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Polynomial Sharing and Uniqueness of Differential-Difference Polynomials of L-functions

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Abstract

In this paper, we study value distributions and uniqueness problems of differential-difference polynomials of L-functions. Considering polynomial sharing of certain differential-difference polynomials of an L-function with that of a meromorphic function we prove a uniqueness theorem which improve and generalize some earlier results due to Hao, Chen [4], Zhu, Chen [16], Mandal, Datta [10] and Datta, Mandal [2].

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1. INTRODUCTION

For the last 150 years the most important open problem in pure mathematics is considered to be the Riemann hypothesis and its extension to the general classes of L-functions. L-functions are most important objects in the modern number theory. Let a function L be defined by the Dirichlet series $L(z) = \sum_{n=1}^{\infty} a(n)/n^z$ with $a_1 = 1$ satisfying the axioms (i) $a(n) \ll n^{\epsilon}$, for every $\epsilon > 0$, (ii) there exists an integer $k \geq 0$

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such that $(z-1)^k L(z)$ is a finite order entire function, (iii) every L-function satisfies the functional equation

$$\lambda_L(z) = \omega\overline{\lambda_L(1-\overline{z})},$$

where

$$\lambda_L(z) = L(z)Q^z \prod_{i=1}^{k} \Gamma(\lambda_i z + \nu_i)$$

with positive real numbers Q, λ_i and complex numbers ν_i, ω with $Re\nu_i \geq 0$ and $\mid \omega \mid = 1$ and (iv) $L(z)$ satisfies $L(z) = \prod_p L_p(z)$, where $L_p(z) = exp(\sum_{k=1}^{\infty} b(p^k)/p^{kz})$ with coefficients $b(p^k)$ satisfying $b(p^k) \ll p^{k\theta}$ for some $\theta < 1/2$ and p denotes prime number. Then L is said to be an L-function in the Selberg class. If L satisfy axioms (i)-(iii) then L is said to be an L-function in the extended Selberg class. Henceforth by an L-function we always mean an L-function in the extended Selberg class.

In this paper, we concentrate our attention on the uniqueness problems of differential-difference polynomial of L-functions. We use the standard definitions and notations of value distribution theory [5].

2. PRELIMINARIES

Let $\alpha \in \mathbf{C} \cup \{\infty\}$ and ξ, ψ be meromorphic functions in the complex plane. The hyper order $\rho_2(\xi)$ of ξ is defined by $\rho_2(\xi) = \limsup_{r\to\infty} \frac{\log\log T(r,\xi)}{\log r}$. We denote by $S(r,\xi)$ any function satisfying $S(r,\xi) = o(T(r,\xi))$ as $r \longrightarrow \infty$, outside a possible exceptional set of finite linear measure.

Definition 2.1. *[6, 7]. Let ξ and ψ be meromorphic functions defined in the complex plane and n be an integer (≥ 0) or infinity . For $\alpha \in \mathbf{C} \cup \{\infty\}$ we denote by $E_n(\alpha;\xi)$ the set of all zeros of $\xi - \alpha$ where a zero of multiplicity k is counted k times if $k \leq n$ and $n+1$ times if $k > n$. If $E_n(\alpha;\xi) = E_n(\alpha;\psi)$, we say that ξ, ψ share the value α with weight n. We say ξ, ψ share (α, n) to mean that ξ, ψ share the value α with weight n.*

Definition 2.2. *[10]. Let ξ be a meromorphic function defined in the complex plane and $P(z)$ be a polynomial or a small function of ξ. Then we denote by $E_{m)}(P;\xi)$, $\overline{E}_{m)}(P;\xi)$ and $E_m(P;\xi)$ the sets $E_{m)}(0;\xi - P)$, $\overline{E}_{m)}(0;\xi - P)$ and $E_m(0;\xi - P)$ respectively. We write ξ, ψ share (P, n) to mean that $\xi - P$, $\psi - P$ share the value 0 with weight n.*

Liu, Li and Yi [9] in 2017 proved the following uniqueness theorem.

Theorem 2.1. *[9]. let L be an L-function and ξ be a nonconstant meromorphic function. If $j \geq 1$, $k \geq 1$ be integers such that $j > 3k + 6$ and $\{\xi^j\}^{(k)}(z)$, $\{L^j\}^{(k)}(z)$ share (z, ∞) then $\xi \equiv \alpha L$ for some nonconstant α satisfying $\alpha^j = 1$.*

Considering differential polynomials in 2018, Hao and Chen [4] proved the following uniqueness theorems.

