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Existence of a Bilinear Delay Differential Realization of Nonlinear Neurodynamic Process in the Constructions of Entropic Rayleigh–Ritz Operator

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Abstract

On the basis of tensor production of the real Hilbert spaces, functional-geometric conditions (necessary and sufficient) are given for the existence of a differential realization model for experimental data of an "input–output" type describing the dynamic behavior of the "black box" in the class of controllable bilinear nonstationary ordinary differential equations of the second order with delay (including non-autonomous quasi-linear hyperbolic models) in the separable Hilbert space. Incidentally, the topological-metric conditions of continuity of the projectivization of the entropic Rayleigh–Ritz operator are substantiated with calculating the fundamental group of its image. The results obtained give incentives to develop a qualitative theory of non-linear structural identification of polylinear non-autonomous differential systems of higher orders with delay, as the tools of mathematical modeling for the weakly-structured neurodynamic processes.

Keywords: mathematical neuroscience, inverse problems of nonlinear neurodynamics, tensor analysis, bilinear delay differential realization, entropic Rayleigh–Ritz operator.

1. INTRODUCTION

R. Kalman, one of the founders of the qualitative theory of differential realization (QTDR), when stating that the realization task plays a pivotal role in the general theory of dynamic systems, formulated the following approach [1]: to consider the task of realization as an attempt to guess the equations of the motion of the dynamic system by the behavior of its input and output signals or as construction of the physical model, explaining experimental data of an "*input–output*" type presented in the form of a cybernetic "*black box*" model. The current period of intensive development of the qualitative theory of differential realization in an infinite-dimensional formulation is largely related to the creation of a new mathematical language – *the entropic theory of extensions of* M_2*-operators* [2]. This theory has substantially reconstructed and strengthened theoretical and system foundations of QTDR and provided harmonic connection of purely geometric ideas of M_2-extendibility with the methods of a posterior modeling of differential equations of higher orders [3, 4] in infinite-dimensional spaces, with an emphasis on the application [5, 6] and not on achieving maximum generality of presentation.

The latter circumstance could not but leave its mark on the content of this work, namely, since in many practically important problems of mathematical neurophysiology the realization model of differential representation of temporal processes of a "*reaction*, *controlling action*" type requires taking into account non-linear non-stationary relationship as from the reaction itself and its flow speed, and from the controlling action, then below, the main attention is focused on the study of the differential realization model depending on *five* nonstationary bilinear structures. The first of these depends on the reaction (response as function of time), the second bilinear operator depends on the reaction and its speed, and the third bilinear operator depends only on the speed of this reaction, and the other two take these variables into account due to the impact of the delay parameter and the signal of the software controlling action.

In addition, the qualitative theory of differential realization, considered in line with infinite-dimensional formulation of the inverse problems of mathematical physics, is more complex, more interesting, deeper in application and is very important for understanding the basic properties of differential models themselves. Its geometric structures can serve as the starting points of the modern development of the general (axiomatic) theory of systems (in the line of [5]), incidentally creating a reputation for these structures as a very useful mathematical tool in the precision a posterior modeling of complex infinitely dimensional dynamic models.

2. PROBLEM FORMULATION

Furthermore, $(X,\|\cdot\|_X)$, $(Y,\|\cdot\|_Y)$ are the real separable Hilbert spaces (i.e. standards fulfill the "parallelogram condition" [7, p. 162]; with that, below we use [7, p. 176] linear isometry (preserving the norm) $E:Y\to X$ of spaces Y and X. As usual, the $L(\mathcal{B}',\mathcal{B}'')$ is the Banach space (with operator norm) of all linear continuous operators for two Banach spaces $\mathcal{B}'$ and $\mathcal{B}''$, $\mathcal{L}(X^2,X)$ is the space of all continuous bilinear mappings from the Cartesian square $X\times X$ into the space X, below we actively use the linear isometry [7, p. 650] of spaces $\mathcal{L}(X^2,X)$ and $L(X,L(X,X))$.

Let us write down the segment of the number scale R with the Lebesgue measure μ as $T := [t_0, t_1]$, the σ-algebra of all μ-measurable subsets out of T as $\wp_\mu$, notation $S \underset{\mathrm{mod}\,\mu}{\subseteq} Q$ ($S, Q \in \wp_\mu$) means $\mu(S \setminus Q) = 0$. Moreover, let us accept that $AC^1(T, X)$ is a set of all the functions $\varphi : T \to X$, the first derivative of which is the absolutely continuous function (with respect to measure μ) at the interval T.

If there is some Banach space below $(\mathcal{B}, \|\cdot\|)$, then, as usual, let $\mathrm{L}_p(T, \mathcal{B})$, $p \in [1, \infty)$ denote the Banach space of all classes of μ-equivalence of Bochner integrable [8] mappings $f : T \to \mathcal{B}$ with norm $(\int_T \|f(\tau)\|^p \mu(d\tau))^{1/p}$, respectively, let $\mathrm{L}_\infty(T, \mathcal{B})$ denote a Banach space of these classes with norm $\operatorname{ess\,sup}_T \|f\|$. In this context we agree that

$$\mathbb{L}_2 := \mathrm{L}_2(T, L(X, X)) \times \mathrm{L}_2(T, L(X, X)) \times \mathrm{L}_2(T, L(Y, X)) \times$$

$$\times \mathrm{L}_2(T, \mathcal{L}(X^2, X)) \times \mathrm{L}_2(T, \mathcal{L}(X^2, X)) \times \mathrm{L}_2(T, \mathcal{L}(X^2, X)) \times \mathrm{L}_2(T, \mathcal{L}(X^2, X)) \times \mathrm{L}_2(T, \mathcal{L}(X^2, X)),$$

$$\mathbb{L}^* := L(X, X) \times L(X, X) \times L(Y, X) \times \mathcal{L}(X^2, X) \times \mathcal{L}(X^2, X) \times \mathcal{L}(X^2, X) \times \mathcal{L}(X^2, X) \times \mathcal{L}(X^2, X).$$

