymbol{w} = \boldsymbol{\nabla}\phi$ in Ω, then $\boldsymbol{w} \in \boldsymbol{W}^{1,p}(\Omega)$ satisfies $\mathrm{curl}\, \boldsymbol{w} = \boldsymbol{0}, \mathrm{div}\, \boldsymbol{w} = \psi$ in Ω, Since $\boldsymbol{n} \times \nabla$ contains only the tangential derivatives, $\boldsymbol{n} \times \boldsymbol{w} = \boldsymbol{n} \times \nabla \phi = \mathbf{0}$ on Γ. Here let $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_I\}$ be a basis of $\mathbb{K}_N^p(\Omega)$ such that $\langle \boldsymbol{n} \cdot \boldsymbol{e}_i, 1 \rangle_{\Gamma_k} = \delta_{jk}$, and define

$$\boldsymbol{v}_\psi = \boldsymbol{w} - \sum_{i=1}^{I} \langle \boldsymbol{w} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} \boldsymbol{e}^i.$$

Then clearly $\langle \boldsymbol{v}_\psi \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_k} = 0$ for $k = 1, \dots, I$. Hence $\boldsymbol{v}_\psi \in \mathbb{X}_N^p(\Omega)$ and $\boldsymbol{v}_\psi \in \mathbb{K}_\varphi$. □

We are in a position to state one of the main theorems of this paper.

Theorem 3.4. *Assume that* $\boldsymbol{f} \in \mathbb{X}_N^p(\Omega)'$. *Then the variational inequality* (3.1) *has a unique solution* $(\boldsymbol{u}, \pi) \in \widehat{\mathbb{K}}_\varphi \times L^{p'}(\Omega)$*, and there exists a constant* $C > 0$ *depending only on* p, λ, Λ *and* Ω *such that*

$$\|\boldsymbol{u}\|_{\mathbb{V}_N^p(\Omega)}^p + \|\pi\|_{L^{p'}(\Omega)}^{p'} \le C \|\boldsymbol{f}\|_{\mathbb{X}_N^p(\Omega)'}^{p'}. \tag{3.7}$$

Proof. Let $\boldsymbol{u}_\varepsilon$ be a unique solution of (3.6). Then from (3.5), we can see that $\operatorname{div} \boldsymbol{u}_\varepsilon \to 0$ strongly in $L^p(\Omega)$ as $\varepsilon \to +0$. Define

$$\pi_\varepsilon = -\frac{1}{\varepsilon} S_t(x, (\operatorname{div} \boldsymbol{u}_\varepsilon)^2) \operatorname{div} \boldsymbol{u}_\varepsilon.$$

From (3.4), $\{\boldsymbol{u}_\varepsilon\}$ is bounded in $\mathbb{X}_N^p(\Omega)$. Passing to a subsequence, we can assume that $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}$ weakly in $\mathbb{X}_N^p(\Omega)$ for some $\boldsymbol{u} \in \mathbb{X}_N^p(\Omega)$ and strongly in $\boldsymbol{L}^p(\Omega)$. Since $\operatorname{div} \boldsymbol{u}_\varepsilon \to \operatorname{div} \boldsymbol{u}$ in $\boldsymbol{\mathcal{D}}'(\Omega)$, we have $\operatorname{div} \boldsymbol{u} = 0$ in Ω. Hence $\boldsymbol{u} \in \mathbb{V}_N^p(\Omega)$. Since $\mathbb{K}_\varphi$ is weakly closed subset of $\mathbb{X}_N^p(\Omega)$, we have $\boldsymbol{u} \in \widehat{\mathbb{K}}_\varphi$, and from (3.4),

$$\|\boldsymbol{u}\|_{\mathbb{V}_N^p(\Omega)}^p = \|\boldsymbol{u}\|_{\mathbb{X}_N^p(\Omega)}^p \le \liminf_{\varepsilon \to +0} \|\boldsymbol{u}_\varepsilon\|_{\mathbb{X}_N^p(\Omega)}^p \le C \|\boldsymbol{f}\|_{\mathbb{X}_N^p(\Omega)'}^{p'}. \tag{3.8}$$

We show that $\{\pi_\varepsilon\}_{\varepsilon \in (0,1]}$ is bounded in $L^{p'}(\Omega)$. To show this we note that for any $\psi \in L^p(\Omega)$, there exists $\boldsymbol{v}_\psi \in \boldsymbol{W}^{1,p}(\Omega)$ as in Lemma 3.3. Taking $\boldsymbol{v} = M \boldsymbol{v}_\psi$ $(M > 0)$ as a test function in (3.6) and using (3.4), we have

$$M \int_\Omega \pi_\varepsilon \psi dx \le C \|\boldsymbol{f}\|_{\mathbb{X}_N^p(\Omega)'}^{p'} + M \|\boldsymbol{f}\|_{\mathbb{X}_N^p(\Omega)'} \|\boldsymbol{v}_\psi\|_{\mathbb{X}_N^p(\Omega)}.$$

If we divide this inequality by M and letting $M \to \infty$, we can see that

$$\int_\Omega \pi_\varepsilon \psi dx \le C \|\boldsymbol{f}\|_{\mathbb{X}_N^p(\Omega)'} \|\psi\|_{L^p(\Omega)} \text{ for all } \psi \in L^p(\Omega).$$

This implies that

$$\left| \int_\Omega \pi_\varepsilon \psi dx \right| \le C \|\boldsymbol{f}\|_{\mathbb{X}_N^p(\Omega)'} \|\psi\|_{L^p(\Omega)} \text{ for all } \psi \in L^p(\Omega).$$

So, it follows that

$$\|\pi_\varepsilon\|_{L^{p'}(\Omega)} \le C\|\boldsymbol{f}\|_{\mathbb{X}^p_N(\Omega)'}.$$

