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Finite Continuous Ridgelet Transforms with Applications to Telegraph and Heat Conduction

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Abstract

In this paper, finite continuous Ridgelet transforms and its inversion formula is studied. Using Sturm-Liouville theory with self-adjoint operator; the operational calculus of finite continuous Ridgelet transform is discussed. In the concluding section, engineering applications like telegraph and heat conduction are demonstrated.
Keywords: Finite continuous Ridgelet transform, Fourier-Ridgelet expansion, adjoint operator, testing spaces, inversion..
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1. INTRODUCTION

Let $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth univariate function with sufficient decay and vanishing mean given by $\int \psi(t)\, dt = 0$.
Candès [2] demonstrated continuous Ridgelet transform: $\forall\, a > 0, b \in \mathbb{R}$ and $\theta \in [0, 2\pi)$, bivariate function $\psi_{a,b,\theta} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined as:

$$\psi_{a,b,\theta}(x, y) = a^{-1/2}\, \psi \left[\frac{(x \cos\theta + y \sin\theta - b)}{a} \right] \tag{1.1}$$

where ridges in (1.1) represent

$$x \cos\theta + y \sin\theta = C \tag{1.2}$$

where $C = constant$.

The continuous Ridgelet transforms for bivariate function $f(x, y)$ was defined in [2] as

$$\Re_f(a, b, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, \psi_{a,b,\theta}(x, y)\, dxdy. \tag{1.3}$$

along with reconstruction formula and Parseval relation.
Murata in [12] independently studied and analysed Ridgelet integral representations. In 1998, Donoho [7] broadened the notion of ridgelet and studied orthonormal ridgelets whose elements can be specified in closed form. In [3] author extended the ridgelets to higher dimensions. The discrete finite ridgelet transforms was introduced in 2003 by Do and Vetterli [6]. In [9], using the convolution of quaternion-valued functions authors defined the ridgelet transform on square integrable quaternion-valued functions. Authors have conducted a series of computational experiments to show that there exists an interesting similarity between the scatter plot of hidden parameters in a shallow neural network after the BP training and the spectrum of the ridgelet transform in [8].
In this paper, classical work of finite continuous Ridgelet transform is been introduced. Using Sturm-Liouville theory [1] developed the study of finite continuous Ridgelet transforms. Fourier-Ridgelet type of series expansion is also demonstrated analogous to [10]. The self-adjoint operator and operational calculus are derived using [1] and [13]. The testing function spaces, inversion formula and uniqueness condition have been developed by using the method analogous to [10, 14] in this study. The operational calculus thus generated in the context is used in solving certain partial differential equations with boundary value problems [11] in the concluding section.

2. PRELIMINARY RESULTS

Consider Sturm-Liouville theory analogues as in [1]

$$(\Omega_{x,y,\theta})\, \psi(x, y) = 0 \tag{2.1}$$

where

$$\Omega_{x,y,\theta} = \left(\sin^2\theta\right) \Omega_x - \left(\cos^2\theta\right) \Omega_y. \tag{2.2}$$

for the differential operator considered as

$$\Omega_x = \frac{\partial^2}{\partial x^2} \tag{2.3}$$

and

$$\Omega_y = \frac{\partial^2}{\partial y^2}\ . \tag{2.4}$$

Also $\alpha,\ \beta$ are real constants and $-\alpha \leq x \leq \alpha;\ -\beta \leq y \leq \beta$ satisfies homogeneous separated boundary conditions [1, pp. 43-44]:

$$i\eta\cos\theta\psi\left(-\alpha,y\right)+\psi'\left(-\alpha,y\right)=0;\ i\eta\cos\theta e^{2i\eta\alpha\,\mathrm{Cos}\,\theta}\psi\left(\alpha,y\right)+\psi'\left(\alpha,y\right)=0. \quad (2.5)$$

$$i\eta\sin\theta\psi\left(x,-\beta\right)+\psi'\left(x,-\beta\right)=0;\ i\eta\sin\theta e^{2i\eta\beta\sin\theta}\psi\left(x,\beta\right)+\psi'\left(x,\beta\right)=0. \quad (2.6)$$

Assume from [1, p. 118] follows:

$$\psi\left(x,y\right)=a^{1/2}\,e^{b/a}\,X_p\left(x\right)Y_q\left(y\right), \quad (2.7)$$

where $a,\ b,\ p$ and q are integers.

Using (2.7) and setting each side $-\eta^2$, by variable separable method (2.1) can be written as [1]

$$\frac{X_p{''}\left(x\right)}{\cos^2\theta X_p\left(x\right)}=\frac{Y''_q\left(y\right)}{\sin^2\theta Y_q\left(y\right)}=-\eta^2. \quad (2.8)$$

Using boundary conditions (2.5) and (2.6), (2.8) can be obtained as

$$X_p\left(x\right)=c_2e^{-i\eta_p x\cos\theta}, \quad (2.9)$$

and

$$Y_q\left(y\right)=c_4e^{-i\eta_q y\sin\theta} \quad (2.10)$$

assuming $\eta_p=\dfrac{p'\pi}{\alpha\cos\theta}$ for $0<p'<\infty$ and $\eta_q=\dfrac{q'\pi}{\beta\sin\theta}$ for $0<q'<\infty$ are eigenvalues of (2.8) respectively. Here c_2 and c_4 arbitary constants.