Further we assume that on the time interval T the behaviour of the studied (simulated) behavioral system [5] is recorded *a posteriori* in the form of non-restricted by power nonlinear cluster of N observable multidimensional dynamic processes (behavioral model of "*black box*"), represented by couples of temporal vector functions of a "*reaction*, *action*" type, i.e. formally:

$$N \subset \{(x, u) : x \in AC^1(T, X),\ u \in \mathrm{L}_2(T, Y)\},\ \ \mathrm{Card}\ N \le \exp \aleph_0,$$

where (using terminology [1]) (x, u) is the "*trajectory, program control*" pair, $\aleph_0$ is the aleph zero, $\exp \aleph_0$ is the continuum; above (and below) the term "*a nonlinear bundle*" means that for the trajectory curves of this bundle, it is *a priori* not assumed that there is a presence of the *superposition principle* (when the dependency of output quantities on input actions is *linear* [1]). In addition, let a statement function with the second derivative (in the realization model) from the $t \mapsto x(t)$ trajectory of the type below be specified as an inertia-mass characteristic of the simulated system:

$$\hat{A} \in \mathrm{L}_\infty(T, L(X, X)),$$

$$\mu\{t \in T :\ \hat{A}(t) = 0 \in L(X, X)\} = 0;$$

with that, violation of the condition of *equivalence to the normal system*, namely:

$$\mu\{t \in T :\ \mathrm{Ker}\ \hat{A}(t) = \{0\} \subset X\} \ne 0,$$

is permissible; in this context, see Note 1 below.

Let us consider the problem: for a fixed pair $(N, \hat{A})$ we need to define necessary and sufficient conditions, expressed in terms of a non-linear bundle of dynamic processes N and statement function $\hat{A}$, of the existence of an ordered set from eight statement functions

$$(A_1, A_0, B, \mathbb{D}_1, \mathbb{D}_2, \mathbb{D}_3, \mathbb{D}_4, \mathbb{D}_5) \in \mathrm{L}_2,$$

for which the second-order bilinear differential realization (BDR) is implementable with a delay $\hat{\tau} = \text{constant} \geq 0$, which has an analytical representation in the form of:[1]

$$\begin{gathered} \hat{A}\ddot{x} + A_1\dot{x} + A_0 x = \\ = Bu + \mathbb{D}_1(x,x) + \mathbb{D}_2(x,\dot{x}) + \mathbb{D}_3(\dot{x},\dot{x}) + \\ + \mathbb{D}_4(E(u),y) + \mathbb{D}_5(E(u),\dot{y}),\ \forall (x,u) \in N, \end{gathered} \tag{1}$$

$$t \mapsto y(t) := \begin{cases} x(t-\hat{\tau}), \text{ if } t_0 + \hat{\tau} \leq t \leq t_1; \\ 0 \in X, \text{ if } t_0 \leq t < t_0 + \hat{\tau}. \end{cases} \tag{1*}$$

If the simulated operators of the BDR-system (1) are supposed to be searched for in the class of *stationary* ones, then we will build them in the class of *continuous* ones, and at the same time write

$$(A_1, A_0, B, \mathbb{D}_1, \mathbb{D}_2, \mathbb{D}_3, \mathbb{D}_4, \mathbb{D}_5) \in \mathrm{L}^*.$$

In connection with the above mathematical formulation, we note that each area of mathematics, as a rule, contains its major problems which are so difficult that their complete solution is not even expected, but they stimulate a constant flow of work and serve as the main milestones to progress in this area. Within the framework of QTDR studies, such a problem is that of a classification of continuous behavioristic systems, considered as if they exactly coincided with the solutions of idealized differential models, including those of higher orders. In its strongest form it presupposes the classification of such systems within the accuracy of a corresponding class of differential realization models, in particular, and a class of non-stationary BDR-models (1). This is substantiated below in Theorems 1–3 (and its corollaries) that allows us, in the classification defined, to get significantly closer to the ideal combination of functional transparency and geometric clarity [9].

3. THE CHARACTERISTIC FEATURE OF THE BDR-MODEL: Equivalent Formulations

We will now describe an analytical diagram for solving the issue of solvability (or unsolvability) of the BDR-problem (1). So let $Z := X \otimes X$ be a Hilbert tensor product [10, p. 54) of Hilbert spaces X and X with the cross-norm $\|\cdot\|_Z$, defined by the scalar

[1]Equality in (1) is regarded as an identity in the $\mathrm{L}_1(T, X)$.

product [8] in X. Moreover, we will introduce new designations , which will be widely used in the future:

$$U := X \times X \times Y \times Z \times Z \times Z \times Z \times Z,$$

$$\|(\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot)\|_U := (\|\cdot\|_X^2 + \|\cdot\|_X^2 + \|\cdot\|_Y^2 + \|\cdot\|_Z^2 + \|\cdot\|_Z^2 + \|\cdot\|_Z^2 + \|\cdot\|_Z^2 + \|\cdot\|_Z^2)^{1/2};$$

$$\mathbf{L}_2 := \mathrm{L}_2(T, L(X,X)) \times \mathrm{L}_2(T, L(X,X)) \times \mathrm{L}_2(T, L(Y,X)) \times$$

$$\times \mathrm{L}_2(T, L(Z,X)) \times \mathrm{L}_2(T, L(Z,X)) \times \mathrm{L}_2(T, L(Z,X)) \times \mathrm{L}_2(T, L(Z,X)) \times \mathrm{L}_2(T, L(Z,X));$$

it is clear that functional space $\mathbf{L}_2$ (with product topology) is linearly homeomorphic to Banach space $\mathrm{L}_2(T, L(U,X))$.