Theorem 2.2. *[4] Let ξ be a nonconstant meromorphic function and L be an L-function such that $[\xi^n(\xi - 1)^m]^{(\tau)}$ and $[L^n(L - 1)^m]^{(\tau)}$ share $(1, \infty)$, where $n, m, \tau \in \mathbb{Z}^+$. If $n > m + 3\tau + 6$ and $\tau \geq 2$, then, $\xi \equiv L$ or, $\xi^n(\xi - 1)^m \equiv L^n(L - 1)^m$.*

Theorem 2.3. *[4] Let ξ be a nonconstant meromorphic function and L be an L-function such that $[\xi^n(\xi - 1)^m]^{(\tau)}$ and $[L^n(L - 1)^m]^{(\tau)}$ share $(1, 0)$, where $n, m, \tau \in \mathbb{Z}^+$. If $n > 4m + 7\tau + 11$ and $\tau \geq 2$, then $\xi \equiv L$ or, $\xi^n(\xi - 1)^m \equiv L^n(L - 1)^m$.*

Using truncated sharing in 2019 W. Q. Zhu and J. F. Chen [16] proved the following theorem.

Theorem 2.4. *[16] Let L be an L-function and ξ be a transcendental meromorphic function defined in the complex plane **C**. Also let n, $k(\geq 2)$, $l(\geq 2)$ be positive integers such that $n \geq 7k + 17$. If $\overline{E}_{l)}(1, (\xi^n(\xi - 1))^{(k)}) = \overline{E}_{l)}(1, (L^n(L - 1))^{(k)})$ then $\xi \equiv L$.*

Definition 2.3. *[8]. Let ξ be a meromorphic function defined in the complex plane. Let $k \geq 1$ be an integer and $\alpha \in \mathbf{C} \cup \{\infty\}$. By $N(r, \alpha; \xi \mid \leq k)$ we denote the counting function of the α points of ξ with multiplicity not greater than k and by $\overline{N}(r, \alpha; \xi \mid \leq k)$ the reduced counting function. Also by $N(r, \alpha; \xi \mid \geq k)$ we denote the counting function of the α points of ξ with multiplicity not less than k and by $\overline{N}(r, \alpha; \xi \mid \geq k)$ the reduced counting function. We define*

$$N_k(r, \alpha; \xi) = \overline{N}(r, \alpha; \xi) + \overline{N}(r, \alpha; \xi \mid \geq 2) + \cdots + \overline{N}(r, \alpha; \xi \mid \geq k).$$

Considering small function sharing in 2020 Mandal and Datta [10] proved the following theorem.

Theorem 2.5. *[10]. Let L be a nonconstant L-function and ρ be a small function of L such that $\rho \not\equiv 0, \infty$. If $\overline{E}_{4)}(\rho; L) = \overline{E}_{4)}(\rho; (L^m)^{(k)})$, $E_{2)}(\rho; L) = E_{2)}(\rho; (L^m)^{(k)})$ and*

$$2N_{2+k}(r, 0; L^m) \leq (\sigma + o(1))T(r, L),$$

where $m \geq 1$, $k \geq 1$ are integers and $0 < \sigma < 1$, then $L \equiv (L^m)^{(k)}$.

Using weighted sharing Datta and Mandal [2] proved the following theorem.

Theorem 2.6. *[2]. Let f be a nonconstant meromorphic function and L be a nonconstant L-function. If $E_0(0, f) = E_0(0, L)$, $E_1(1, f) = E_1(1, L)$ and $\overline{N}(r, 0; f) + \overline{N}(r, \infty; f) = S(r, f)$ then either $L \equiv f$ or $T(r, L) = N(r, 0; L \mid\leq 2) + S(r, L)$ and $T(r, f) = N(r, 0; L' \mid\leq 1) + S(r, L)$.*

Now the following questions come naturally.

Question 2.1. *If we consider polynomial sharing in theorem 2.1, 2.2, 2.3, 2.4, 2.5 and 2.6 then what will be the results?*

Question 2.2. *If we consider differential-difference polynomials in place of differential polynomials in theorem 2.1, 2.2, 2.3, 2.4 and 2.5 then what will be the results?*

Definition 2.4. *[6]. Let two nonconstant meromorphic functions ξ and ψ share a value α IM. We denote by $\overline{N}_*(r, \alpha; \xi, \psi)$ the counting function of the α-points of ξ and ψ with different multiplicities, where each α-point is counted only once.*

Clearly $\overline{N}_*(r, \alpha; \xi, \psi) \equiv \overline{N}_*(r, \alpha; \psi, \xi)$.

Definition 2.5. *Let two nonconstant meromorphic functions ξ and ψ share a value α IM. We denote by $\overline{N}(r, \alpha; \xi| > \psi)$ the counting function of the α-points of ξ and ψ with multiplicities with respect to ξ is greater than the multiplicities with respect to ψ, where each α-point is counted once only.*

Definition 2.6. *Let two nonconstant meromorphic functions ξ and ψ share a value α IM. We denote by $\overline{N}_E(r, \alpha; \xi, \psi| > m)$ the counting function of the α-points of ξ and ψ with multiplicities greater than m and the multiplicities with respect to ξ is equal to the multiplicities with respect to ψ, where each α-point is counted once only.*