Thus $\{\pi_\varepsilon\}$ is bounded in $L^{p'}(\Omega)$. Passing to a subsequence, we can assume that $\pi_\varepsilon \to \pi$ weakly in $L^{p'}(\Omega)$ for some $\pi \in L^{p'}(\Omega)$ and

$$\|\pi\|^{p'}_{L^{p'}(\Omega)} \le \liminf_{\varepsilon\to+0} \|\pi_\varepsilon\|^{p'}_{L^{p'}(\Omega)} \le C\Lambda\|\boldsymbol{f}\|^{p'}_{\mathbb{X}^p_N(\Omega)'}. \tag{3.9}$$

By the monotonicity of S_t, for any $\boldsymbol{v} \in \mathbb{K}_\varphi$,

$$\int_\Omega S_t(x, |\mathrm{curl}\,\boldsymbol{u}_\varepsilon|^2)\mathrm{curl}\,\boldsymbol{u}_\varepsilon \cdot \mathrm{curl}\,(\boldsymbol{u}_\varepsilon - \boldsymbol{v})dx \ge \int_\Omega S_t(x, |\mathrm{curl}\,\boldsymbol{v}|^2)\mathrm{curl}\,\boldsymbol{v} \cdot \mathrm{curl}\,(\boldsymbol{u}_\varepsilon - \boldsymbol{v})dx.$$

Hence we have

$$\int_\Omega S_t(x, |\mathrm{curl}\,\boldsymbol{v}|^2)\mathrm{curl}\,\boldsymbol{v} \cdot \mathrm{curl}\,(\boldsymbol{v} - \boldsymbol{u}_\varepsilon)dx - \int_\Omega \pi_\varepsilon \mathrm{div}\,(\boldsymbol{v} - \boldsymbol{u}_\varepsilon)dx \ge \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}_\varepsilon\rangle_{\mathbb{X}^p_N(\Omega)', \mathbb{X}^p_N(\Omega)}. \tag{3.10}$$

Since $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}$ weakly in $\mathbb{X}^p_N(\Omega)$, $\pi_\varepsilon \to \pi$ weakly in $L^{p'}(\Omega)$ and $\mathrm{div}\,\boldsymbol{u}_\varepsilon \to 0$ strongly in $L^p(\Omega)$, letting $\varepsilon \to +0$ in (3.10), we can derive

$$\int_\Omega S_t(x, |\mathrm{curl}\,\boldsymbol{v}|^2)\mathrm{curl}\,\boldsymbol{v} \cdot \mathrm{curl}\,(\boldsymbol{v} - \boldsymbol{u})dx - \int_\Omega \pi \mathrm{div}\,\boldsymbol{v} dx \ge \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}\rangle_{\mathbb{X}^p_N(\Omega)', \mathbb{X}^p_N(\Omega)} \text{ for all } \boldsymbol{v} \in \mathbb{K}_\varphi. \tag{3.11}$$

For any $\boldsymbol{w} \in \mathbb{K}_\varphi$, taking $\boldsymbol{v} = (1-\mu)\boldsymbol{u} + \mu\boldsymbol{w} = \boldsymbol{u} + \mu(\boldsymbol{w} - \boldsymbol{u}), 0 < \mu < 1$ as a test function of (3.11), we have

$$\int_\Omega S_t(x, |\mathrm{curl}\,(\boldsymbol{u} + \mu(\boldsymbol{w} - \boldsymbol{u}))|^2)\mathrm{curl}\,(\boldsymbol{u} + \mu(\boldsymbol{w} - \boldsymbol{u})) \cdot \mu\mathrm{curl}\,(\boldsymbol{w} - \boldsymbol{u})dx - \mu\int_\Omega \pi \mathrm{div}\,\boldsymbol{w} dx \ge \mu\langle \boldsymbol{f}, \boldsymbol{w} - \boldsymbol{u}\rangle_{\mathbb{X}^p_N(\Omega)', \mathbb{X}^p_N(\Omega)}.$$

If we divide both hand sides by μ, and let $\mu \to +0$, then we have

$$\int_\Omega S_t(x, |\mathrm{curl}\,\boldsymbol{u}|^2)\mathrm{curl}\,\boldsymbol{u} \cdot \mathrm{curl}\,(\boldsymbol{w} - \boldsymbol{u})dx - \int_\Omega \pi \mathrm{div}\,\boldsymbol{w} dx \ge \langle \boldsymbol{f}, \boldsymbol{w} - \boldsymbol{u}\rangle_{\mathbb{X}^p_N(\Omega)', \mathbb{X}^p_N(\Omega)} \text{ for all } \boldsymbol{w} \in \mathbb{K}_\varphi. \tag{3.12}$$

This means that the inequality (3.1) holds.

Finally, we show the uniqueness of the solution. Let $(\boldsymbol{u}_1, \pi_1), (\boldsymbol{u}_2, \pi_2) \in \widehat{\mathbb{K}}_\varphi \times L^{p'}(\Omega)$ be two solution of (3.1). Then since $\operatorname{div} \boldsymbol{u}_i = 0$ in Ω for $i = 1, 2$, we have

$$\int_\Omega S_t(x, |\operatorname{curl} \boldsymbol{u}_i|^2)\operatorname{curl} \boldsymbol{u}_i \cdot \operatorname{curl}(\boldsymbol{u}_j - \boldsymbol{u}_i)dx \geq \langle \boldsymbol{f}, \boldsymbol{u}_j - \boldsymbol{u}_i\rangle_{\mathbb{X}^p_N(\Omega)', \mathbb{X}^p_N(\Omega)} \text{ for } i \neq j.$$

Hence

$$\int_\Omega \big(S_t(x, |\operatorname{curl} \boldsymbol{u}_1|^2)\operatorname{curl} \boldsymbol{u}_1 - S_t(x, |\operatorname{curl} \boldsymbol{u}_2|^2)\operatorname{curl} \boldsymbol{u}_2\big) \cdot \operatorname{curl}(\boldsymbol{u}_1 - \boldsymbol{u}_2)dx \leq 0.$$