Let a universal bilinear mapping $\pi : X \times X \to X \otimes X$ denote π. In the category language, the morphism π defines tensor product as a universal repelling object [10, p. 40]. The versatility of the bilinear mapping π is also consists in the fact that

$$\pi : X \times X \to X \otimes X,$$

$$(\mathrm{x}_1, \mathrm{x}_2) \mapsto \pi(\mathrm{x}_1, \mathrm{x}_2) = \mathrm{x}_1 \otimes \mathrm{x}_2,$$

$$\|\mathrm{x}_1 \otimes \mathrm{x}_2\|_Z = \|\mathrm{x}_1\|_X \, \|\mathrm{x}_2\|_X ;$$

these ratios are important for determining the structure of the non-linear functional Rayleigh–Ritz operator in terms of the specification of the norm $\|\cdot\|_U$.

Further, we believe that the Cartesian square $X^2 = X \times X$ is endowed with the norm $(\|\cdot\|_X^2 + \|\cdot\|_X^2)^{1/2}$. In this formulation $\pi \in \mathcal{L}(X^2, Z)$ and, taking into consideration Theorem 2 [7, p. 245], for any bilinear mapping $\mathcal{D} \in \mathcal{L}(X^2, X)$ there will always be a linear continuous operator $\mathrm{D} \in L(Z, X)$, such that $\mathcal{D} = \mathrm{D} \circ \pi$, with that, inclusions with be performed for any pair $(x,u) \in N$ (according to (1^*)):

$$\pi(x,x),\ \pi(x,\dot{x}),\ \pi(\dot{x},\dot{x}) \in \mathrm{L}_\infty(T,Z),$$

$$\pi(E(u),y),\ \pi(E(u),\dot{y}) \in \mathrm{L}_2(T,Z).$$

These constructions are summed up by the following statement.

Lemma 1. *For any set* $(A_1, A_0, B, \mathbb{D}_1, \mathbb{D}_2, \mathbb{D}_3, \mathbb{D}_4, \mathbb{D}_5) \in \mathbb{L}_2$ *and mapping*

$$F : \mathrm{L}_2(T,X) \times \mathrm{L}_2(T,X) \times \mathrm{L}_2(T,Y) \times$$

$$\times \mathrm{L}_2(T,X^2) \times \mathrm{L}_2(T,X^2) \times \mathrm{L}_2(T,X^2) \times \mathrm{L}_2(T,X^2) \times \mathrm{L}_2(T,X^2) \to \mathrm{L}_1(T,X),$$

$$(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) \mapsto F(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) :=$$

$$= A_1 y_1 + A_0 y_2 + B y_3 + \mathbb{D}_1 y_4 + \mathbb{D}_2 y_5 + \mathbb{D}_3 y_6 + \mathbb{D}_4 y_7 + \mathbb{D}_5 y_8,$$

there is a single 8-tuple of statement functions

$$(D_1, D_2, D_3, D_4, D_5, D_6, D_7, D_8) \in \mathbf{L}_2$$

and, respectively, the only linear mapping

$$M : \mathrm{L}_2(T,U) \to \mathrm{L}_1(T,X),$$

with an analytical representation of the form

$$(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8) \mapsto M(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8) :=$$

$$= D_1 z_1 + D_2 z_2 + D_3 z_3 + D_4 z_4 + D_5 z_5 + D_6 z_6 + D_7 z_7 + D_8 z_8,$$

such that the following functional equality is satisfied:

$$(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) \mapsto F(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) =$$

$$= M(y_1, y_2, y_3, \pi(y_4), \pi(y_5), \pi(y_6), \pi(y_7), \pi(y_8)),$$

which, in turn, induces the following operator equations for statement functions from the constructions of mappings F and M :

$$A_1 = D_1,\ A_0 = D_2,\ B = D_3,$$

$$\mathbb{D}_1 = D_4 \circ \pi,\ \mathbb{D}_2 = D_5 \circ \pi,\ \mathbb{D}_3 = D_6 \circ \pi,\ \mathbb{D}_4 = D_7 \circ \pi,\ \mathbb{D}_5 = D_8 \circ \pi.$$

Everywhere further (in the context of the BDR-problem (1)) we will accept that

$$V_N := \mathrm{Span}\{(\dot{x}, x, u, \pi(x,x), \pi(x,\dot{x}), \pi(\dot{x},\dot{x}),$$

$$\pi(E(u),y), \pi(E(u),\dot{y})) \in \mathrm{L}_2(T,U) : (x,u) \in N\}.$$

The next Lemma generalizes the behaviorist condition (7) [11].

Lemma 2. *Let*

$$S := \{t \in T : (g(t), w(t), v(t), q(t), s(t), h(t), \hat{u}(t), \tilde{u}(t)) = 0 \in U\},$$

$$Q := \{t \in T : \dot{g}(t) = 0 \in X\},$$

where $(g, w, v, q, s, h, \hat{u}, \tilde{u}) \in V_N$. Then $S \underset{\mathrm{mod}\,\mu}{\subseteq} Q$ takes place.