Using (2.7), (2.9), (2.10), and substituting in (2.1) follows:

$$\psi\left(x,y\right)=\psi_{p,q}\left(x,y\right)=c_2c_4a^{1/2}\,e^{b/a}\,e^{\frac{-ip'\pi}{\alpha}x}e^{\frac{-iq'\pi}{\beta}y}. \quad (2.11)$$

Let $ip'=p$ and $iq'=q$, $0<\dfrac{p}{i}<\infty$ and $0<\dfrac{q}{i}<\infty$, then (2.11) becomes

$$\psi_{p,q}\left(x,y\right)=c_2c_4a^{1/2}\,e^{-\left[\frac{(pa\pi x/\alpha\ )+(qa\pi y/\beta\ )-b}{a}\right]}. \quad (2.12)$$

Then (2.8) can be written [10] as

$$\left(\Omega_x+\eta_p^2\cos^2\theta\right)X_p\left(x\right)=0. \quad (2.13)$$

$$\left(\Omega_y+\eta_q^2\sin^2\theta\right)Y_q\left(y\right)=0. \quad (2.14)$$

And the boundary conditions are given in (2.5) and (2.6), where $X_p\left(x\right)$ and $Y_q\left(y\right)$ are the eigenfunctions of (2.8). Hence (2.12) is eigenfunction of the problem (2.1)-(2.6) which corresponds to the non-zero eigenvalues η_p and η_q.

Then the orthogonality and orthonormality condition of (2.12) [1, p. 94] is given by can be written as

$$\left|\left\langle\psi_{p_1,q_1}\left(x,y\right),\psi_{p_2,q_2}\left(x,y\right)\right\rangle\right|^2=\begin{cases}4ae^{2b/a}\left(c_2c_4\right)^2\alpha\beta & ;\quad p_1=p_2,\ q_1=q_2\\ 0 & ;\quad p_1\neq p_2, q_1\neq q_2\end{cases}. \quad (2.15)$$

3. MAIN RESULTS (THE FOURIER-RIDGELET SERIES AND CLASSICAL FINITE CONTINUOUS RIDGELET TRANSFORMATION)

Assume $f(x,y)$ is a square integrable function over rectangle [1, p. 122], Fourier-Ridgelet series expansion follows from (2.15), [1, p. 124] and using (2.12):

$$f(x,y) = \psi_{p,q}(x,y) = c_{p,q} a^{1/2} \left(1 + \sum_{p,q=1}^{\infty} e^{((pa\pi x/\alpha)+(qa\pi y/\beta)-b)/a}\right)^{-1}. \tag{3.1}$$

Multiplying $a^{-1/2}\left(1 + \sum_{p,q=1}^{\infty} e^{-((pa\pi x/\alpha)+(qa\pi y/\beta)-b)/a}\right)^{-1}$ to (3.1); solving double integral w.r.t x, y in $[-\alpha,\alpha]$, $[-\beta,\beta]$ respectively yields:

$$\int_{-\beta}^{\beta}\int_{-\alpha}^{\alpha} f(x,y)\, a^{-1/2} \left(1 + \sum_{p,q=1}^{\infty} e^{-((pa\pi x/\alpha)+(qa\pi y/\beta)-b)/a}\right)^{-1} dxdy = 4\alpha\beta c_{p,q}.$$

Thus

$$c_{p,q} = \frac{1}{4\alpha\beta}\int_{-\beta}^{\beta}\int_{-\alpha}^{\alpha} f(x,y)\, a^{-1/2} \left(1 + \sum_{p,q=1}^{\infty} e^{-((pa\pi x/\alpha)+(qa\pi y/\beta)-b)/a}\right)^{-1} dxdy. \tag{3.2}$$

From [4], $\left(1 + \sum_{p,q=1}^{\infty} e^{-((pa\pi x/\alpha)+(qa\pi y/\beta)-b)/a}\right)^{-1}$ represents the regularized sigmoid function which is a ridge function or ridgelet with parameters:
i) a; the scale of the ridge function
ii) b; location of the ridge function
iii) $[(pa\pi x/\alpha) + (qa\pi y/\beta)]$; its orientation
and can be represented by:

$$a^{-1/2}\left(1 + \sum_{p,q=1}^{\infty} e^{-((pa\pi x/\alpha)+(qa\pi y/\beta)-b)/a}\right)^{-1} = a^{-1/2}\,\psi\left(\left((pa\pi x/\alpha) + (qa\pi y/\beta) - b\right)/a\right) \tag{3.3}$$

Hence (3.2) becomes

$$c_{p,q} = \frac{1}{4\alpha\beta}\int_{-\beta}^{\beta}\int_{-\alpha}^{\alpha} f(x,y)\, a^{-1/2}\,\psi\left(\left((pa\pi x/\alpha) + (qa\pi y/\beta) - b\right)/a\right) dxdy. \tag{3.4}$$

The convergence of the series (3.1) is straightforward by [13, p. 433] and the following theorem 3.1 [13, pp. 425-432].

Theorem 3.1. *Let $f(x,y)$ be a function defined and absolute integrable on the rectangle $\{(x,y): -\alpha < x < \alpha, -\beta < y < \beta\}$, then*

$$c_{p,q} = \frac{1}{4\alpha\beta} \int_{-\beta}^{\beta} \int_{-\alpha}^{\alpha} f(x,y)\, a^{-1/2}\, \psi\left(\left(\left(pa\pi x/\alpha\right) + \left(qa\pi y/\beta\right) - b\right)/a\right) dxdy$$

at each point of the open interval $[-\alpha,\alpha] \times [-\beta,\beta]$ at which $f(x,y)$ is continuous. At any point of the interval at which $f(x,y)$ has a finite discontinuity the symbol $f(x,y)$ is taken to mean $\frac{1}{2}[f(x+,y+) + f(x-,y-)]$ and at point $x = -\alpha$, $y = -\beta$ or $x = \alpha$, $y = \beta$ it is taken to mean $\frac{1}{2}[f(-\alpha,-\beta) + f(\alpha,\beta)]$.