By the monotonicity of S_t (Lemma 2.2), we have $\operatorname{curl} \boldsymbol{u}_1 = \operatorname{curl} \boldsymbol{u}_2$ in Ω, so $\boldsymbol{u}_1 = \boldsymbol{u}_2$. Let $\boldsymbol{v} \in \mathbb{K}_\varphi$. From (3.12) with $\boldsymbol{w} = \boldsymbol{v}$ and $\boldsymbol{w} = -\boldsymbol{v}$, we have

$$\int_\Omega S_t(x, |\operatorname{curl} \boldsymbol{u}_1|^2)\operatorname{curl} \boldsymbol{u}_1 \cdot \operatorname{curl}(\boldsymbol{v} - \boldsymbol{u}_1)dx - \int_\Omega \pi_1 \operatorname{div} \boldsymbol{v} dx \\ \geq \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}_1\rangle_{\mathbb{X}^p_N(\Omega)', \mathbb{X}^p_N(\Omega)}$$

and

$$\int_\Omega S_t(x, |\operatorname{curl} \boldsymbol{u}_1|^2)\operatorname{curl} \boldsymbol{u}_1 \cdot \operatorname{curl}(-\boldsymbol{v} - \boldsymbol{u}_1)dx + \int_\Omega \pi_2 \operatorname{div} \boldsymbol{v} dx \\ \geq \langle \boldsymbol{f}, -\boldsymbol{v} - \boldsymbol{u}_1\rangle_{\mathbb{X}^p_N(\Omega)', \mathbb{X}^p_N(\Omega)}.$$

Therefore, we have

$$\int_\Omega (\pi_1 - \pi_2)\operatorname{div} \boldsymbol{v} dx \leq 2\langle \boldsymbol{f}, \boldsymbol{u}_1\rangle_{\mathbb{X}^p_N(\Omega)', \mathbb{X}^p_N(\Omega)} \\ - 2\int_\Omega S_t(x, |\operatorname{curl} \boldsymbol{u}_1|^2)|\operatorname{curl} \boldsymbol{u}_1|^2 dx \text{ for all } \boldsymbol{v} \in \mathbb{K}_\varphi. \quad (3.13)$$

From (3.12) with $\boldsymbol{w} = \boldsymbol{0}$, we see that

$$c := \langle \boldsymbol{f}, \boldsymbol{u}_1\rangle_{\mathbb{X}^p_N(\Omega)', \mathbb{X}^p_N(\Omega)} - \int_\Omega S_t(x, |\operatorname{curl} \boldsymbol{u}_1|^2)|\operatorname{curl} \boldsymbol{u}_1|^2 dx \geq 0.$$

For any $\psi \in C_0^\infty(\Omega)$, choose $\boldsymbol{v}_\psi$ as in Lemma 3.3. Then from (3.13), we have

$$\int_\Omega (\pi_1 - \pi_2)\operatorname{div} \boldsymbol{v}_\psi dx \leq 2c.$$

For large $M > 0$, since $M\boldsymbol{v}_\psi \in \mathbb{K}_\varphi$, we see that

$$\int_\Omega (\pi_1 - \pi_2)\operatorname{div} \boldsymbol{v}_\psi dx \leq \frac{2c}{M}.$$

Letting $M \to \infty$, we have

$$\int_\Omega (\pi_1 - \pi_2)\psi dx \le 0 \text{ for all } \psi \in C_0^\infty(\Omega).$$

This implies that

$$\int_\Omega (\pi_1 - \pi_2)\psi dx = 0 \text{ for all } \psi \in C_0^\infty(\Omega).$$

By the celebrated Du Bois Raymond Lemma, we have $\pi_1 = \pi_2$ a.e. in Ω. This completes the proof of Theorem 3.4. □

4. CONTINUOUS DEPENDENCE ON THE DATA

In this section, we show the continuous dependence of the solution obtained in section 3 to problem (3.1) on the data. Let $\boldsymbol{f} \in \mathbb{X}_N^p(\Omega)'$ and $\varphi \in L^\infty(\Omega)$. For solution $\boldsymbol{u}$ of (3.1), we consider the following variational inequality.

$$\begin{aligned}\int_\Omega S_t(x, |\mathrm{curl}\, \boldsymbol{u}|^2)\mathrm{curl}\, \boldsymbol{u} \cdot \mathrm{curl}\, (\boldsymbol{v} - \boldsymbol{u})dx &\ge \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}\rangle_{\mathbb{X}_N^p(\Omega)', \mathbb{X}_N^p(\Omega)} \\ &= \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}\rangle_{\mathbb{V}_N^p(\Omega)', \mathbb{V}_N^p(\Omega)} \quad (4.1)\end{aligned}$$

for all $\boldsymbol{v} \in \widehat{\mathbb{K}}_\varphi$.

Lemma 4.1. *If* $(\boldsymbol{u}, \pi) \in \widehat{\mathbb{K}}_\varphi \times L^{p'}(\Omega)$ *is a unique solution of* (3.1)*, then* $\boldsymbol{u} \in \widehat{\mathbb{K}}_\varphi$ *is a unique solution of* (4.1).