Next, let $(\mathrm{L}_+(T,R), \leq_{\mathrm{L}})$ be a positive cone [12] of classes of μ-equivalence of all real non-negative μ-measurable at the interval of T functions with quasi-ordering $\leq_{\mathrm{L}}$, at which $\xi' \leq_{\mathrm{L}} \xi''$ if and only if $\xi'(t) \leq \xi''(t)$ μ is almost everywhere in T. At that, for the given subset $W \subset \mathrm{L}_+(T,R)$, let $\sup_{\mathrm{L}} W$ denote the smallest upper bound of the subset W if this bound exists in the cone $\mathrm{L}_+(T,R)$ in the structure of quasi-ordering $\leq_{\mathrm{L}}$, in particular, it is easy to establish that there is a ratio

$$\sup_{\mathrm{L}}\{\xi', \xi''\} = \xi' \vee \xi'' := 2^{-1}(\xi' + \xi'' + |\xi' - \xi''|).$$

In this formulation, consider the lattice [12]:

$$\mathcal{R}(W) := \{\xi \in L_+(T,R) : \xi \le_L \sup_L W\}.$$

Then $(\mathcal{R}(W), \le_L)$ is the lattice with the smallest $\chi_\varnothing \in L_+(T,R)$ and the largest $\sup_L W \in L_+(T,R)$ elements; here and on, $\chi_\varnothing$ is the "zero function" of the cone $L_+(T,R)$. In the context of the Theorem 17 [7, p. 68] and Corollary 1 [7, p. 69] is easy to extract a more general statement; below $\inf_L$ is the largest lower $\le_L$-bound.

Lemma 3. *The lattice $\mathcal{R}(W)$ is full, that is,*

$$\inf_L V,\ \sup_L V \in \mathcal{R}(W)\ \ \forall V \subseteq \mathcal{R}(W).$$

Let $\Psi : V_N \to L_+(T,R)$ be an entropic Rayleigh–Ritz operator [2]:

$$t \mapsto \Psi(\varphi)(t) := \begin{cases} \left\|\hat{A}(t)\dot{g}(t)\right\|_X / \|\varphi(t)\|_U, \text{ if } \varphi(t) \ne 0 \in U; \\ 0 \in R, \text{ if } \varphi(t) = 0 \in U; \end{cases}$$

where $\varphi := (g, w, v, q, s, h, \hat{u}, \tilde{u}) \in V_N$. It is clear that the following equality holds:

$$\|(g(t), w(t), v(t), q(t), s(t), h(t), \hat{u}(t), \tilde{u}(t))\|_U :=$$

$$= (\|g(t)\|_X^2 + \|w(t)\|_X^2 + \|v(t)\|_Y^2 + \|q(t)\|_Z^2 + \|s(t)\|_Z^2 + \|h(t)\|_Z^2 + \|\hat{u}(t)\|_Z^2 + \|\tilde{u}(t)\|_Z^2)^{1/2},$$

at the same time, let us call the functions

$$(g, w, v) \mapsto \eta_L(g, w, v) := \| g, w, v, 0, \ldots, 0 \|_U^2,$$

$$(q, s, h, \hat{u}, \tilde{u}) \mapsto \eta_B(q, s, h, \hat{u}, \tilde{u}) := \| 0, 0, 0, q, s, h, \hat{u}, \tilde{u} \|_U^2,$$

respectively, the linear and bilinear characteristics of the Rayleigh–Ritz operator.

By virtue of Lemma 2, the following is performed at the time interval T

$$\operatorname{supp}\Psi(\varphi) = \operatorname{supp}\left\|\hat{A}\dot{g}\right\|_X \pmod{\mu};$$

here, in the definition of the supp-construction of the function support, we follow [7, p. 137] (i.e. the support is determined within the accuracy of a zero to a set of measure zero).

Entropic operator Ψ satisfies very simple (but important) ratios

$$\chi_\varnothing \le_L \Psi(\varphi) = \Psi(r\varphi),$$

where $r \in R^* := R \setminus \{0\}$, $\varphi \in V_N$; in the designations below we will distinguish the image of the point $\Psi(\varphi)$ and the image of the set $\Psi[\{\varphi\}]$.

The theory of the Rayleigh–Ritz operator needs an exact functional-geometric language, which makes us pay special attention to this language. Therefore, it will be useful to introduce additional terminology before proceeding further. Specifically, the functional operator Ψ induces the mapping $P\Psi : P_N \to \mathrm{L}_+(T, R)$, which, according to the established tradition in representation theory [10, p. 239], we will call the *projectivization* of the Rayleigh–Ritz operator:

$$P\Psi(\gamma) := \Psi[\gamma],\ \gamma \in P_N\ \ (\gamma \subset V_N),$$

where P_N is the real projective space, associated with the linear manifold V_N (with topology, induced from space $\mathrm{L}_2(T, U)$); i.e. P_N is a set of orbits of a multiplicative group R^*, acting on $V_N \setminus \{0\}$. In this geometric interpretation, the key point is the topological properties of space P_N, $\dim P_N < \aleph_0$, of course, first of all, (in the context of Theorem 2), its compactness, in particular, if $\dim V_N = 3$, takes place, then a compact 2-manifold P_N is arranged like the Mobius strip, to which a circle is glued along its border [13, p. 162]. We mention in passing that on P_N we can introduce a geometric structure of a *CW*-complex [13, p. 140], which, in its turn, simplifies regarding the issue of the geometric realization of the manifold P_N – Theorem 9.7 [13, p. 149].

Theorem 1 (existence of differential model of "black box"). *Each of the following three conditions results in the other two*:

(i) *the BDR-problem* (1) *is solvable with respect to*

$$(A_1, A_0, B, \mathbb{D}_1, \mathbb{D}_2, \mathbb{D}_3, \mathbb{D}_4, \mathbb{D}_5) \in \mathbb{L}_2;$$

(ii) $\exists \theta \in \mathrm{L}_2(T, R) : \Psi(\varphi) \leq_{\mathrm{L}} \theta\ \ \forall \varphi \in V_N$;

(iii) $\exists \sup_{\mathrm{L}} P\Psi[P_N]$: $\sup_{\mathrm{L}} P\Psi[P_N] \in \mathrm{L}_2(T, R)$,

at the same time, to perform $(A_1, A_0, B, \mathbb{D}_1, \mathbb{D}_2, \mathbb{D}_3, \mathbb{D}_4, \mathbb{D}_5) \in \mathbb{L}^*$, *it is essential that*

(iv) $\mathcal{R}(P\Psi[P_N]) \subset \mathrm{L}_\infty(T, R)$.