Definition 2.7. *[8]. Let ξ be a meromorphic function defined in the complex plane and P be a small function of ξ or a polynomial of z. Then we denote by $N(r, P; \xi \mid\leq k)$, $\overline{N}(r, P; \xi \mid\leq k)$, $N(r, P; \xi \mid\geq k)$, $\overline{N}(r, P; \xi \mid\geq k)$, $N_k(r, P; \xi)$ etc. the counting functions $N(r, 0; \xi - P \mid\leq k)$, $\overline{N}(r, 0; \xi - P \mid\leq k)$, $N(r, 0; \xi - P \mid\geq k)$, $\overline{N}(r, 0; \xi - P \mid\geq k)$, $N_k(r, 0; \xi - P)$ etc. respectively.*

3. MAIN RESULTS

Let L be a nonconstant L-function, ξ be a transcendental meromorphic function and $P(z)$ be a polynomial of z. Also let $\tau, n, \eta, \mu_j (j = 1, 2, ..., \eta), \lambda = \sum_{j=1}^{\eta} \mu_j$ be positive integers and $\omega_j \in \mathbf{C} - \{0\}$ $(j = 1, 2, ..., \eta)$ be distinct constants. Henceforth we denote by ϕ, ψ, Φ, Ψ the following functions $\phi(z) = \prod_{j=1}^{\eta} \xi(z+\omega_j)^{\mu_j}$, $\psi(z) = \prod_{j=1}^{\eta} L(z+\omega_j)^{\mu_j}$, $\Phi(z) = \frac{\xi(z)^n(\phi(z))^{(\tau)}}{P(z)}$ and $\Psi(z) = \frac{L(z)^n(\psi(z))^{(\tau)}}{P(z)}$.

Using the concept of weighted sharing we try to solve Questions 2.1, 2.2 and prove the following theorem.

Theorem 3.1. *Let L be a nonconstant L-function and ξ be a transcendental meromorphic function such that $\rho_2(L) < 1$, $\rho_2(\xi) < 1$ and ξ, L share $(\infty, 0)$. If $L(z)^n[\psi(z)]^{(\tau)}$ and $\xi(z)^n[\phi(z)]^{(\tau)}$ share $(P(z), l)$, where $0 \le l < \infty$ and $P(z)$ is a polynomial of z, then one of the following holds*

(i) $L(z)^n[\psi(z)]^{(\tau)} \equiv \xi(z)^n[\phi(z)]^{(\tau)}$

(ii) $L = e^{a(z)}$ *and* $\xi = e^{b(z)}$, *where $a(z)$ and $b(z)$ are entire functions*

if

(i) $l = 0$ *and* $n > \max\{\lambda + \eta(5\tau + 7) + 7, c_1, c_2\}$

(ii) $l = 1$ *and* $n > \max\{\lambda + \frac{1}{2}(\eta(5\tau + 9) + 7), c_1, c_2\}$

(iii) $l \ge 2$ *and* $n > \max\{\lambda + \eta(2\tau + 4) + 4, c_1, c_2\}$, *where* $c_1 = \sum_{j=1}^{\eta} a_j\mu_j$ *and* $c_2 = \sum_{j=1}^{\eta} b_j\mu_j$, *$a_j$ and b_j denotes maximum orders of zeros of $\xi(z+\omega_j)$ and $L(z+\omega_j)$ respectively for $j = 1, 2, ..., \eta$.*

4. LEMMAS

In this section we present some lemmas which will be needed in the proof of our results. Henceforth we denote by $\Omega_{\Phi,\Psi}$ the function defined by

$$\Omega_{\Phi,\Psi} = \left(\frac{\Phi''}{\Phi'} - \frac{2\Phi'}{\Phi - 1}\right) - \left(\frac{\Psi''}{\Psi'} - \frac{2\Psi'}{\Psi - 1}\right)$$

Lemma 4.1. *[12]. Let L be an L-function with degree q. Then*

$$T(r, L) = \frac{q}{\pi} r \log r + O(r).$$

Lemma 4.2. *[10]. Let L be an L-function.Then $N(r, \infty; L) = S(r, L) = O(\log r)$.*

Lemma 4.3. *Let ξ be a transcendental meromorphic function and L be an L-function. If ξ and L share $(\infty, 0)$ then $N(r, \infty; \xi) = O(\log r) = S(r, \xi)$ and $N(r, \infty; \xi) = O(\log r) = S(r, L)$.*

Proof. Since ξ and L share $(\infty, 0)$ therefore ξ has finitely many poles.
Hence $N(r, \infty; \xi) = O(\log r)$.
Again since ξ and L are transcendental meromorphic functions therefore $N(r, \infty; \xi) = O(\log r) = S(r, \xi)$ and $N(r, \infty; \xi) = O(\log r) = S(r, L)$. This completes the proof. □