Proof. It is clear that $\boldsymbol{u} \in \widehat{\mathbb{K}}_\varphi$ is a solution of (4.1), since $\boldsymbol{v} \in \widehat{\mathbb{K}}_\varphi$ satisfies $\mathrm{div}\, \boldsymbol{v} = 0$ in Ω. Let $\boldsymbol{u}, \widetilde{\boldsymbol{u}}$ be two solutions of (4.1). Then

$$\int_\Omega S_t(x, |\mathrm{curl}\, \boldsymbol{u}|^2)\mathrm{curl}\, \boldsymbol{u} \cdot \mathrm{curl}\, (\widetilde{\boldsymbol{u}} - \boldsymbol{u})dx \ge \langle \boldsymbol{f}, \widetilde{\boldsymbol{u}} - \boldsymbol{u}\rangle_{\mathbb{V}_N^p(\Omega)', \mathbb{V}_N^p(\Omega)}$$

and

$$\int_\Omega S_t(x, |\mathrm{curl}\, \widetilde{\boldsymbol{u}}|^2)\mathrm{curl}\, \widetilde{\boldsymbol{u}} \cdot \mathrm{curl}\, (\boldsymbol{u} - \widetilde{\boldsymbol{u}})dx \ge \langle \boldsymbol{f}, \boldsymbol{u} - \widetilde{\boldsymbol{u}}\rangle_{\mathbb{V}_N^p(\Omega)', \mathbb{V}_N^p(\Omega)}.$$

Therefore, we have

$$\int_\Omega \big(S_t(x, |\mathrm{curl}\, \boldsymbol{u}|^2)\mathrm{curl}\, \boldsymbol{u} - S_t(x, |\mathrm{curl}\, \widetilde{\boldsymbol{u}}|^2)\mathrm{curl}\, \widetilde{\boldsymbol{u}}\big) \cdot \mathrm{curl}\, (\boldsymbol{u} - \widetilde{\boldsymbol{u}})dx \le 0.$$

By the monotonicity lemma (Lemma 2.2), we have $\boldsymbol{u} = \widetilde{\boldsymbol{u}}$ in $\mathbb{V}_N^p(\Omega)$. □

Now, we give the second main theorem of this paper.

Theorem 4.2. *Assume that $F : \mathbb{R} \to [0, \infty)$ is a continuous function satisfying that there exists a constant $\nu > 0$ such that $\nu \le F(s)$ for all $s \in \mathbb{R}$. Let $\boldsymbol{f}_n, \boldsymbol{f} \in \mathbb{X}^p_N(\Omega)' (\subset \mathbb{V}^p_N(\Omega)')$ and $\varphi_n, \varphi \in L^\infty(\Omega)$, and let $(\boldsymbol{u}_n, \pi_n) \in \widehat{\mathbb{K}}_{\varphi_n} \times L^{p'}(\Omega)$ and $(\boldsymbol{u}, \pi) \in \widehat{\mathbb{K}}_\varphi \times L^{p'}(\Omega)$ be unique solutions of* (3.1) *with $\varphi = \varphi_n$ and $\varphi = \varphi$, respectively. If $\boldsymbol{f}_n \to \boldsymbol{f}$ in $\mathbb{X}^p_N(\Omega)'$ (so in $\mathbb{V}^p_N(\Omega)'$) and $\varphi_n \to \varphi$ in $L^\infty(\Omega)$ as $n \to \infty$, then $\boldsymbol{u}_n \to \boldsymbol{u}$ strongly in $\mathbb{V}^p_N(\Omega)$ and $\pi_n \to \pi$ weakly in $L^{p'}(\Omega)$.*

Proof. In order to show that $\boldsymbol{u}_n \to \boldsymbol{u}$ strongly in $\mathbb{V}^p_N(\Omega)$, we apply the result of Mosco [12, Theorem A]. Define an operator $S : \mathbb{V}^p_N(\Omega) \to \mathbb{V}^p_N(\Omega)'$ by

$$\langle S\boldsymbol{u}, \boldsymbol{v}\rangle = \int_\Omega S_t(x, |\mathrm{curl}\, \boldsymbol{u}|^2)\mathrm{curl}\, \boldsymbol{u} \cdot \mathrm{curl}\, \boldsymbol{v} dx \text{ for } \boldsymbol{u}, \boldsymbol{v} \in \mathbb{V}^p_N(\Omega).$$

By the Hölder inequality, since

$$\begin{aligned} &\left| \int_\Omega S_t(x, |\mathrm{curl}\, \boldsymbol{u}|^2)\mathrm{curl}\, \boldsymbol{u} \cdot \mathrm{curl}\, \boldsymbol{v} dx \right| \\ &\qquad \le \left(\int_\Omega |S_t(x, |\mathrm{curl}\, \boldsymbol{u}|^2)\mathrm{curl}\, \boldsymbol{u}|^{p'} dx \right)^{1/p'} \left(\int_\Omega |\mathrm{curl}\, \boldsymbol{v}|^p dx \right)^{1/p} \\ &\qquad \le \Lambda \|\boldsymbol{u}\|^{p-1}_{\mathbb{V}^p_N(\Omega)} \|\boldsymbol{v}\|_{\mathbb{V}^p_N(\Omega)}, \end{aligned} \tag{4.2}$$

the operator S is well defined. Furthermore, define operators T_n and T from $\mathbb{V}^p_N(\Omega)$ to $\mathbb{V}^p_N(\Omega)'$ by

$$T_n \boldsymbol{u} = S\boldsymbol{u} - \boldsymbol{f}_n \text{ and } T\boldsymbol{u} = S\boldsymbol{u} - \boldsymbol{f}.$$

We check conditions I, II and III in Theorem A of Mosco [12] in the following lemmas.

First we check Mosco's condition I.

Lemma 4.3. *The above operators T_n and T are monotone hemi-continuous mappings from $\mathbb{V}^p_N(\Omega)$ to $\mathbb{V}^p_N(\Omega)'$, and $\{T_n\}$ is uniformly bounded in $\mathbb{V}^p_N(\Omega)$ and satisfies*

$$G(T) \subset s\text{-}\underline{\mathrm{Lim}}\ G(T_n) \text{ in } \mathbb{V}^p_N(\Omega) \times \mathbb{V}^p_N(\Omega)', \tag{4.3}$$

where $G(T)$ and $G(T_n)$ denote the graphs of T and T_n, respectively. Here we say that $\{T_n\}$ is uniformly bounded on $\mathbb{V}^p_N(\Omega)$, if for any bounded subset B of $\mathbb{V}^p_N(\Omega)$, there exists a bounded subset B' of $\mathbb{V}^p_N(\Omega)'$ such that $T_n B \subset B'$ for each n.