Note 1. Theorem 1 can be seen as an initial step in the study of the problem[2], when the bundle of controllable trajectory curves N from *the implicit* differential equation of the higher order is required to be paired with *the explicit* nonstationary bilinear differential system of the second order with the same bundle of controllable trajectory curves N.

[2] In particular, this formulation is appropriate, when a BDR-problem that is solvable for the pair $(N,\ \hat{A}_1)$ must be reduced to a solvable BDR-problem for the pair $(N,\ \hat{A}_2)$ in such a position, when $\mu\{t \in T : \mathrm{Ker}\ \hat{A}_1(t) = 0 \in X\} \neq 0$, $\hat{A}_2$ – a homothetic operator with coefficient 1; the nature of the accompanying calculations is illustrated in Examples 1, 2.

Proof. We will use the ideas of work [2]. Adhering to Definition 1 [2], we will introduce into consideration the structure of the M_2-operator $M: \mathrm{L}_2(T,U) \to \mathrm{L}_1(T,X)$ of the form

$$\exists(D_1, D_2, D_3, D_4, D_5, D_6, D_7, D_8) \in \mathbf{L}_2 : M(g, w, v, q, s, h, \hat{u}, \tilde{u}) :=$$

$$= D_1 g + D_2 w + D_3 v + D_4 q + D_5 s + D_6 h + D_7 \hat{u} + D_8 \tilde{u}$$

$$\forall (g, w, v, q, s, h, \hat{u}, \tilde{u}) \in \mathrm{L}_2(T,U).$$

The rest of the proof details with minor clarifications (taking into account Lemmas 1–3 for the lattice $\mathcal{R}(P\Psi[P_N]))$ is included in a diagram of M_2-continuability in the form of Corollary 2 [2] and Theorem 3 [2] (maximum entropy principle). At this, the necessary condition for the

$$(A_1, A_0, B, \mathbb{D}_1, \mathbb{D}_2, \mathbb{D}_3, \mathbb{D}_4, \mathbb{D}_5) \in \mathbb{L}^*$$

is set up by modifying the proof of Theorem 3 [11]. ■

Note 2. It should be noted that even in the case of $1 < \mathrm{Card}\, N < \aleph_0$ there is a proposition of $\mathrm{Card}\, P_N = \exp \aleph_0$; but it can be shown that there is (Theorem 17 [7, p. 68]) a countable set $G \subset P_N$ such that if $\sup_{\mathrm{L}} P\Psi[P_N]$ lies in space $\mathrm{L}_+(T, R)$, then the real function $\zeta := \sup_{\mathrm{L}} P\Psi[P_N]$ is performed by the following $\sup$-structure:

$$t \mapsto \zeta(t) = \sup\{P\Psi(\gamma)(t) \in R : \gamma \in G\}.$$

Note 3. The proof of Theorem 1 can be easily modified to formulate an analogue of Theorem 3 [12], expressing in terms of the angular distance in the Hilbert space the conditions for the existence of a bilinear system (1) that implements bundles N_1, N_2, each of which has its BDR-model, alongside with that, in a mathematical formulation [11], when the simulated operators of the differential system (1) are stationary, i.e.

$$(A_1, A_0, B, \mathbb{D}_1, \mathbb{D}_2, \mathbb{D}_3, \mathbb{D}_4, \mathbb{D}_5) \in \mathbb{L}^*,$$

in particular, with the optimal norm [14].

The following particular case is also important in specific discussions:

Corollary 1. *If* $\dim V_N < \aleph_0$, $\Psi[V_N] \subset \mathrm{L}_2(T,R)$ *and there is a* $p \in [1, \infty)$, *at which*

$$\Psi(\varphi_1 + \varphi_2) \leq_{\mathrm{L}} p\Psi(\varphi_1) + p\Psi(\varphi_2), \quad (\varphi_1, \varphi_2) \in V_N \times V_N,$$

then the BDR-problem (1) *is solvable.*

Note that at $p = 1$, this property (in the context of quasi-ordering $\leq_{\mathrm{L}}$) is akin to the property of "sub-linearity" [12] of functional operators.

4. CONTINUITY OF THE RAYLEIGH–RITZ OPERATOR IN THE BDR-PROBLEM SOLVABILITY ANALYSIS

In the case of compactness of the projective manifold P_N (equivalently, $\dim P_N < \aleph_0$), it is natural to try to connect this property with the problem of constructing a lattice $\mathcal{R}(P\Psi[P_N])$ in the context of the continuity conditions for the projectivization of the Rayleigh–Ritz operator (see also [15]); below, when choosing a metric structure in a cone $\mathrm{L}_+(T,R)$, in Theorem 2], we have resorted to Theorems 15, 16 [7, pp. 65, 67] (in this formulation, $\mathrm{L}_+(T,R)$ is a complete separable metric space).