Remark 3.2. Let $U_{p,q} = \left(\frac{\pi ap}{\alpha} + \frac{\pi aq}{\beta}\right) \in \mathbb{R}^2$ and $a, b \in \mathbb{R}$.
Subsituting

$$I\left(U_{p,q}, a, b\right) = \left\{(x,y) \in \mathbb{R}^2 \middle| U_{p,q} \cdot (x,y) = \left(\frac{\pi ap}{\alpha}x + \frac{\pi aq}{\beta}y\right) = \frac{b}{a}\right\},$$

where $I\left(U_{p,q}, a, b\right)$ is a hyperplane, more precisely: $I\left(U_{p,q}, a, b\right)$ is the right in the rectangle $[-\alpha,\alpha] \times [-\beta,\beta] \subset \mathbb{R}^2 \cdot \mathbb{P}^2$, $\left\{I\left(U_{p,q}, a, b\right) \middle| (U, a, b) \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}\right\}$ is the differentiable variate.
Let $S\left(\mathbb{P}^2\right)$ be the Schwartz space (3.1) and (3.4) and theorem 3.1 suggest introducing the finite continuous Ridgelet transform analogous to [13, p. 425] as follows:

$$\Re f\left(I\left(U_{p,q}, a, b\right)\right) = \frac{a^{-1/2}}{4\alpha\beta} \int_{-\beta}^{\beta} \int_{-\alpha}^{\alpha} f(x,y)\, \psi\left(\left(U_{p,q} \cdot (x,y) - b\right)/a\right) dxdy \tag{3.5}$$

where $U_{p,q} \cdot (x,y) = \left(\frac{\pi ap}{\alpha}x + \frac{\pi aq}{\beta}y\right)$. Hence $\Re : T\left(\mathbb{R}^2\right) \to T\left(\mathbb{P}^2\right) : f \to \Re f$.
The inversion formula of (3.5) is given by

$$\Re^{-1}\left(\Re f\left(I\left(U_{p,q}, a, b\right)\right)\right) = f(x,y) = a^{1/2} \sum_{p,q=1}^{\infty} \Re f\left(I\left(U_{p,q}, a, b\right)\right) \psi\left(\left(b - U_{p,q} \cdot (x,y)\right)/a\right). \tag{3.6}$$

We point out the following operational rules:

(1) If $f(x,y) \in \mathbb{R}^2\left([-\alpha,\alpha] \times [-\beta,\beta]\right)$,

$$\Re\left\{\frac{\partial^2 f}{\partial x^2}\right\}\left(I\left(U_{p,q}, a, b\right)\right) = \frac{a^{-1/2}}{4\alpha\beta} \int_{-\beta}^{\beta} \int_{-\alpha}^{\alpha} \left\{\frac{\partial^2 f}{\partial x^2}\right\} \psi\left(\left(U_{p,q} \cdot (x,y) - b\right)/a\right) dxdy,$$

Integrating by parts we get,

$$\Re\left\{\frac{\partial^2 f}{\partial x^2}\right\}(I(U_{p,q},a,b)) = \frac{a^{-1/2}}{4\alpha\beta}\int_{-\beta}^{\beta}\left\{\begin{array}{l} f'(\alpha,y)\,\psi((U_{p,q}\cdot(\alpha,y)-b)/a) \\ -f'(-\alpha,y)\,\psi((U_{p,q}\cdot(\alpha,y)-b)/a) \\ -f(\alpha,y)\,\psi'((U_{p,q}\cdot(\alpha,y)-b)/a) \\ +f(-\alpha,y)\,\psi'((U_{p,q}\cdot(-\alpha,y)-b)/a) \\ +\int_{-\alpha}^{\alpha} f(x,y)\,\frac{\partial^2\psi((U_{p,q}\cdot(x,y)-b)/a)}{\partial x^2}dx \end{array}\right\}dy. \tag{3.7}$$

Assuming $f(x,y)$ vanishes on the boundary of $[-\alpha,\alpha]\times[-\beta,\beta]$ as in [13],

$$f(\alpha,y) = f(-\alpha,y) = f'(\alpha,y) = f'(\alpha,y) = 0.$$

Hence (3.7) becomes

$$\Re\left\{\frac{\partial^2 f}{\partial x^2}\right\}(I(U_{p,q},a,b)) = \frac{a^{-1/2}}{4\alpha\beta}\int_{-\beta}^{\beta}\int_{-\alpha}^{\alpha} f(x,y)\,\frac{\partial^2\psi((U_{p,q}\cdot(x,y)-b)/a)}{\partial x^2}dxdy. \tag{3.8}$$

But from (3.3),

$$\frac{\partial^2\psi\left(\frac{U_{p,q}\cdot(x,y)-b}{a}\right)}{\partial x^2} = \frac{\partial^2}{\partial x^2}\left[\left(1+\sum_{p,q=1}^{\infty} e^{-((pa\pi x/\alpha)+(qa\pi y/\beta)-b)/a}\right)^{-1}\right]$$

$$= \frac{\partial^2 e^{((pa\pi x/\alpha)+(qa\pi y/\beta)-b)/a}}{\partial x^2},$$

$$\frac{\partial^2\psi\left(\frac{U_{p,q}\cdot(x,y)-b}{a}\right)}{\partial x^2} = \left(\frac{p\pi}{\alpha}\right)^2 e^{((pa\pi x/\alpha)+(qa\pi y/\beta)-b)/a},$$

$$\frac{\partial^2\psi\left(\frac{U_{p,q}\cdot(x,y)-b}{a}\right)}{\partial x^2} = \left(\frac{p\pi}{\alpha}\right)^2\left(1+\sum_{p,q=1}^{\infty} e^{-((pa\pi x/\alpha)+(qa\pi y/\beta)-b)/a}\right)^{-1}. \tag{3.9}$$