Lemma 4.4. *[15]. Let $\xi(z) = \frac{\alpha_0 + \alpha_1 z + ... + \alpha_n z^n}{\beta_0 + \beta_1 z + ... + \beta_m z^m}$ be a nonconstant rational function defined in the complex plane **C**, where $\alpha_0, \alpha_1, ..., \alpha_n (\neq 0)$ and $\beta_0, \beta_1, ..., \beta_m (\neq 0)$ are complex constants. Then*

$$T(r, \xi) = max\{m, n\} \log r + O(1).$$

Lemma 4.5. *[13]. Let ξ be a transcendental meromorphic function of hyper order $\rho_2(\xi) < 1$. Then for any $\alpha \in$ **C** $- \{0\}$*

$$T(r, \xi(z + \alpha)) = T(r, \xi(z)) + S(r, \xi(z))$$

$$N(r, \infty; \xi(z + \alpha)) = N(r, \infty; \xi(z)) + S(r, \xi(z))$$

$$N(r, 0; \xi(z + \alpha)) = N(r, 0; \xi(z)) + S(r, \xi(z))$$

Lemma 4.6. *[1] Let f and g be two nonconstant meromorphic functions sharing $(1, l)$ and $(\infty, 0)$ where $2 \leq l < \infty$. If $\Omega_{f,g} \not\equiv 0$ then*

$$\begin{aligned} T(r, f) &\leq N_2(r, 0; f) + N_2(r, 0; g) + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + \overline{N}_*(r, \infty; f, g) \\ &- m(r, 1; g) - N_E(r, 1; f, g| > 3) - \overline{N}(r, 1; g| > f) + S(r, f) + S(r, g) \end{aligned}$$

$$\begin{aligned} T(r, g) &\leq N_2(r, 0; f) + N_2(r, 0; g) + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + \overline{N}_*(r, \infty; f, g) \\ &- m(r, 1; f) - N_E(r, 1; g, f| > 3) - \overline{N}(r, 1; f| > g) + S(r, f) + S(r, g) \end{aligned}$$

Lemma 4.7. *[11] Let f and g be two nonconstant meromorphic functions sharing $(1,1)$ and $(\infty,0)$. If $\Omega_{f,g} \not\equiv 0$ then*

$$\begin{aligned} T(r,f) &\leq N_2(r,0;f)+N_2(r,0;g)+\frac{3}{2}\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}_*(r,\infty;f,g) \\ &+ \frac{1}{2}\overline{N}(r,0;f)+S(r,f)+S(r,g) \end{aligned}$$

$$\begin{aligned} T(r,g) &\leq N_2(r,0;f)+N_2(r,0;g)+\frac{3}{2}\overline{N}(r,\infty;g)+\overline{N}(r,\infty;f)+\overline{N}_*(r,\infty;f,g) \\ &+ \frac{1}{2}\overline{N}(r,0;g)+S(r,f)+S(r,g) \end{aligned}$$

Lemma 4.8. *[11] Let f and g be two nonconstant meromorphic functions sharing $(1,0)$ and $(\infty,0)$. If $\Omega_{f,g} \not\equiv 0$ then*

$$\begin{aligned} T(r,f) &\leq N_2(r,0;f)+N_2(r,0;g)+3\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+\overline{N}_*(r,\infty;f,g) \\ &+ 2\overline{N}(r,0;f)+\overline{N}(r,0;g)+S(r,f)+S(r,g) \end{aligned}$$

$$\begin{aligned} T(r,g) &\leq N_2(r,0;f)+N_2(r,0;g)+3\overline{N}(r,\infty;g)+2\overline{N}(r,\infty;f)+\overline{N}_*(r,\infty;f,g) \\ &+ 2\overline{N}(r,0;g)+\overline{N}(r,0;f)+S(r,f)+S(r,g) \end{aligned}$$

Lemma 4.9. *[14] Let f be a nonconstant meromorphic function and k, p be two positive integers. Then*

$$T(r,f^{(k)}) \leq T(r,f)+k\overline{N}(r,\infty;f)+S(r,f)$$

$$N_p(r,0;f^{(k)}) \leq T(r,f^{(k)})-T(r,f)+N_{p+k}(r,0;f)+S(r,f)$$

$$N_p(r,0;f^{(k)}) \leq N_{p+k}(r,0;f)+k\overline{N}(r,\infty;f)+S(r,f)$$

$$N(r,0;f^{(k)}) \leq N(r,0;f)+k\overline{N}(r,\infty;f)+S(r,f)$$

Lemma 4.10. *[3] Let ξ be a transcendental meromorphic function of hyper order $\rho_2(\xi)<1$. Then*

$$(n-\lambda)T(r,\xi)+S(r,\xi) \leq T(r,\xi^n\phi) \leq (n+\lambda)T(r,\xi)+S(r,\xi)$$

Lemma 4.11. *Let ξ be a transcendental meromorphic function of hyper order $\rho_2(\xi)<1$. If ξ and an L-function L share $(\infty,0)$ then*

$$(n-\lambda)T(r,\xi)+S(r,\xi) \leq T(r,\xi^n\phi^{(\tau)}) \leq (n+\lambda)T(r,\xi)+S(r,\xi)$$