Theorem 2. *Let* $\dim P_N < \aleph_0$ *and the cone* $\mathrm{L}_+(T,R)$ *be endowed with a topology induced by convergence in measure* μ, *or, equivalently, by an invariant* [3] *metric*

$$\rho_T(f_1, f_2) := \int_T | f_1(\tau) - f_2(\tau) | (1 + | f_1(\tau) - f_2(\tau) |)^{-1} \mu(d\tau), \quad f_1, f_2 \in \mathrm{L}_+(T,R).$$

Then the operator $P\Psi : P_N \to \mathrm{L}_+(T,R)$ *will be continuous if the bundle* N *is such that*

$$\forall \varphi \in V_N \setminus \{0\} : \mathrm{supp} \|\varphi\|_U = T \ (\mathrm{mod}\ \mu), \tag{2}$$

in particular, if

$$\forall \gamma \in P_N : \mathrm{supp} P\Psi(\gamma) = T \ (\mathrm{mod}\ \mu). \tag{3}$$

Note that Theorem 2 is the development of the Theorem 3 [16], which confirms its methodological importance in the a posterior simulation of complex dynamic systems [2–6]. One of the applications of this result is the following statement.

Corollary 2. *If, when performing* (2) *or* (3) *the* $P\Psi$ *is one-to-one, then* $P\Psi$ *is homeomorphism, and the fundamental group of the metric space* $(P\Psi[P_N], \rho_T)$ *is isomorphic to the additive group of integers* $\mathbb{Z}$ *at* $\dim \mathrm{Span} N = 2$ *and the residue group* $\mathbb{Z}_2$ *at* $\dim \mathrm{Span}\ N \geq 3$, *moreover, the space* $(P\Psi[P_N], \rho_T)$ *is orientable if the dimension of the linear envelope* $\mathrm{Span} N$ *is even and non-orientable if this dimension is odd.*

Taking into account that the continuous real function on the compact space reaches its highest and lowest values, we come to the conclusion that in the formulation of Corollary 2 and Theorem 5 [7, p. 28], for the case when $1 \leq \dim P_N < \aleph_0$ and with $\sup_{\mathrm{L}} P\Psi[P_N]$, there will be such points $\gamma', \gamma'' \in P_N$ that

$$\rho_T(P\Psi(\gamma'), \chi_\varnothing) = \sup\{\rho_T(P\Psi(\gamma), \chi_\varnothing) : \gamma \in P_N\} \leq \rho_T(\sup_{\mathrm{L}} P\Psi[P_N], \chi_\varnothing) < \mu(T),$$

$$\rho_T(P\Psi(\gamma''), \sup_{\mathrm{L}} P\Psi[P_N]) = \inf\{\rho_T(P\Psi(\gamma), \sup_{\mathrm{L}} P\Psi[P_N]) : \gamma \in P_N\} \geq 0.$$

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[3]The invariance assumes $\rho_T(f+q, g+q) = \rho_T(f,g)$ for any of $f, g, q \in \mathrm{L}_+(T,R)$. A version of a non-invariant incomplete metric that ensures the continuity of the Rayleigh–Ritz operator, is considered in [15].

It should be noted that the inclusion of $P\Psi(\gamma') \in \mathrm{L}_2(T,R)$ does not guarantee the embedding of $\mathcal{R}(P\Psi[P_N]) \subset \mathrm{L}_2(T,R)$ (see Example 1 from [16]). At the same time, it should be noted that $\dim P_N = 0$ results in the proposition

$$\sup_{\mathrm{L}} P\Psi[P_N] = P\Psi[P_N] = \left\|\hat{A}\ddot{x}\right\|_X / (\|\dot{x}\|_X^2 + \|x\|_X^2 + \|u\|_Y^2 + \|x\|_X^4 +$$

$$+\|x\|_X^2 \|\dot{x}\|_X^2 + \|\dot{x}\|_X^4 + \|E(u)\|_X^2 \|y\|_X^2 + \|E(u)\|_X^2 \|\dot{y}\|_X^2)^{1/2}. \qquad (4)$$

In the context of Theorems 1, 2 we can clarify the terms of the existence of the lattice $\mathcal{R}(P\Psi[P_N])$. As a starting point, we introduce an auxiliary structure: for a natural n, let W_n denote some finite n^{-1}-dense subset in metric space $(P\Psi[P_N], \rho_T)$; the subset W_n exists by virtue of Theorem 2. Below, $\mathrm{Lim}_{\rho_T}\{\xi_n\}$ means the limit of the sequence $\{\xi_n\} \subset \mathrm{L}_+(T,R)$ in the topology, induced by a metric ρ_T.

Theorem 3. *Let* $\{W_i\}_{i=1,\dots,n}$, $W_i = \{\zeta_1,\dots,\zeta_{k_i}\} \subset P\Psi[P_N]$ *and*

$$f_n := \xi_1 \vee \ldots \vee \xi_n,\ \xi_i = \zeta_1 \vee \ldots \vee \zeta_{k_i},\ 1 \le i \le n.$$

Then the cone $\mathrm{L}_+(T,R)$ *contains the lattice* $\mathcal{R}(P\Psi[P_N])$ *if and only if*

$$\rho_T(f_n, f_m) \to 0\ (n, m \to \infty),$$

besides, the BDR-solvability takes the following form: the pair $(N, \hat{A})$ *has a differential realization* (1) *if and only if* $\mathrm{Lim}_{\rho_T}\{f_n\} \in \mathrm{L}_2(T,R)$, *which is equivalent to*

$$\mathcal{R}(P\Psi[P_N]) \subset \mathrm{L}_2(T,R).$$

In conclusion, we give examples which disconfirm a possible view that everywhere above we laid emphasis solely on the ideological aspect of each concept, thereby unwittingly neglecting its consideration from a computational point of view; everywhere below we believe that the simulation is carried out with a zero delay $\hat{\tau} = 0$ (i.e. $y(\cdot) = x(\cdot)$; according to (1^*)).