Using (3.8), (3.9) becomes

$$\Re\left\{\frac{\partial^2 f}{\partial x^2}\right\}(I(U_{p,q},a,b))$$
$$= \frac{a^{-1/2}}{4\alpha\beta}\left(\frac{p\pi}{\alpha}\right)^2\int_{-\beta}^{\beta}\int_{-\alpha}^{\alpha} f(x,y)\left(1+\sum_{p,q=1}^{\infty} e^{-((pa\pi x/\alpha)+(qa\pi y/\beta)-b)/a}\right)^{-1}dxdy$$

$$\Re\left\{\frac{\partial^2 f}{\partial x^2}\right\}(I(U_{p,q},a,b)) = \frac{a^{-1/2}}{4\alpha\beta}\left(\frac{p\pi}{\alpha}\right)^2\int_{-\beta}^{\beta}\int_{-\alpha}^{\alpha} f(x,y)\,\psi((U_{p,q}\cdot(x,y)-b)/a)\,dxdy.$$

Using (3.5), we get

$$\Re\left\{\frac{\partial^2 f}{\partial x^2}\right\}(I(U_{p,q},a,b)) = \left(\frac{p\pi}{\alpha}\right)^2 \Re f(I(U_{p,q},a,b)). \tag{3.10}$$

(2) If $f(x,y)$ at boundary is zero, and

$$f^s(-\alpha,y) = f^s(\alpha,y) = 0,$$

then

$$\Re\left\{\frac{\partial^s f}{\partial x^s}\right\}(I(U_{p,q},a,b)) = \left(\frac{-p\pi}{\alpha}\right)^s \Re f(I(U_{p,q},a,b)) \tag{3.11}$$

where s being a positive integer.

(3) If $f(x,y) \in \mathbb{R}^2([-\alpha,\alpha]\times[-\beta,\beta])$, upon integrating by parts, we deduce the relation

$$\Re\left\{\frac{\partial^2 f}{\partial y^2}\right\}(I(U_{p,q},a,b)) = \left(\frac{q\pi}{\beta}\right)^2 \Re f(I(U_{p,q},a,b)). \tag{3.12}$$

(4) If $f(x,y)$ at boundary is zero and

$$f^s(x,-\beta) = f^s(x,\beta) = 0,$$

then

$$\Re\left\{\frac{\partial^s f}{\partial y^s}\right\}(I(U_{p,q},a,b)) = \left(\frac{-q\pi}{\beta}\right)^s \Re f(I(U_{p,q},a,b)). \tag{3.13}$$

(5) If $f(x,y)$ at boundary is zero as
$f(-\alpha,y) = f(\alpha,y) = 0$ and $f(x,-\beta) = f(x,\beta) = 0$.

$$\Re\left\{\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right\}(I(U_{p,q},a,b)) = \left[\left(\frac{p\pi}{\alpha}\right)^2 + \left(\frac{q\pi}{\beta}\right)^2\right] \Re f(I(U_{p,q},a,b)). \tag{3.14}$$

And also

$$\Re\left\{\frac{\partial^s f}{\partial x^s}+\frac{\partial^s f}{\partial y^s}\right\}(I(U_{p,q},a,b)) = \left[\left(\frac{-p\pi}{\alpha}\right)^s + \left(\frac{-q\pi}{\beta}\right)^s\right] \Re f(I(U_{p,q},a,b)). \tag{3.15}$$

4. THE TESTING FUNCTION SPACE A_θ AND A_θ^* AND THEIR DUALS

In this, we employ the same notation and terminology as those used in [14]. Thus $I_{\alpha,\beta}$ denote the interval $[-\alpha,\alpha]\times[-\beta,\beta]$. $L_2(I_{\alpha,\beta})$ and $L_2^*(I_{\alpha,\beta})$ represent the space of equivalence class of functions that are quadratically integrable on $I_{\alpha,\beta}$.

A mixed inner product is defined on $L_2(I_{\alpha,\beta}) \times L_2^*(I_{\alpha,\beta})$ for $f \in L_2(I_{\alpha,\beta}),\ g \in L_2^*(I_{\alpha,\beta})$ [1] as follows:

$$\langle f(X), g(X)\rangle = \int_{I_{\alpha,\beta}} f(X)\overline{g(X)}dX \tag{4.1}$$

where $(x,y) = X,\ \overline{g(X)}$ denotes the complex conjugate of $g(X)$.
This definition is consistent with the inner product on $L_2(I_{\alpha,\beta})$ and $L_2^*(I_{\alpha,\beta})$. And note that $f(X)$ and $g(X)$ both are in $L_2(I_{\alpha,\beta})$ and $L_2^*(I_{\alpha,\beta})$. The symbol $D(I_{\alpha,\beta})$ will denote the space of infinitely differential function on $I_{\alpha,\beta} = [-\alpha,\alpha] \times [-\beta,\beta]$, which have compact support on $I_{\alpha,\beta}$.
The topology of $D(I_{\alpha,\beta})$ is that which makes its dual $D'(I_{\alpha,\beta})$ of Schwartz's distribution. $E(I_{\alpha,\beta})$ will denote the space of distribution with compact support.
The adjoint of Ω_x from (2.3) if it exists, is the operator Ω_x^* which satisfies [1, p. 55],

$$\langle \Omega_x X_p, u\rangle = \langle X_p, \Omega_x^* u\rangle, \tag{4.2}$$

where u is the dummy function having every characteristics of X_p.
If $\Omega_x^* = \Omega_x$, then Ω_x is said to be self-adjoint operator [1, p. 55].
The proof is as follows

$$\langle \Omega_x X_p, u\rangle = \int_{-\alpha}^{\alpha} u\frac{d^2X_p}{dx^2}dx. \tag{4.3}$$

On integrating by parts, we get

$$\langle \Omega_x X_p, u\rangle = J(u, X_p) + \int_{-\alpha}^{\alpha} X_p\Omega_x^* u dx, \tag{4.4}$$

where $J(u, X_p) = \left[u\frac{dX_p}{dx}\right]_{-\alpha}^{\alpha} - \left[\frac{du}{dx}X_p\right]_{-\alpha}^{\alpha}$ is called Bi-linear concomitant.