Proof. Since ξ and L share $(\infty, 0)$ therefore by lemma 4.3 we get $N(r, \infty; \xi) = S(r, \xi)$,.
Hence by lemma 4.5 we have

$$\begin{aligned} T(r, \xi^n \phi^{(\tau)}) &\leq T(r, \xi^n) + T(r, \phi^{(\tau)}) + S(r, \xi) \\ &\leq nT(r, \xi) + T(r, \phi) + \tau \overline{N}_(r, \infty, \phi) + S(r, \xi) \\ &\leq (n + \lambda)T(r, \xi) + S(r, \xi). \end{aligned} \tag{4.1}$$

Also by lemma 4.5 we have

$$\begin{aligned} nT(r, \xi) &= T(r, \xi^n) + S(r, \xi) \\ &\leq T(r, \frac{\xi^n \phi^{(\tau)}}{\phi^{(\tau)}}) + S(r, \xi) \\ &\leq T(r, \xi^n \phi^{(\tau)}) + T(r, \phi^{(\tau)}) + S(r, \xi) \\ &\leq T(r, \xi^n \phi^{(\tau)}) + T(r, \phi) + \tau \overline{N}_(r, \infty, \phi) + S(r, \xi) \\ &\leq T(r, \xi^n \phi^{(\tau)}) + \lambda T(r, \xi) + S(r, \xi). \end{aligned} \tag{4.2}$$

We get from (4.2)

$$(n - \lambda)T(r, \xi) \leq T(r, \xi^n \phi^{(\tau)}) + S(r, \xi). \tag{4.3}$$

Hence we get from (4.1) and (4.3)

$$(n - \lambda)T(r, \xi) + S(r, \xi) \leq T(r, \xi^n \phi^{(\tau)}) \leq (n + \lambda)T(r, \xi) + S(r, \xi)$$

This completes the proof of the lemma. □

Lemma 4.12. *Let L be a nonconstant L-function and ξ be a transcendental meromorphic function such that $\rho_2(L) < 1$, $\rho_2(\xi) < 1$ and ξ, L share $(\infty, 0)$. Also let $L(z)^n[\psi(z)]^{(\tau)}$ and $\xi(z)^n[\phi(z)]^{(\tau)}$ share $(P(z), 0)$ and , where $P(z)$ is a polynomial of z. If $\Omega_{\Phi,\Psi} \not\equiv 0$ then $n \leq \lambda + \eta(5\tau + 7) + 7$.*

Proof. Since ξ, L share $(\infty, 0)$ and $\xi(z)^n \phi(z)^{(\tau)}$, $L(z)^n \psi(z)^{(\tau)}$ share $(P(z), 0)$ therefore Φ and Ψ share $(1, l)$ except zeros of $P(z)$ and share $(\infty, 0)$.
By lemma 4.1 it is clear that L is a transcendental meromorphic function.
Since L and ξ are transcendental meromorphic functions therefore P is a small function of L and ξ.

Hence by lemma 4.2, lemma 4.3 and lemma 4.8 we have

$$\begin{aligned}
T(r, L^n\psi^{(\tau)}) &= T(r,\Psi) + S(r,L)\\
&\leq N_2(r,0;\Phi) + N_2(r,0;\Psi) + 3\overline{N}(r,\infty;\Psi) + 2\overline{N}(r,\infty;\Phi)\\
&+ \overline{N}_*(r,\infty;\Phi,\Psi) + 2\overline{N}(r,0;\Psi) + \overline{N}(r,0;\Phi) + S(r,\Phi) + S(r,L)\\
&\leq N_2(r,0;\xi^n) + N_2(r,0;\phi^{(\tau)}) + N_2(r,0;L^n) + N_2(r,0;\psi^{(\tau)})\\
&+ 2\overline{N}(r,0;L^n) + 2\overline{N}(r,0;\psi^{(\tau)}) + \overline{N}(r,0;\xi^n) + \overline{N}(r,0;\phi^{(\tau)})\\
&+ S(r,\Phi_1) + S(r,L)\\
&\leq 2T(r,L) + N_2(r,0;\psi^{(\tau)}) + 2\overline{N}(r,0;L^n) + 2\overline{N}(r,0;\psi^{(\tau)})\\
&+ 2T(r,\xi) + N_2(r,0;\phi^{(\tau)}) + \overline{N}(r,0;\xi^n) + \overline{N}(r,0;\phi^{(\tau)})\\
&+ S(r,\Phi_1) + S(r,L)\\
&\leq 2T(r,L) + T(r,\psi^{(\tau)}) - T(r,\psi) + N_{2+\tau}(r,0;\psi)\\
&+ 2\overline{N}(r,0;L^n) + 2\overline{N}(r,0;\psi^{(\tau)})\\
&+ 2T(r,\xi) + N_2(r,0;\phi^{(\tau)}) + \overline{N}(r,0;\xi^n) + \overline{N}(r,0;\phi^{(\tau)})\\
&+ S(r,\Phi_1) + S(r,L)\\
&\leq 2T(r,L) + T(r,L^n\psi^{(\tau)}) - T(r,L^n\psi) + \eta(2+\tau)T(r,L)\\
&+ 2\overline{N}(r,0;L^n) + 2\overline{N}(r,0;\psi^{(\tau)})\\
&+ 2T(r,\xi) + \eta(2+\tau)T(r,\xi) + \overline{N}(r,0;\xi^n) + \overline{N}(r,0;\phi^{(\tau)})\\
&+ S(r,\xi) + S(r,L)\\
&\leq 2T(r,L) + T(r,L^n\psi^{(\tau)}) - T(r,L^n\psi) + \eta(2+\tau)T(r,L)\\
&+ 2T(r,L) + 2N_{\tau+1}(r,0;\psi)\\
&+ 2T(r,\xi) + \eta(2+\tau)T(r,\xi) + T(r,\xi) + N_{\tau+1}(r,0;\phi) + S(r,\xi) + S(r,L)\\
&\leq (\eta(3\tau+4)+4)T(r,L) + (\eta(2\tau+3)+3)T(r,\xi) + T(r,L^n\psi^{(\tau)})\\
&- T(r,L^n\psi) + S(r,\xi) + S(r,L) \qquad (4.4)
\end{aligned}$$