Example 1. Let $T = [0,10]$, $Y := X$, $\hat{A}$ be an operator of homothetic [12] with a unit coefficient, $A_1 = 0 \in L(X,X)$, $\mathbb{D}_1 = \mathbb{D}_3 = \mathbb{D}_4 = \mathbb{D}_5 = 0 \in \mathcal{L}(X^2, X)$, $e \in X$, $\|e\|_X = 1$,

$$t \mapsto x(t) = (t \sin t)e,$$

$$t \mapsto u(t) = 0 \in \mathrm{L}_2(T,X).$$

Then the function $f := \sup_{\mathrm{L}} P\Psi(P_N) = \|\ddot{x}\|_X (\|x\|_X^2 + \|x\|_X^2 \|\dot{x}\|_X^2)^{-1/2}$ (see Fig. 1) doesn't belong to space $\mathrm{L}_2(T,R)$ and, consequently, according to Theorem 1 and formula (4), realization (1) for an uncontrollable process $N = \{(x,u)\}$ *does not exist.*

Example 2. Let us change the formulation of the Example 1 by

$$t \mapsto u(t) = (t\sin^2 t + 2^{-1}t^2 \sin 2t + \cos t)e.$$

Then (see Fig. 2) $f := \sup_{\mathrm{L}} P\Psi(P_N) = \|\ddot{x}\|_X (\|x\|_X^2 + \|x\|_X^2 \|\dot{x}\|_X^2 + \|u\|_Y^2)^{-1/2} \in \mathrm{L}_2(T,R)$ and it means that realization (1) for a controllable process $N = \{(x,u)\}$ *exists*; it's easy to establish that

$$\ddot{x} + x = 2u - 2\mathbb{D}_2(x,\dot{x}),$$

where $\mathbb{D}_2 = <\cdot,\cdot>_X e$, $<\cdot,\cdot>_X$ – is the scalar product in X.

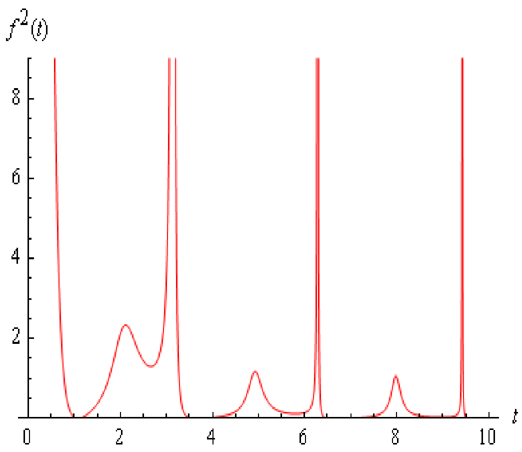

Fig. 1. $f^2(t) = (2\cos t - t\sin t)^2 \times \times ((t\sin t)^2 + (t\sin t)^2(\sin t + t\cos t)^2)^{-1}$

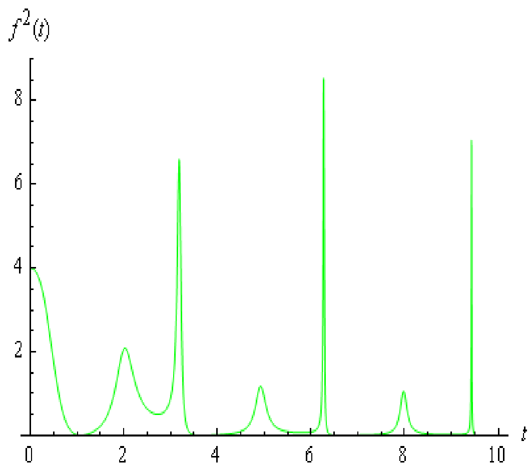

Fig. 2. $f^2(t) = (2\cos t - t\sin t)^2 \times ((t\sin t)^2 + +(t\sin t)^2(\sin t + t\cos t)^2 + (t\sin^2 t + 2^{-1}t^2\sin 2t + \cos t)^2)^{-1}$

Note that for the more complex versions of setting the pair ($t \mapsto x(t)$, $t \mapsto u(t)$) symbolic calculations of the function $f^2(\cdot)$ (similar to Fig. 1, 2) can be carried out by means of computer algebra of mathematical physics [17]. In this context, the scheme for analyzing the solvability of the BDR-problem in Examples 1, 2 can be modified to qualitatively analyze the reduction of exact multidimensional diffusion solutions with power nonlinearities to the Cauchy problem for a countable system of ordinary differential equations with polylinear structure.

The following example characterizes the fact of the presence of a property of *the nonstationary* of the BDR-model as an endogenous factor in the differential realization of the bundle N (see paragraph (iv) of Theorem 1).

Example 3. Let $T=[0,1]$ and a coefficient of homothetic of the operator $\hat{A}$ be equal to 5 while

$$N=\{(x,u)\},$$

$$t\mapsto x(t)=t^{1,6}e,\ \ t\mapsto u(t)=\chi_T(t)e.$$

Then (as it is easy to establish) the function $f:=\sup_{\mathrm{L}} P\Psi(P_N)$ according to formula (4) satisfies the inequalities

$$0{,}6t^{-0,4}\le \sup_{\mathrm{L}} P\Psi(P_N)\le 4{,}8t^{-0,4},$$

where the second inequality guarantees (by virtue of paragraph (iii) of Theorem 1) the existence of some BDR-model implementing the bundle process N, with that, the first inequality, according to paragraph (iv) of Theorem 1, shows that this model cannot be a *stationary one*.

Let us give an example that illustrates the situation when the *linearity* property of the differential realization model is insufficient to the precisely construct the equations of the N bundle dynamics.