$$\begin{aligned} J(u, X_p) &= \left[u\frac{dX_p}{dx}\right]_{-\alpha}^{\alpha} - \left[\frac{du}{dx}X_p\right]_{-\alpha}^{\alpha} \\ &= u(\alpha)\frac{dX_p(\alpha)}{dx} - u(-\alpha)\frac{dX_p(-\alpha)}{dx} - \frac{du(\alpha)}{dx}X_\alpha(\alpha) + \frac{du(-\alpha)}{dx}X_p(-\alpha). \end{aligned}$$

Applying the boundary conditions from (2.5), we obtain

$$J(u, X_p) = \frac{dX_p(\alpha)}{dx}[u(\alpha) - u(-\alpha)] - X_p(\alpha)[u'(\alpha) - u'(-\alpha)]. \tag{4.5}$$

In order to get the Bi-linear concomitant to vanish in (4.5), we assume
a) $u(\alpha) - u(-\alpha) = 0$

b) and $u'(\alpha) - u'(-\alpha) = 0$.
Therefore (4.5) becomes

$$J(u, X_p) = 0. \tag{4.6}$$

And (4.4) becomes

$$\langle \Omega_x X_p, u \rangle = \int_{-\alpha}^{\alpha} X_p \Omega_x^* u dx,$$

which can be written as

$$\langle \Omega_x X_p, u \rangle = \langle X_p, \Omega_x^* u \rangle ,$$

thus is the proof of (4.2). Hence the operator Ω_x is a self-adjoint operator.
Thus (2.9) gives eigenfunctions and $(\frac{p\pi}{\alpha \cos\theta})$ are eigenvalues (positive roots of (2.13)).
Similarly from (2.4) it follows:

$$\langle \Omega_y Y_q, v \rangle = \left\langle Y_q, \Omega_y^* v \right\rangle \tag{4.7}$$

for $\Omega_y^* = \Omega_y$, where v is the dummy function has every characteristic of Y_q, which means adjoint of Ω_y exists and is also a self-adjoint operator.
Using (2.10) eigenfunctions of Ω_y^* are obtained. Positive roots of (2.4) are given by are eigenvalues $\dfrac{q\pi}{\beta \sin\theta}$.
Since Ω_x and Ω_y are self-adjoint operator, $\Omega_{x,y,\theta}$ is also a self-adjoint operator whose eigenfunctions by (3.1) are given by:

$$\psi_{p,q}(x, y) = a^{1/2} \sum_{p,q=1}^{\infty} c_{p,q} \psi\left((b - U_{p,q} \cdot (x, y))/a\right). \tag{4.8}$$

This is equivalent to say that $\{\psi_{p,q}(X)\}_{p,q=1}^{\infty}$ is orthogonal function of differential operator using (2.15) as

$$\Omega_{x,y,\theta} \psi_{p,q}(X) = 0. \tag{4.9}$$

A_θ is defined as the testing function space of all infinitely differentiable complex-valued functions $\psi(X)$ on $I_{\alpha,\beta}$ such that by [14, p. 252].

(i) $\varsigma_k \psi(X) = \left[\int_{I_{\alpha,\beta}} \left| \Omega_{x,y,\theta}^k \psi(X) \right|^2 dX \right]^{\frac{1}{2}}$ exist for every $k = 0, 1, 2 \cdots$

(ii) For each p, q and k.

$$\left\langle \Omega_{x,y,\theta}^k \psi, \psi_{p,q} \right\rangle = \int_{I_{\alpha,\beta}} \Omega_{x,y,\theta}^k \psi(X)\, \psi_{p,q} dX$$

$$= \int_{I_{\alpha,\beta}} \psi(X)\, \Omega^k_{x,y,\theta} \psi_{p,q} dX$$

$$\left\langle \Omega^k_{x,y,\theta} \psi, \psi_{p,q} \right\rangle = \left\langle \psi, \Omega^k_{x,y,\theta} \psi_{p,q} \right\rangle. \tag{4.10}$$

A_θ is the countable multinormed space having the topology generated by $\{\varsigma\}$. A_θ is also complete. Consequently A_θ is a Fréchet space.
In our context we can establish a result analogous to [14].

Theorem 4.1. *Every member $\psi \in A_\theta$ can be expanded into a series of the form*

$$\psi = \sum_{p,q=1}^{\infty} \langle \psi, \psi_{p,q} \rangle \psi_{p,q}, \tag{4.11}$$

where converges in A_θ.

Proof. Note that $\Omega^k_{x,y,\theta} \psi \in L_2(I_{\alpha,\beta})$. Hence by (4.10) and (4.11), we have

$$\Omega^k_{x,y,\theta} \psi = \sum_{p,q=1}^{\infty} \left\langle \Omega^k_{x,y,\theta} \psi, \psi_{p,q} \right\rangle \psi_{p,q},$$

$$= \sum_{p,q=1}^{\infty} \left\langle \psi, \Omega^k_{x,y,\theta} \psi_{p,q} \right\rangle \psi_{p,q},$$

$$= \sum_{p,q=1}^{\infty} \langle \psi, \psi_{p,q} \rangle\, \Omega^k_{x,y,\theta} \psi_{p,q},$$

where the series involved converge in $L_2(I_{\alpha,\beta})$. Therefore

$$\varsigma_k \left[\psi - \sum_{p,q=1}^{\infty} \langle \psi, \psi_{p,q} \rangle \psi_{p,q} \right] \to 0,$$

as $p, q \to \infty$. This complete the proof of theorem 4.1.
A'_θ is dual space of A_θ and also complete. □

We now list some of the properties of these spaces:

(a) $D(I_{\alpha,\beta}) \subset A_\theta \subset E(I_{\alpha,\beta})$. $E'(I_{\alpha,\beta})$ is a space of A'_θ.