Proof. That the operator S is monotone follows from Lemma 2.2 and S is clearly hemi-continuous since S is a Carathéodory function. For any $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{V}^p_N(\Omega)$, from (4.2),

$$\begin{aligned} |\langle T_n \boldsymbol{v}, \boldsymbol{w}\rangle_{\mathbb{V}^p_N(\Omega)', \mathbb{V}^p_N(\Omega)}| &= \Big| \int_\Omega S_t(x, |\mathrm{curl}\, \boldsymbol{v}|^2)\mathrm{curl}\, \boldsymbol{v} \cdot \mathrm{curl}\, \boldsymbol{w} dx \\ &\qquad - \langle \boldsymbol{f}_n, \boldsymbol{w}\rangle_{\mathbb{V}^p_N(\Omega)', \mathbb{V}^p_N(\Omega)} \Big| \\ &\le (\Lambda \|\boldsymbol{v}\|^{p-1}_{\mathbb{V}^p_N(\Omega)} + \|\boldsymbol{f}_n\|_{\mathbb{V}^p_N(\Omega)'}) \|\boldsymbol{w}\|_{\mathbb{V}^p_N(\Omega)}. \end{aligned}$$

Since $\boldsymbol{f}_n \to \boldsymbol{f}$ in $\mathbb{V}^p_N(\Omega)'$, we can assume that there exists a constant $C_0 > 0$ such that $\|\boldsymbol{f}_n\|_{\mathbb{V}^p_N(\Omega)'} \leq C_0$. Hence

$$\|T_n \boldsymbol{v}\|_{\mathbb{V}^p_N(\Omega)'} \leq \Lambda \|\boldsymbol{v}\|^{p-1}_{\mathbb{V}^p_N(\Omega)} + C_0.$$

Thus $\{T_n\}$ is uniformly bounded in $\mathbb{V}^p_N(\Omega)$. The inclusion (4.3) means that for every $\boldsymbol{v} \in \mathbb{V}^p_N(\Omega)$, there exists $\boldsymbol{v}_n \in \mathbb{V}^p_N(\Omega)$ such that $\boldsymbol{v}_n \to \boldsymbol{v}$ strongly in $\mathbb{V}^p_N(\Omega)$ and $T_n \boldsymbol{v}_n \to T\boldsymbol{v}$ strongly in $\mathbb{V}^p_N(\Omega)'$. We show this. For every $\boldsymbol{v} \in \mathbb{V}^p_N(\Omega)$, let $\boldsymbol{v}_n = \boldsymbol{v}$. For any $\boldsymbol{w} \in \mathbb{V}^p_N(\Omega)$,

$$\begin{aligned}
&|\langle T_n \boldsymbol{v}_n - T\boldsymbol{v}, \boldsymbol{w}\rangle_{\mathbb{V}^p_N(\Omega)', \mathbb{V}^p_N(\Omega)}| \\
&\quad = |\langle S\boldsymbol{v}_n - S\boldsymbol{v}, \boldsymbol{w}\rangle_{\mathbb{V}^p_N(\Omega)', \mathbb{V}^p_N(\Omega)} - \langle \boldsymbol{f}_n - \boldsymbol{f}, \boldsymbol{w}\rangle_{\mathbb{V}^p_N(\Omega)', \mathbb{V}^p_N(\Omega)}| \\
&\quad = |\langle \boldsymbol{f}_n - \boldsymbol{f}, \boldsymbol{w}\rangle_{\mathbb{V}^p_N(\Omega)', \mathbb{V}^p_N(\Omega)}| \\
&\quad \leq \|\boldsymbol{f}_n - \boldsymbol{f}\|_{\mathbb{V}^p_N(\Omega)'} \|\boldsymbol{w}\|_{\mathbb{V}^p_N(\Omega)}.
\end{aligned}$$

Thus it follows from the hypothesis of the Theorem that

$$\|T_n \boldsymbol{v}_n - T\boldsymbol{v}\|_{\mathbb{V}^p_N(\Omega)'} \leq \|\boldsymbol{f}_n - \boldsymbol{f}\|_{\mathbb{V}^p_N(\Omega)} \to 0 \text{ as } n \to \infty.$$

This completes the proof of Lemma 4.3. □

Next we check Mosco's condition II.

Lemma 4.4. *If $\varphi_n \to \varphi$ in $L^\infty(\Omega)$, then*

$$\widehat{\mathbb{K}}_\varphi = \operatorname{Lim} \widehat{\mathbb{K}}_{\varphi_n} \textit{ in the sense of Mosco}.$$

This means that if $\boldsymbol{v}_n \in \widehat{\mathbb{K}}_{\varphi_n}$ and $\boldsymbol{v}_n \to \boldsymbol{v}$ weakly in $\mathbb{V}^p_N(\Omega)$, then $\boldsymbol{v} \in \widehat{\mathbb{K}}_\varphi$, and for given $\boldsymbol{v} \in \widehat{\mathbb{K}}_\varphi$, there exists $\boldsymbol{v}_n \in \widehat{\mathbb{K}}_{\varphi_n}$ such that $\boldsymbol{v}_n \to \boldsymbol{v}$ strongly in $\mathbb{V}^p_N(\Omega)$.

Proof. Let $\boldsymbol{v}_n \in \widehat{\mathbb{K}}_{\varphi_n}$ and $\boldsymbol{v}_n \to \boldsymbol{v}$ weakly in $\mathbb{V}^p_N(\Omega)$. Then $\operatorname{curl} \boldsymbol{v}_n \to \operatorname{curl} \boldsymbol{v}$ weakly in $\boldsymbol{L}^p(\Omega)$ from Lemma 2.5. For any measurable subset $\omega \subset \Omega$,

$$\int_\omega |\operatorname{curl} \boldsymbol{v}|^p dx \leq \liminf_{n\to\infty} \int_\omega |\operatorname{curl} \boldsymbol{v}_n|^p dx \leq \liminf_{n\to\infty} \int_\omega F(\varphi_n)^p dx = \int_\omega F(\varphi)^p dx.$$

This implies that $|\operatorname{curl} \boldsymbol{v}| \leq F(\varphi)$ a.e. in Ω, so $\boldsymbol{v} \in \widehat{\mathbb{K}}_\varphi$.