Hence from (4.4) we have

$$\begin{aligned}
T(r,L^n\psi) &\leq (\eta(3\tau+4)+4)T(r,L) + (\eta(2\tau+3)+3)T(r,\xi)\\
&+ S(r,\xi) + S(r,L) \qquad (4.5)
\end{aligned}$$

By lemma 4.10 we have from (4.5)

$$\begin{aligned}
(n-\lambda)T(r,L) &\leq (\eta(3\tau+4)+4)T(r,L) + (\eta(2\tau+3)+3)T(r,\xi)\\
&+ S(r,\xi) + S(r,L) \qquad (4.6)
\end{aligned}$$

Similarly we have

$$\begin{aligned}(n-\lambda)T(r,\xi) &\leq (\eta(3\tau+4)+4)T(r,\xi)+(\eta(2\tau+3)+3)T(r,L)\\ &+ S(r,\xi)+S(r,L)\end{aligned} \tag{4.7}$$

From (4.6) and (4.7) we have

$$(n-(\lambda+\eta(5\tau+7)+7)(T(r,L)+T(r,\xi))\leq S(r,\xi)+S(r,L) \tag{4.8}$$

From (4.8) we get $n\leq\lambda+\eta(5\tau+7)+7$.

□

Lemma 4.13. *Let L be a nonconstant L-function and ξ be a transcendental meromorphic function such that $\rho_2(L)<1$, $\rho_2(\xi)<1$ and ξ, L share $(\infty,0)$. Also let $L(z)^n[\psi(z)]^{(\tau)}$ and $\xi(z)^n[\phi(z)]^{(\tau)}$ share $(P(z),1)$ and , where $P(z)$ is a polynomial of z. If $\Omega_{\Phi,\Psi}\not\equiv 0$ then $n\leq\lambda+\frac{1}{2}(\eta(5\tau+9)+7)$.*

Proof. Using lemma 4.2, lemma 4.3, lemma 4.7 and proceeding as lemma 4.12 we can prove this lemma.

□

Lemma 4.14. *Let L be a nonconstant L-function and ξ be a transcendental meromorphic function such that $\rho_2(L)<1$, $\rho_2(\xi)<1$ and ξ, L share $(\infty,0)$. Also let $L(z)^n[\psi(z)]^{(\tau)}$ and $\xi(z)^n[\phi(z)]^{(\tau)}$ share $(P(z),l)$, where $2\leq l<\infty$ and $P(z)$ is a polynomial of z. If $\Omega_{\Phi,\Psi}\not\equiv 0$ then $n\leq\lambda+\eta(2\tau+4)+4$.*

Proof. Using lemma 4.2, lemma 4.3, lemma 4.6 and proceeding as lemma 4.12 we can prove this lemma.

□

Lemma 4.15. *Let L be a nonconstant L-function and ξ be a transcendental meromorphic function such that $\rho_2(L)<1$, $\rho_2(\xi)<1$ and ξ, L share $(\infty,0)$. Also let $L(z)^n[\psi(z)]^{(\tau)}$ and $\xi(z)^n[\phi(z)]^{(\tau)}$ share $(P(z),0)$, where $P(z)$ is a polynomial of z. If $\Omega_{\Phi,\Psi}\equiv 0$ and $n>\lambda+(2\eta(\tau+1)+2)$ then either $L(z)^n[\psi(z)]^{(\tau)}\xi(z)^n[\phi(z)]^{(\tau)}\equiv P(z)^2$ or $L(z)^n[\psi(z)]^{(\tau)}\equiv\xi(z)^n[\phi(z)]^{(\tau)}$.*