(b) It can be seen that $\psi_{p,q}$, given by (4.8) belongs to A_θ.

(c) The operation $\psi \to \Omega^k_{x,y,\theta} \psi$ is a continuous linear mapping of A_θ into itself.

Consequently, the operation $f \to \Omega^k_{x,y,\theta} f$ defined on A'_θ by

$$\langle \Omega_{x,y,\theta} f, \psi \rangle = \langle f, \Omega_{x,y,\theta} \psi \rangle . \tag{4.12}$$

is also a continuous linear mapping of A'_θ into itself.
It is important to note that operator $\Omega_{x,y,\theta}$ is self-adjoint on A_θ. Therefore A_θ is equivalent to A^*_θ and A'_θ is equivalent to A'^*_θ by [14, p. 255].

Remark 4.2. Since the $\{\psi\}$ is an orthogonal system on $I_{\alpha,\beta}$, verifying the orthogonality condition (2.15), we propose to consider finite continuous Ridgelet transform

$$\Re f\left(I\left(U_{p,q}, a, b\right)\right) = \frac{a^{-1/2}}{4\alpha\beta} \int_{-\beta}^{\beta} \int_{-\alpha}^{\alpha} f\left(x, y\right) \psi\left(\left(U_{p,q} \cdot \left(x, y\right) - b\right)/a\right) dxdy. \tag{4.13}$$

The inversion formula given by

$$f\left(x, y\right) = a^{1/2} \sum_{p,q=1}^{\infty} \Re f\left(I\left(U_{p,q}, a, b\right)\right) \psi\left(\left(b - U_{p,q} \cdot \left(x, y\right)\right)/a\right) \tag{4.14}$$

can be known inversion formula for finite continuous Ridgelet transform.

Remark 4.3. A_θ may be identified with a subspace of A'_θ that is $A_\theta \subset A'_\theta$.
Indeed, every member $f \in A_\theta$ generates a regular distribution in A'_θ represented by

$$(f, \psi) = \int_{I_{\alpha,\beta}} f\left(X\right) \psi\left(X\right) dX, \;\; \psi \in A_\theta.$$

Since $|(f, \psi)| \le \varsigma_0\left(\psi\right) \varsigma_0\left(\psi\right)$.
Furthermore, two members of A_θ which give rise to the same member of A'_θ must be identical.
In similar way A_θ can be considered as a subspace of A'_θ.

5. INVERSION FORMULA

The main result of this section can be sated as follows:

Theorem 5.1. *Every member $f \in A'_\theta$, then*

$$f = \sum_{p,q=1}^{\infty} \langle f, \psi_{p,q} \rangle \psi_{p,q} \tag{5.1}$$

where the series converges in A'_θ.

Proof. By virtue of theorem 4.1, it is inferred that

$$\begin{aligned}\langle f,\psi\rangle &= \left\langle f,\sum_{p,q=1}^{\infty}\langle f,\psi_{p,q}\rangle\,\psi_{p,q}\right\rangle\\ &= \sum_{p,q=1}^{\infty}\langle f,\psi_{p,q}\rangle\,\overline{\langle\psi,\psi_{p,q}\rangle}\\ &= \sum_{p,q=1}^{\infty}\langle f,\psi_{p,q}\rangle\,\langle\psi,\psi_{p,q}\rangle\end{aligned}$$

$\forall\ \psi\in A_\theta$. This implies (5.1) converges in A'_θ.
In the view of theorem 5.1, the distributional finite continuous Ridgelet transform of $f\in A'_\theta$ is defined by

$$\Re' f\left(I\left(U_{p,q},a,b\right)\right)=\frac{a^{-1/2}}{4\alpha\beta}\int_{-\beta}^{\beta}\int_{-\alpha}^{\alpha} f\left(x,y\right)\psi\left(\left(U_{p,q}\cdot\left(x,y\right)-b\right)/a\right)dxdy. \tag{5.2}$$

Its corresponding inversion formula is supplied by theorem 5.1 and can be expressed as

$$\Re'^{-1}\left(\Re f\left(I\left(U_{p,q},a,b\right)\right)\right)=f\left(x,y\right)=a^{1/2}\sum_{p,q=1}^{\infty}\Re f\left(I\left(U_{p,q},a,b\right)\right)\psi\left(\left(b-U_{p,q}\cdot\left(x,y\right)\right)/a\right). \tag{5.3}$$

We invoke (4.11) to get

$$\begin{aligned}&\frac{a^{-1/2}}{4\alpha\beta}\int_{-\beta}^{\beta}\int_{-\alpha}^{\alpha}\left\{\Omega^k_{x,y,\theta}f\left(x,y\right)\right\}\psi\left(\left(U_{p,q}\cdot\left(x,y\right)-b\right)/a\right)dxdy\\ &=\left(-\eta^2\right)^k\left[\sin^2\theta\cos^{2k}\theta-\cos^2\theta\sin^{2k}\theta\right]\frac{a^{-1/2}}{4\alpha\beta}\int_{-\alpha}^{\alpha}\int_{-\beta}^{\beta} f\left(x,y\right)\psi\left(\left(U_{p,q}\cdot\left(x,y\right)-b\right)/a\right)dxdy\end{aligned}$$

$\forall f\in A'_\theta$ and $k=0,1,2,\cdots$.
Thus the result:

$$\begin{aligned}&\frac{a^{-1/2}}{4\alpha\beta}\int_{-\beta}^{\beta}\int_{-\alpha}^{\alpha}\left\{\Omega^k_{x,y,\theta}f\left(x,y\right)\right\}\psi\left(\left(U_{p,q}\cdot\left(x,y\right)-b\right)/a\right)dxdy\\ &=\left(-\eta^2\right)^k\left[\sin^2\theta\cos^{2k}\theta-\cos^2\theta\sin^{2k}\theta\right]\Re' f\left(I\left(U_{p,q},a,b\right)\right)\end{aligned} \tag{5.4}$$