Next, put $\lambda_n = \|F(\varphi_n) - F(\varphi)\|_{L^\infty(\Omega)}$, then $\lambda_n \to 0$ as $n \to \infty$. For given $\boldsymbol{v} \in \widehat{\mathbb{K}}_\varphi$, define $\boldsymbol{v}_n = \boldsymbol{v}/\mu_n$ with $\mu_n = 1 + \lambda_n/\nu$. Then we have

$$|\operatorname{curl} \boldsymbol{v}_n| = \frac{1}{\mu_n} |\operatorname{curl} \boldsymbol{v}| \leq \frac{1}{\mu_n} F(\varphi) \leq F(\varphi_n)$$

since

$$\mu_n = 1 + \frac{\|F(\varphi_n) - F(\varphi)\|_{L^\infty(\Omega)}}{\nu} \geq 1 + \frac{F(\varphi) - F(\varphi_n)}{F(\varphi_n)} = \frac{F(\varphi)}{F(\varphi_n)}.$$

Thus $\boldsymbol{v}_n \in \widehat{\mathbb{K}}_{\varphi_n}$ and

$$\|\boldsymbol{v}_n - \boldsymbol{v}\|^p_{\mathbb{V}^p_N(\Omega)} = \int_\Omega |\mathrm{curl}\,(\boldsymbol{v}_n - \boldsymbol{v})|^p dx = \big(1 - \frac{1}{\mu_n}\big) \int_\Omega |\mathrm{curl}\,\boldsymbol{v}|^p dx \to 0$$

as $n \to \infty$. This completes the proof of Lemma 4.4. □

Finally we check Mosco's condition III.

Lemma 4.5. *For any $\boldsymbol{w} \in \widehat{\mathbb{K}}_\varphi$, there exists a continuous strictly increasing function $\beta : [0, \infty] \to [0, \infty]$ with $\beta(0) = 0$ such that*

$$\beta(\|\boldsymbol{v} - \boldsymbol{w}\|_{\mathbb{V}^p_N(\Omega)}) \leq \liminf_{n\to\infty} \langle T_n\boldsymbol{v} - T\boldsymbol{w}, \boldsymbol{v} - \boldsymbol{w}\rangle_{\mathbb{V}^p_N(\Omega)',\mathbb{V}^p_N(\Omega)}$$

for all $\boldsymbol{v} \in \mathbb{V}^p_N(\Omega)$ uniformly as $\boldsymbol{v}$ varies in a bounded set.

Proof. It follows from Lemma 2.2 that

$$\begin{aligned}
&\langle T_n\boldsymbol{v} - T\boldsymbol{w}, \boldsymbol{v} - \boldsymbol{w}\rangle_{\mathbb{V}^p_N(\Omega)',\mathbb{V}^p_N(\Omega)} \\
&\quad = \langle S\boldsymbol{v} - S\boldsymbol{w}, \boldsymbol{v} - \boldsymbol{w}\rangle_{\mathbb{V}^p_N(\Omega)',\mathbb{V}^p_N(\Omega)} - \langle \boldsymbol{f}_n - \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{w}\rangle_{\mathbb{V}^p_N(\Omega)',\mathbb{V}^p_N(\Omega)} \\
&\quad \geq \begin{cases} c\int_\Omega |\mathrm{curl}\,(\boldsymbol{v} - \boldsymbol{w})|^p dx - \|\boldsymbol{f}_n - \boldsymbol{f}\|_{\mathbb{V}^p_N(\Omega)'}\|\boldsymbol{v} - \boldsymbol{w}\|_{\mathbb{V}^p_N(\Omega)} & \text{if } p \geq 2, \\ c\int_\Omega (|\mathrm{curl}\,\boldsymbol{v}| + |\mathrm{curl}\,\boldsymbol{w}|)^{p-2}|\mathrm{curl}\,(\boldsymbol{v} - \boldsymbol{w})|^2 dx & \\ \qquad -\|\boldsymbol{f}_n - \boldsymbol{f}\|_{\mathbb{V}^p_N(\Omega)'}\|\boldsymbol{v} - \boldsymbol{w}\|_{\mathbb{V}^p_N(\Omega)} & \text{if } 1 < p < 2. \end{cases}
\end{aligned}$$

When $p \geq 2$, using the Young inequality, for some constant $C > 0$ we have

$$\begin{aligned}
&c\int_\Omega |\mathrm{curl}\,(\boldsymbol{v} - \boldsymbol{w})|^p dx - \|\boldsymbol{f}_n - \boldsymbol{f}\|_{\mathbb{V}^p_N(\Omega)'}\|\boldsymbol{v} - \boldsymbol{w}\|_{\mathbb{V}^p_N(\Omega)} \\
&\qquad\qquad \geq \frac{c}{2}\|\boldsymbol{v} - \boldsymbol{w}\|^p_{\mathbb{V}^p_N(\Omega)} - C\|\boldsymbol{f}_n - \boldsymbol{f}\|^{p'}_{\mathbb{V}^p_N(\Omega)'}.
\end{aligned}$$