Proof. Since $\Omega_{\Phi,\Psi}\equiv 0$ therefore $(\frac{\Phi''}{\Phi'}-\frac{2\Phi'}{\Phi-1})-(\frac{\Psi''}{\Psi'}-\frac{2\Psi'}{\Psi-1})\equiv 0$.
Integrating we have

$$\frac{1}{\Phi-1}\equiv\frac{A-B(\Psi-1)}{\Psi-1}, \tag{4.9}$$

where $A(\neq 0)$ and B are constants.
Now we have to consider the following two cases

Case 1 Let $B = 0$. Then from (4.9) we have

$$\frac{1}{1-\Phi} \equiv \frac{A}{1-\Psi}, . \tag{4.10}$$

If possible let $A \neq 1$, then from (4.10) we have

$$\overline{N}(r,0;\Phi) = \overline{N}(r,1-A;\Psi) \tag{4.11}$$

By lemma 4.2, lemma 4.3 lemma 4.9 using second fundamental theorem we have

$$\begin{aligned}
T(r, L^n\psi^{(\tau)}) &= T(r,\Psi) + S(r,L) \\
&\leq \overline{N}(r,0;\Psi) + \overline{N}(r,1-A;\Psi) + \overline{N}(r,\infty;\Psi) + S(r,\Psi) \\
&\leq \overline{N}(r,0;\Psi) + \overline{N}(r,0;\Phi) + S(r,\Psi) \\
&\leq \overline{N}(r,0;\xi^n\phi^{(\tau)}) + \overline{N}(r,0;L^n\psi^{(\tau)}) + S(r,L) \\
&\leq \overline{N}(r,0;\xi^n) + \overline{N}(r,0;\phi^{(\tau)}) + \overline{N}(r,0;L^n) + \overline{N}(r,0;\psi^{(\tau)}) + S(r,L) \\
&\leq T(r,L) + \tau\overline{N}(r,\infty;\psi) + N_{\tau+1}(r,0;\psi) + T(r,\xi) + \tau\overline{N}(r,\infty;\phi) \\
&+ N_{\tau+1}(r,0;\phi) + S(r,\xi) + S(r,L) \\
&\leq (\eta(\tau+1)+1)T(r,L) + (\eta(\tau+1)+1)T(r,\xi) \\
&+ S(r,\xi) + S(r,L)
\end{aligned} \tag{4.12}$$

Hence we get from (4.12)

$$\begin{aligned}
T(r, L^n\psi^{(\tau)}) &\leq (1+\eta(\tau+1))T(r,L) + (1+\eta(\tau+1))T(r,\xi) \\
&+ S(r,\xi) + S(r,L)
\end{aligned} \tag{4.13}$$

Using lemma 4.11 we get from (4.13)

$$\begin{aligned}
(n-\lambda)T(r,L) &\leq (1+\eta(\tau+1))T(r,L) + (1+\eta(\tau+1))T(r,\xi) \\
&+ S(r,\xi) + S(r,L)
\end{aligned} \tag{4.14}$$

Similarly we have

$$\begin{aligned}
(n-\lambda)T(r,\xi) &\leq (1+\eta(\tau+1))T(r,\xi) + (1+\eta(\tau+1))T(r,L) \\
&+ S(r,\xi) + S(r,L)
\end{aligned} \tag{4.15}$$

From (4.14) and (4.15) we have

$$(n-\lambda-(2\eta(\tau+1)+2))(T(r,L)+T(r,\xi))\leq S(r,\xi)+S(r,L),$$

which contradicts $n>\lambda+2\eta(\tau+1)+2$.
Hence $A=1$ and therefore we get from (4.10)

$$L(z)^n[\psi(z)]^{(\tau)}\equiv\xi(z)^n[\phi(z)]^{(\tau)}$$

Case 2 Let $B\neq 0$. If possible let $A\neq -B$.
If $B=1$, then from (4.9) we have

$$\frac{1}{\Phi}\equiv\frac{1}{A}(1+A-\Psi) \tag{4.16}$$

Using lemma 4.3 we have from (4.16)

$$\overline{N}(r,A+1;\Psi)=\overline{N}(r,\infty;\Phi)=S(r,L)$$

Proceeding as Case 1 we arrive at a contradiction.
If $B\neq 1$, then from (4.9) we have

$$\frac{1}{\Phi-(1-\frac{1}{B})}\equiv\frac{B^2}{A}(\frac{A+B}{B}-\Psi).$$

Hence we get by lemma 4.3

$$\overline{N}(r,\frac{A+B}{B};\Psi)=\overline{N}(r,\infty;\Phi)=S(r,L)$$

Proceeding as Case 1 we arrive at a contradiction.
Hence $A=-B$.
If $B=1$, then from (4.9) we have $\Phi\Psi\equiv 1$. Hence

$$L(z)^n[\psi(z)]^{(\tau)}\xi(z)^n[\phi(z)]^{(\tau)}\equiv P(z)^2. \tag{4.17}$$

If $B\neq 1$, then from (4.9) we have $\frac{1}{\Phi}=\frac{-B\Psi}{(1-B)\Psi-1}$.
Hence $\overline{N}(r,0;\Phi)=\overline{N}(r,\frac{1}{1-B};\Psi)$.
Proceeding as Case 1 we arrive at a contradiction.