□

Theorem 5.2 (The Uniqueness Theorem)**.** *Let $f, g \in A'_\theta$ and let finite continuous Ridgelet transform of f and g be $\Re f\left(I\left(U_{p,q}, a, b\right)\right)$ and $\Re g\left(I\left(U_{p,q}, a, b\right)\right)$ respectively, as defined by (4.13). If $\Re f\left(I\left(U_{p,q}, a, b\right)\right) = \Re g\left(I\left(U_{p,q}, a, b\right)\right)$, then $f = g$ in the sense of equality in $D'\left(I_{\alpha,\beta}\right)$.*

Proof. By (4.14)

$$f - g = a^{1/2} \sum_{p,q=1}^{\infty} \left[\Re f\left(I\left(U_{p,q}, a, b\right)\right) - \Re\left(I\left(U_{p,q}, a, b\right)\right)\right] \psi\left(\left(b - U_{p,q} \cdot (x, y)\right)/a\right),$$

$$f - g = 0.$$

as

$$\Re f\left(I\left(U_{p,q}, a, b\right)\right) = \Re g\left(I\left(U_{p,q}, a, b\right)\right).$$

Hence $f = g$ in the sense of equality in $D'\left(I_{\alpha,\beta}\right)$. □

6. APPLICATION OF FINITE CONTINUOUS RIDGELET TRANSFORM TO SOLVE BOUNDARY VALUE PROBLEM.

Example 6.1 (The Telegraph Equation)**.** The Finite continuous Ridgelet transform can also be used to solve boundary-value problems for partial differential equations. Consider the equation

$$u_{xx} = Au_{tt} + Bu_t + Cu, \tag{6.1}$$

for every $0 < t < \infty$ and where A, B and C are nonnegative constant. This equation, known as the telegraph equation, describes an electromagnetic signal $u\left(x, t\right)$ such as an electric current or voltage, traveling along a transmission line. The constant A, B, C are determined by the distributed inductance, resistance and capacitance (per unit length) along the line [5]. If the transmission line extends over $l < t < -l$, then two initial conditions at $t = 0$ on u and u_t are sufficient to specify u. Hence the initial condition becomes

$$u\left(x, y, 0\right) = f\left(x, y\right) \quad and \quad u_t\left(x, y, 0\right) = 0. \tag{6.2}$$

Taking finite continuous Ridgelet transform on both sides of (6.1)

$$\Re\left[\frac{\partial^2 u}{\partial x^2}\right]\left(I\left(U_{p,q}, a, b\right)\right) = \Re\left[A\frac{\partial^2 u}{\partial t^2} + B\frac{\partial u}{\partial t} + Cu\right]\left(I\left(U_{p,q}, a, b\right)\right)$$

From (3.10)

$$\left(\frac{p\pi}{\alpha}\right)^2 \bar{U} = A\frac{\partial^2 \bar{U}}{\partial t^2} + B\frac{\partial \bar{U}}{\partial t} + C\bar{U}, \tag{6.3}$$

where

$$\bar{U} = \Re u\left(I\left(U_{p,q}, a, b\right)\right) = \frac{a^{-1/2}}{4\alpha\beta}\int_{-\beta}^{\beta}\int_{-\alpha}^{\alpha} u\left(x, y\right)\psi\left(\left(U_{p,q}\cdot\left(x, y\right) - b\right)/a\right)dxdy.$$

Thus

$$A\frac{\partial^2\bar{U}}{\partial t^2} + B\frac{\partial\bar{U}}{\partial t} + \left(C - \left(\frac{p\pi}{\alpha}\right)^2\right)\bar{U} = 0, \tag{6.4}$$

here $\sigma = \left(C - \left(\frac{p\pi}{\alpha}\right)^2\right)$.

Similarly taking finite continuous Ridgelet transform on both of (6.2) gives

$$\bar{U}\left(x, y, 0\right) = \Re f\left(x, y\right)\left(I\left(U_{p,q}, a, b\right)\right) \quad and \quad \bar{U}_t\left(x, y, 0\right) = 0. \tag{6.5}$$

The solution of (6.4); second order differential equation is given by

$$\bar{U} = A_1 e^{\sigma_1 t} + B_1 e^{\sigma_2 t}, \tag{6.6}$$

where $\sigma_1 = \dfrac{-B + \sqrt{B^2 - 4A\left(C - \left(\frac{p\pi}{\alpha}\right)^2\right)}}{2A}$ and $\sigma_2 = \dfrac{-B - \sqrt{B^2 - 4A\left(C - \left(\frac{p\pi}{\alpha}\right)^2\right)}}{2A}$.

Applying initial conditions (6.5), we get

$$\bar{U} = \frac{\sigma_2}{\sigma_2 - \sigma_1}\Re f\left(x, y\right)\left(I\left(U_{p,q}, a, b\right)\right)e^{\sigma_1 t} + \frac{\sigma_1}{\sigma_1 - \sigma_2}\Re f\left(x, y\right)\left(I\left(U_{p,q}, a, b\right)\right)e^{\sigma_2 t} \tag{6.7}$$

and applying inversion formula (4.14) gives

$$u\left(x, y, t\right) = a^{1/2}\sum_{p=1,q=1}^{\infty}\frac{\sigma_2}{\sigma_2 - \sigma_1}\Re f\left(x, y\right)\left(I\left(U_{p,q}, a, b\right)\right)e^{\sigma_1 t}\psi\left(\left(b - U_{p,q}\cdot\left(x, y\right)\right)/a\right)$$
$$+ \frac{\sigma_1}{\sigma_1 - \sigma_2}\sum_{p=1,q=1}^{\infty}\frac{\sigma_1}{\sigma_1 - \sigma_2}\Re f\left(x, y\right)\left(I\left(U_{p,q}, a, b\right)\right)e^{\sigma_2 t}\psi\left(\left(b - U_{p,q}\cdot\left(x, y\right)\right)/a\right), \tag{6.8}$$

which gives the solution of (6.1).