When $1 < p < 2$, we recall the reverse Hölder inequality (cf. Sobolev [15, p. 8]). Let $0 < s < 1$ and $s' = s/(s-1)$. If $F \in L^s(\Omega), FG \in L^1(\Omega)$ and $\int_\Omega |G(s)|^{s'} dx < \infty$, then

$$\left(\int_\Omega |F(s)|^s dx\right)^{1/s} \leq \int_\Omega |F(x)G(x)|dx \left(\int_\Omega |G(x)|^{s'} dx\right)^{-1/s'},$$

and we apply it, with $s = p/2$ (so $s' = p/(p-2)$), $F = |\mathrm{curl}\,(\boldsymbol{v}-\boldsymbol{w})|^2, G = (|\mathrm{curl}\,\boldsymbol{v}| + |\mathrm{curl}\,\boldsymbol{w}|)^{p-2}$, in $\widehat{\Omega} = \{x \in \Omega; |\mathrm{curl}\,\boldsymbol{v}(x)| + |\mathrm{curl}\,\boldsymbol{w}(x)| \neq 0\}$. So,

$$\left(\int_{\widehat{\Omega}} (|\mathrm{curl}\,(\boldsymbol{v}-\boldsymbol{w})|^2)^{p/2} dx\right)^{2/p}$$
$$\leq \int_{\widehat{\Omega}} |\mathrm{curl}\,(\boldsymbol{v}-\boldsymbol{w})|^2 (|\mathrm{curl}\,\boldsymbol{v}| + |\mathrm{curl}\,\boldsymbol{w}|)^{p-2} dx \left(\int_{\widehat{\Omega}} (|\mathrm{curl}\,\boldsymbol{v}| + |\mathrm{curl}\,\boldsymbol{w}|)^p dx\right)^{(2-p)/2},$$

and so,

$$\int_{\widehat{\Omega}} |\mathrm{curl}\,(\boldsymbol{v}-\boldsymbol{w})|^2 (|\mathrm{curl}\,\boldsymbol{v}| + |\mathrm{curl}\,\boldsymbol{w}|)^{p-2} dx$$
$$\geq \left(\int_{\widehat{\Omega}} (|\mathrm{curl}\,(\boldsymbol{v}-\boldsymbol{w})|^2)^{p/2} dx\right)^{2/p} \left(\int_{\widehat{\Omega}} (|\mathrm{curl}\,\boldsymbol{v}| + |\mathrm{curl}\,\boldsymbol{w}|)^p dx\right)^{(p-2)/2}.$$

Since $\boldsymbol{v}$ varies in a bounded set in $\mathbb{V}^p_N(\Omega)$, we can assume that $\int_\Omega |\mathrm{curl}\,\boldsymbol{v}|^p dx \leq C_1$, so there exists a constant $c_2 > 0$ depending only on p and $\boldsymbol{w}$ such that

$$\int_{\widehat{\Omega}} |\mathrm{curl}\,(\boldsymbol{v}-\boldsymbol{w})|^2 (|\mathrm{curl}\,\boldsymbol{v}| + |\mathrm{curl}\,\boldsymbol{w}|)^{p-2} dx \geq c_2 \|\boldsymbol{v}-\boldsymbol{w}\|^2_{\mathbb{V}^p_N(\Omega)}.$$

Thus we have

$$c\int_\Omega (|\mathrm{curl}\,\boldsymbol{v}| + |\mathrm{curl}\,\boldsymbol{w}|)^{p-2} |\mathrm{curl}\,(\boldsymbol{v}-\boldsymbol{w})|^2 dx - \|\boldsymbol{f}_n - \boldsymbol{f}\|_{\mathbb{V}^p_N(\Omega)'} \|\boldsymbol{v}-\boldsymbol{w}\|_{\mathbb{V}^p_N(\Omega)}$$
$$\geq \frac{c}{2}\|\boldsymbol{v}-\boldsymbol{w}\|^2_{\mathbb{V}^p_N(\Omega)} - C\|\boldsymbol{f}_n - \boldsymbol{f}\|^2_{\mathbb{V}^p_N(\Omega)'}.$$

Therefore, we can derive

$$\liminf_{n\to\infty} \langle T_n\boldsymbol{v} - T\boldsymbol{w}, \boldsymbol{v}-\boldsymbol{w}\rangle_{\mathbb{V}^p_N(\Omega)', \mathbb{V}^p_N(\Omega)} \geq \begin{cases} \frac{c}{2}\|\boldsymbol{v}-\boldsymbol{w}\|^p_{\mathbb{V}^p_N(\Omega)} & \text{if } p \geq 2, \\ \frac{c}{2}\|\boldsymbol{v}-\boldsymbol{w}\|^2_{\mathbb{V}^p_N(\Omega)} & \text{if } 1 < p < 2. \end{cases}$$

If we put $\beta(s) = \frac{c}{2}s^{p\vee 2}$, where $p\vee 2 = \max\{p, 2\}$, the conclusion holds. This completes the proof of Lemma 4.5. □

We continue the proof of Theorem 4.2. From Lemma 4.3, 4.4 and 4.5, the hypotheses of [12, Theorem A] hold, and we can conclude that $\boldsymbol{u}_n \to \boldsymbol{u}$ strongly in $\mathbb{V}^p_N(\Omega)$.

Finally, we show that $\pi_n \to \pi$ weakly in $L^{p'}(\Omega)$.

Lemma 4.6. *If $\boldsymbol{v}_j \to \boldsymbol{v}$ strongly in $\mathbb{V}^p_N(\Omega)$ as $j \to \infty$, then*

$$S_t(x, |\mathrm{curl}\,\boldsymbol{v}_j|^2)\mathrm{curl}\,\boldsymbol{v}_j \to S_t(x, |\mathrm{curl}\,\boldsymbol{v}|^2)\mathrm{curl}\,\boldsymbol{v}$$

strongly in $\boldsymbol{L}^{p'}(\Omega)$ as $j \to \infty$.