□

5. PROOF OF THE MAIN RESULT

Proof of Theorem 3.1

Since ξ, L share $(\infty, 0)$ and $\xi(z)^n[\phi(z)]^{(\tau)}$, $L(z)^n[\psi(z)]^{(\tau)}$ share $(P(z), 0)$ therefore Φ and Ψ share $(1, l)$ except zeros of $P(z)$ and share $(\infty, 0)$.
By lemma 4.1 it is clear that L is a transcendental meromorphic function.
Since L and ξ are transcendental meromorphic functions therefore P is a small function of L and ξ.
Now we have to consider the following two cases

Case 1 Let $\Omega_{\Phi,\Psi} \not\equiv 0$.
By lemma 4.12, lemma 4.13 and lemma 4.14 we arrive at a contradiction.

Case 2 Let $\Omega_{\Phi,\Psi} \equiv 0$.
Hence by lemma 4.15 one of the following holds
(i) $L(z)^n[\prod_{j=1}^{\eta} L(z+\omega_j)^{\mu_j}]^{(\tau)}\xi(z)^n[\prod_{j=1}^{\eta} \xi(z+\omega_j)^{\mu_j}]^{(\tau)} \equiv P(z)^2$;
(ii) $L(z)^n[\prod_{j=1}^{\eta} L(z+\omega_j)^{\mu_j}]^{(\tau)} \equiv \xi(z)^n[\prod_{j=1}^{\eta} \xi(z+\omega_j)^{\mu_j}]^{(\tau)}$. If

$$L(z)^n[\prod_{j=1}^{\eta} L(z+\omega_j)^{\mu_j}]^{(\tau)}\xi(z)^n[\prod_{j=1}^{\eta} \xi(z+\omega_j)^{\mu_j}]^{(\tau)} \equiv P(z)^2, \tag{5.1}$$

then by lemma 4.4, lemma 4.5 and lemma 4.9 we get from (5.1)

$$\begin{aligned} n[N(r,\infty;L)+N(r,\infty;\xi)] &\leq N(r,0;\phi^{(\tau)})+N(r,0;\psi^{(\tau)}) \\ &\leq N(r,0;\phi)+\tau\overline{N}(r,\infty;\phi)+N(r,0;\psi) \\ &+ \tau\overline{N}(r,\infty;\psi)+S(r,L)+S(r,\xi) \\ &\leq \lambda[N(r,0;L)+N(r,0;\xi)] \\ &+ \tau\eta[N(r,\infty;L)+N(r,\infty:\xi)]+S(r,L)+S(r,\xi). \end{aligned} \tag{5.2}$$

Similarly we have

$$\begin{aligned} n[N(r,0;L)+N(r,0;\xi)] &\leq (\lambda+\tau\eta)[N(r,\infty;L)+N(r,\infty;\xi)] \\ &+ S(r,L)+S(r,\xi). \end{aligned} \tag{5.3}$$

By lemma 4.2 and lemma 4.3 we get from (5.2) and (5.3)

$$(n-\lambda)[N(r,0;L)+N(r,0;\xi)] \leq S(r,L)+S(r,\xi). \tag{5.4}$$

Since $n > \lambda$ therefore from (5.4) we can conclude that L and ξ have finitely many zeros. If possible let z_1 be a pole of ξ of multiplicity m.

Since L and ξ share $(\infty, 0)$ therefore without loss of generality we may assume z_1 is a pole of order k. If z_1 is a zero of $\phi^{(\tau)}$ and $\psi^{(\tau)}$ then we get from (5.1)

$$\begin{aligned} n(m+k) &\leq \sum_{j=1}^{\eta} a_j \mu_j + \sum_{j=1}^{\eta} b_j \mu_j - 2\tau \\ &\leq c_1 + c_2. \end{aligned} \tag{5.5}$$

From (5.5) we have $n \leq \max\{c_1, c_2\}$, which contradicts $n > \max\{\lambda, c_1, c_2\}$.
Similarly we get a contradiction if we assume that z_1 is a zero of $\phi^{(\tau)}$ but not a zero of $\psi^{(\tau)}$ and vice versa.
Hence ξ has no poles.
Since ξ and L share $(\infty, 0)$ therefore L also has no pole. Hence ξ and L are entire functions and so $\phi^{(\tau)}$ and $\psi^{(\tau)}$ are entire functions.
We can also deduce from (5.1) that ξ and L have no zeros.
Hence $L = e^{a(z)}$ and $\xi = e^{b(z)}$, where $a(z)$ and $b(z)$ are entire functions.

This completes the proof of the theorem.

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