Nevertheless, there are number of significant cases of (6.1), where the solution can be obtained explicitly.

(i) If $A = 1/c^2$ and $B = C = 0$, (6.1) reduces to the wave equation $u_{tt} = c^2 u_{xx}$. Using (4.13), the transform function becomes

$$\bar{U} = \sinh\left(\frac{pc\pi}{\alpha}t\right) \Re f(x, y)(I(U_{p,q}, a, b))$$

is easily inverted using (4.14) to

$$u(x, y, t) = a^{1/2} \sum_{p,q=1}^{\infty} \sinh\left(\frac{pc\pi}{\alpha}t\right) \Re f(x, y)(I(U_{p,q}, a, b))\, \psi((b - U_{p,q} \cdot (x, y))/a).$$

(ii) If $A = C = 0$ and $B = 1/k$, where k is a positive constant, we obtain the heat equation $u_t = k u_{xx}$.
In this case the finite continuous Ridgelet transform of u is

$$\Re\left[\frac{\partial u}{\partial t}\right](I(U_{p,q}, a, b)) = k\Re\left(\frac{\partial^2 u}{\partial x^2}\right)(I(U_{p,q}, a, b)).$$

Hence

$$\frac{\partial \bar{U}}{\partial t} = k\left(\frac{p\pi}{\alpha}\right)^2 \bar{U} \tag{6.9}$$

where

$$\bar{U} = \Re u(I(U_{p,q}, a, b)) = \frac{a^{-1/2}}{4\alpha\beta} \int_{-\beta}^{\beta} \int_{-\alpha}^{\alpha} u(x, y)\, \psi((U_{p,q} \cdot (x, y) - b)/a)\, dxdy.$$

Now the solution of (6.9) is given by

$$\log \bar{U} = k\left(\frac{p\pi}{\alpha}\right)^2 t + A' \tag{6.10}$$

where A' is constant of integration.
Applying initial conditions (6.5) to (6.10)

$$\log \bar{U} = k\left(\frac{p\pi}{\alpha}\right)^2 t + \log\left[\Re f(x, y)(I(U_{p,q}, a, b))\right].$$

Hence

$$\bar{U} = \Re f(x, y)(I(U_{p,q}, a, b))\, e^{k\left(\frac{p\pi}{\alpha}\right)^2 t}.$$

From (4.14), follows:

$$u(x, y, t) = a^{1/2} \sum_{p,q=1}^{\infty} \Re f(x, y)(I(U_{p,q}, a, b))\, e^{k\left(\frac{p\pi}{\alpha}\right)^2 t} \psi((b - U_{p,q} \cdot (x, y))/a). \tag{6.11}$$

Example 6.2 (Heat Conduction)**.** A square plate has its faces insulated with sides of length $-\alpha < x < \alpha$ and $-\beta < y < \beta$ and it's sides kept at 0°C. If the initial temperature is specified, then the equation for the subsequent temperature at any point of the plate is given by

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right). \tag{6.12}$$

The boundary conditions are given by

$$|u(x,y,t)| < M,$$

$$u(-\alpha,y,t) = u(\alpha,y,t) = u(x,-\beta,t) = u(x,\beta,t) = 0 \tag{6.13}$$

and

$$u(x,y,0) = f(x,y) \tag{6.14}$$

where $-\alpha < x < \alpha,\ -\beta < y < \beta$ and $t > 0$.

To solve the boundary value problem, applying finite continuous Ridgelet transform on both sides of (6.11), we get

$$\Re\left[\frac{\partial u}{\partial t}\right](I(U_{p,q},a,b)) = k\Re\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)(I(U_{p,q},a,b)),$$

Hence

$$\frac{\partial \bar{U}}{\partial t} = k\left[\left(\frac{p\pi}{\alpha}\right)^2 + \left(\frac{q\pi}{\beta}\right)^2\right]\bar{U}. \tag{6.15}$$

Also finite continuous Ridgelet transform on both sides of (6.13)

$$\bar{U}(x,y,0) = \Re f(x,y)(I(U_{p,q},a,b)). \tag{6.16}$$

Here (6.14) is first order differential equation, whose solution is given by

$$\log\bar{U} = k\left[\left(\frac{p\pi}{\alpha}\right)^2 + \left(\frac{q\pi}{\beta}\right)^2\right]t + A, \tag{6.17}$$

where A is integration constant.

Now applying the condition (6.15) to (6.16), we obtain

$$\log\bar{U} = k\left[\left(\frac{p\pi}{\alpha}\right)^2 + \left(\frac{q\pi}{\beta}\right)^2\right]t + \log\left[\Re f(x,y)(I(U_{p,q},a,b))\right].$$

Hence

$$\bar{U} = \Re f(x,y)(I(U_{p,q},a,b))\, e^{k\left[\left(\frac{p\pi}{\alpha}\right)^2 + \left(\frac{q\pi}{\beta}\right)^2\right]t}. \tag{6.18}$$

We may now invoke the inversion formula (4.14) to provide the required result

$$u(x,y,t) = a^{1/2}\sum_{p,q=1}^{\infty}\Re f(x,y)(I(U_{p,q},a,b))\, e^{k\left[\left(\frac{p\pi}{\alpha}\right)^2 + \left(\frac{q\pi}{\beta}\right)^2\right]t}\psi((b - U_{p,q}\cdot(x,y))/a). \tag{6.19}$$

CONCLUSION

The finite continuous Ridgelet transforms and its inversion formula is studied in this paper. Using Sturm-Liouville theory the operational calculus of finite continuous Ridgelet transforms is analyzed. The study is supported with applications in engineering field dealt by partial differential equations.

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