Proof. From Lemma 2.3, we have

$$|S_t(x, |\mathrm{curl}\, \boldsymbol{v}_j|^2)\mathrm{curl}\, \boldsymbol{v}_j - S_t(x, |\mathrm{curl}\, \boldsymbol{v}|^2)\mathrm{curl}\, \boldsymbol{v}|^{p'} \\ \le \begin{cases} C_1|\mathrm{curl}\, \boldsymbol{v}_j - \mathrm{curl}\, \boldsymbol{v}|^p & \text{if } 1 < p < 2, \\ C_1(|\mathrm{curl}\, \boldsymbol{v}_j| + |\mathrm{curl}\, \boldsymbol{v}|)^{(p-2)p'}|\mathrm{curl}\, \boldsymbol{v}_j - \mathrm{curl}\, \boldsymbol{v}|^{p'} & \text{if } p \ge 2. \end{cases}$$

When $1 < p < 2$, the conclusion is clear. When $p \ge 2$, using Hölder inequality, we have

$$\int_\Omega |S_t(x, |\mathrm{curl}\, \boldsymbol{v}_j|^2)\mathrm{curl}\, \boldsymbol{v}_j \to S_t(x, |\mathrm{curl}\, \boldsymbol{v}|^2)\mathrm{curl}\, \boldsymbol{v}|^{p'} dx \\ \le \left(\int_\Omega (|\mathrm{curl}\, \boldsymbol{v}_j| + |\mathrm{curl}\, \boldsymbol{v}|)^p dx\right)^{(p-p')/p} \left(\int_\Omega |\mathrm{curl}\, \boldsymbol{v}_j - \mathrm{curl}\, \boldsymbol{v}|^p dx\right)^{p'/p} \to 0$$

as $j \to \infty$. Here we used the fact

$$\int_\Omega |\mathrm{curl}\, \boldsymbol{v}_j|^p dx \le C$$

for some constant independent of j since $\boldsymbol{v}_j \to \boldsymbol{v}$ strongly in $\mathbb{V}^p_N(\Omega)$. This completes the proof of Lemma 4.6. □

We continue the proof of Theorem 4.2. For any $\psi \in L^p(\Omega)$, choose $\boldsymbol{v}_\psi \in \boldsymbol{W}^{1,p}(\Omega)$ as in Lemma 3.3. We note that $\boldsymbol{v}_\psi \in \mathbb{K}_{\varphi_n} \cap \mathbb{K}_\varphi$. If we choose $-\boldsymbol{v}_\psi$ as a test function of (3.1), then we have

$$\int_\Omega S_t(x, |\mathrm{curl}\, \boldsymbol{u}_n|^2)\mathrm{curl}\, \boldsymbol{u}_n \cdot \mathrm{curl}\,(-\boldsymbol{v}_\psi - \boldsymbol{u}_n)dx + \int_\Omega \pi_n \mathrm{div}\, \boldsymbol{v}_\psi dx \\ \ge \langle \boldsymbol{f}, -\boldsymbol{v}_\psi - \boldsymbol{u}_n\rangle_{\mathbb{X}^p_N(\Omega)', \mathbb{X}^p_N(\Omega)}.$$

Adding this inequality and (3.1) with $\boldsymbol{v} = \boldsymbol{v}_\psi$, we have

$$\int_\Omega S_t(x, |\mathrm{curl}\, \boldsymbol{u}_n|^2)\mathrm{curl}\, \boldsymbol{u}_n \cdot \mathrm{curl}\,(-\boldsymbol{v}_\psi - \boldsymbol{u}_n)dx \\ + \int_\Omega S_t(x, |\mathrm{curl}\, \boldsymbol{u}|^2)\mathrm{curl}\, \boldsymbol{u} \cdot \mathrm{curl}\,(\boldsymbol{v}_\psi - \boldsymbol{u})dx + \int_\Omega (\pi_n - \pi)\mathrm{div}\, \boldsymbol{v}_\psi dx \\ \ge -\langle \boldsymbol{f}, \boldsymbol{u}_n\rangle_{\mathbb{X}^p_N(\Omega)', \mathbb{X}^p_N(\Omega)} - \langle \boldsymbol{f}, \boldsymbol{u}\rangle_{\mathbb{X}^p_N(\Omega)', \mathbb{X}^p_N(\Omega)}.$$

Taking the lower limit of this inequality and using Lemma 4.6,

$$\liminf_{n\to\infty} \int_\Omega (\pi_n - \pi)\mathrm{div}\, \boldsymbol{v}_\psi dx \\ \ge -2\langle \boldsymbol{f}, \boldsymbol{u}\rangle_{\mathbb{X}^p_N(\Omega)', \mathbb{X}^p_N(\Omega)} - 2\int_\Omega S_t(x, |\mathrm{curl}\, \boldsymbol{u}|^2)|\mathrm{curl}\, \boldsymbol{u}|^2 dx = -2c_1$$

where $c_1 \geq 0$. For any $M > 0$, since $M\boldsymbol{v}_\psi \in \mathbb{K}_\varphi$ and $-M\boldsymbol{v}_\psi \in \mathbb{K}_\varphi$, we have

$$-\frac{2c_1}{M} \leq \liminf_{n\to\infty} \int_\Omega (\pi_n - \pi)\mathrm{div}\,\boldsymbol{v}_\psi dx \leq \limsup_{n\to\infty} \int_\Omega (\pi_n - \pi)\mathrm{div}\,\boldsymbol{v}_\psi dx \leq \frac{2c_1}{M}.$$

Letting $M \to \infty$ and using $\mathrm{div}\,\boldsymbol{v}_\psi = \psi$ in Ω, we have

$$\lim_{n\to\infty} \int_\Omega (\pi_n - \pi)\psi dx = 0 \text{ for all } \psi \in L^p(\Omega). \tag{4.4}$$

Since $\|\pi_n\|_{L^{p'}(\Omega)} \leq C\|\boldsymbol{f}_n\|_{\mathbb{X}_N^p(\Omega)'} \leq C_1$, where C_1 is a constant independent of n since $\boldsymbol{f}_n \to \boldsymbol{f}$ in $\mathbb{X}_N^p(\Omega)'$. This completes the proof of Theorem 4.2. □

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