Example 4. Let $T=[1,2]$, $\hat{A}$ be a homothetic operator with a coefficient of 1, while

$$N=\{(x_1,u_1),(x_2,u_2),(x_3,u_3)\},$$

$$t\mapsto x_1(t)=(t^2+2)e,\ \ t\mapsto u_1(t)=2^{-1}\chi_T(t)e,$$

$$t\mapsto x_2(t)=te,\ \ t\mapsto u_2(t)=\chi_T(t)e,$$

$$t\mapsto x_3(t)=(2-4\sqrt{2})te,\ \ t\mapsto u_3(t)=\chi_T(t)e.$$

First of all, we will show that the dynamic bundle N *cannot* have linear differential realization. To do this, it is sufficient to establish that the linear characteristic η_{L}, which responds to the Rayleigh–Ritz operator, induces the function $t\mapsto \eta_{\mathrm{L}}(g,w,v)(t)$ with zero of order 2. The characteristic condition of this formulation will be the following system of equations (relative to the parameters α,β):

$$\|x_1(t)+\alpha x_2(t)+\beta x_3(t)\|_X^2=((t^2+2)+\alpha t+\beta(2-4\sqrt{2})t)^2=0,$$

$$\|\dot{x}_1(t)+\alpha\dot{x}_2(t)+\beta\dot{x}_3(t)\|_X^2=(2t+\alpha\chi_T(t)+\beta(2-4\sqrt{2})\chi_T(t))^2=0,$$

$$\|u_1(t)+\alpha u_2(t)+\beta u_3(t)\|_X^2=(2^{-1}t+\alpha t+\beta t)^2=0,$$

from where it's easy to calculate that $\alpha=-1,\ \beta=0{,}5$, with that, a zero point is $t=\sqrt{2}$.

Now let us show that a BDR-model (1) for the bundle N *exists*. To do this, it is sufficient to establish that the sum of linear and bilinear characteristics (parametric representation by α,β) of the Rayleigh–Ritz operator for the bundle N is limited from below by some function written as:

$$t \mapsto r\chi_T(t),\ r \in (0,\infty).$$

In doing this, it's easy to see that at any $t \in [1, 2],\ \alpha, \beta \in R$ there will be

$$\eta_{\mathrm{L}}(t,\alpha,\beta) + \eta_{\mathrm{B}}(t,\alpha,\beta) \geq$$

$$\geq \| \dot{x}_1(t) + \alpha\dot{x}_2(t) + \beta\dot{x}_3(t) \|_X^2 + \| u_1(t)\dot{x}_1(t) + \alpha u_2(t)\dot{x}_2(t) + \beta u_3(t)\dot{x}_3(t) \|_X^2 =$$

$$= (2t + \alpha + \beta(2 - 4\sqrt{2}))^2 + (t^2 + \alpha t + \beta(2 - 4\sqrt{2})t)^2 =$$

$$= [(2t + \gamma(\alpha,\beta))^2 + t^2(t + \gamma(\alpha,\beta))^2]\Big|_{\gamma(\alpha,\beta)=\alpha+\beta(2\text{-}4\sqrt{2})} \geq$$

$$\geq [(2t + \gamma(\alpha,\beta))^2 + (t + \gamma(\alpha,\beta))^2]\Big|_{\gamma(\alpha,\beta)=\alpha+\beta(2\text{-}4\sqrt{2})} =$$

$$= [(t + \lambda(t,\gamma))^2 + \lambda^2(t,\gamma)]\Big|_{\lambda(t,\gamma)=t+\gamma(\alpha,\beta)} \geq$$

$$\geq [(t + \theta(t))^2 + \theta^2(t)]\Big|_{\theta(t)=-t/2} = t^2/2 \geq$$

$$\geq 0{,}5\chi_T(t).$$

You can check that the BDR-model (1) for the bundle N has analytical representation:

$$\ddot{x} + x = \mathbb{D}_5(u,\dot{x}),$$

where $\mathbb{D}_5(\cdot,\cdot) = - < \cdot,\cdot >_X e$.

Important model and theoretical conclusions follow directly from Example 4, as its analytical generalization. Any qualitative analysis of the BDR-modeling problem should essentially begin (1) with the verification of these conclusions; at the same time they make it possible to reduce (on the basis of purely engineering ideas (relying on experience and common sense) the gap between theory and practice. The first conclusion is that the final dynamic bundle N can have at the interval T interpolation representation in the class of *polynomial* spline-functions [18]. The second conclusion expresses

Proposition 1. *Let the dynamic bundle N fulfill the following condition*:

$$\exists (g,w,v,q,s,h,\hat{u},\tilde{u}) \in V_N : \| \hat{A}\dot{g} \|_X \left(\sqrt{\eta_{\mathrm{L}}(g,w,v)} + \chi_{S_{\dot{g}}}\right)^{-1} \notin \mathrm{L}_2(T,R), \qquad (5)$$

where $\chi_{S_{\dot{g}}}$ is the characteristic function of the set $S_{\dot{g}} := T \setminus \operatorname{supp}\dot{g}$.

Then, the differential realization of this bundle can't have analytical representation (1) *in which* ***all the bilinear operators*** *are zero.*

Condition (5) can be weakened (constructively strengthened) by reducing it to the search, by means of computer algebra, of zeros of the function $\eta_{\mathrm{L}} + \eta_{\mathrm{B}}$, parameterized by the coefficients of interpolation representation of the bundle N in the class of polynomial spline-functions.

5. CONCLUSION

In order to go further, one can quite confidently point out the theoretical and systemic direction which will constitute the algebraic basis without the excessive building-up of technical means of algebraic geometry of the next stage of development of the qualitative theory of differential realization of higher orders [3, 4, 19], namely, the transition from the *bilinear* structure of nonlinear links *to polylinear* links. Methodologically, this transition consists in using a geometric language of tensor structures of the Fock spaces [20] and projective representations [10, p. 238] in the context of the study of metric properties of the Rayleigh–Ritz operators [15], by means of computer algebra of mathematical physics [17].

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