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Hyperbolic Two Temperature Fractional-Order Thermoelastic Model Subjected to Thermal Loading with Two Relaxation Times

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Abstract

The present work investigates the thermoelastic behaviour of an elastic material occupying the half space subjected to shock wave in the context of the fractional order generalized thermoelasticity associated with two relaxation times. The Laplace transform together with the Laplace transform of Caputo fractional integral has been applied to solve the closed form of the obtained solutions in the Laplace transform domain. The inversion of the dimensionless physical quantities are obtained numericaly using a complex inversion formula of Laplace transform based on a Fourier expansion. The variation of the heat conduction, the distribution of the stress and the strain with the fractional order parameter, second relaxation time and time are studied and the results presented graphically. Comparison between the effects of different parameters has been illustrated graphically.

Keywords: Hyperbolic two-temperature, fractional-order strain, Shock Wave, fractional-order equation of motion, Two relaxation time, Generalized Thermoelasticity.

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NOMENCLATURE

The following notaions will be used through out the present work:

φ^h	conductive temperature in the hyperbolic two-temperature model
φ^p	conductive temperature in the classical two-temperature model
σ^h	principal stress component in the case of hyperbolic two-temperature model
σ^p	principle stress component in the Classical two-temperature model
e^h	cubic dilatation in the hyperbolic two-temperature model
e^p	cubic dilatation in the classical two-temperature model
C_E	specific heat at constant strain.
c_o	longitudinal wave speed.
T	absolute temperature.
T_o	reference temperature
t	time
u_i	components of displacement vector
$\alpha \geq 0$	two-temperature parameter.
α_T	coefficientρnt of linear thermal expansion
ε	dimensionless mechanical coupling constant
λ, μ	Lame's constants
ρ	mass density
τ_o:	relaxation time parameter.
β	fractional-order parameter
Γ	gamma function
K	thermal conductivity
$\theta = T - T_0$	thermodynamic temperature increment such that $\theta/T_0 \ll 1$

1. INTRODUCTION

In Thermoelasticity the heat conduction in deformable bodies arises from the conductive and thermodynamical temperatures [1], [2]. It is seen that in case of time dependent situation when there is no supply of heat the two-temperatures are the same where as in case of time dependent situation the two-temperature are different. Some more details of such studies can be found in [3],[4]. Youssef has defined the variance theory and the uniqueness of the initial boundary value problem in the generalized thermoelasticity with two-temperatures in separate situations [5]-[7]. To remove the paradox of heat conduction present in the two-temperature thermoelasticity theory which admit infinite speed of signals, Youssef enhanced the thermoelasticity theory based on conductive and thermodynamic temperature by assuming a hyperbolic form of the two-temperature relation [8]. The concept of derivative and integral have been generalized to a non-integer order and studied by many researchers [9]-[15]. Various physical process and models have been implemented through the application of fractional-order derivatives. Applications of the fractional-order theory and many other contributions have been published by many researchers [16]-[22]. The fractional-order thermoelasticity becomes more realistic when it relies on the fractional-order operator because the presence of the fractional-order derivatives permits the differential equations of the system to take into consideration the effects of the intermediate as well as the previous states to express the present and the next states of the medium. One of the most famous definitions of fractional-order was introduced by Riemann-Liouville and given by [14]:

$$_{RL}D_t^{\beta} f(t) = \frac{d^n}{dt^n}[\frac{1}{\Gamma(n-\beta)} \int_0^t (t-\tau)^{n-\beta-1} f(\tau)d\tau], \qquad n-1 < \beta < n \qquad (1)$$

The second definition was presented by [14]and given by:

$$_{C}D_t^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \int_0^t (t-\tau)^{n-\beta-1} \frac{d^n f(\tau)}{d\tau^n} d\tau \qquad n-1 < \beta < n \qquad (2)$$

These two definitions are the same if $f(0) = 0$. For more details about the comparison between the two definitions of the fractional-order time derivative introduced by Riemann-Liouville and that of Caputo and various definitions and works of fractional-order derivatives were reported in [22].

Based on the new theory of the hyperbolic two - temperature generalized thermoelasticity by Youssef [8] the present work can be considered as a generalization to the application studied in it and more realistic as the present model contains fractional-order derivatives in both equations of motion as well as the heat equation. In the present work we will use the following equation:

$$L_C D_t^{\beta} f(t) = s^{(\beta-n)} L f^n(t), \qquad n-1 < \beta < n, \qquad (3)$$

as in [21] to investigate the behaviours of a thermoelastic isotropic and homogeneous half-space subjected to a thermal loading a represented by a heavy side step function at the end $x = 0$. In Eq. (3), s is the complex parameter connected to Laplace transform.

2. ONE DIMENSIONAL THERMOELASTIC MODEL

For the present model we presume the following one-dimensional fractional-order system of equations which is capable to describe the overall behaviour of a semi-infinite one-dimensional homogeneous isotropic material occupying the half- space $x \geq 0$ and subjected to thermal loading at the end $x = 0$. The three- dimensional forms of this system can be found in Youssef [8]. We assume that the material is subjected to thermal loading and stress-free at the end $x = 0$. All the field functions are initially set at zero. We also presume that no body force is applied to the medium. When no inner heat sources and any charges are present, the generalized thermoelastic one dimensional system of differential fractional-order equation assumes the following equations:
The conductive heat equation:

$$K\,(\frac{\partial^2\varphi(x,t)}{\partial x^2}) = (\frac{\partial}{\partial t} + \frac{\tau_o^\beta}{\beta!}D_t^{\beta+1})(\rho C_E\,\theta(x,t)) + T_o\gamma(1+\tau^\beta D_t^\beta)e(x,t) \tag{4}$$

$$\rho\frac{\partial^2 e(x,t)}{\partial t^2} = (\lambda+2\mu)\,(1+\tau^\beta D_t^\beta)\frac{\partial^2 e(x,t)}{\partial x^2} - \gamma(1+\frac{\tau_o^\beta}{\beta!}D_t^{\beta+1})\frac{\partial^2\theta(x,t)}{\partial x^2} \tag{5}$$

and the stress-strain constitutive equations take the forms:

$$\sigma(x,t) = (1+\tau^\beta D_t^\beta)(\lambda+2\mu)e(x,t) - \gamma(1+\frac{\tau_o^\beta}{\beta!}D_t^{\beta+1})\theta(x,t) \tag{6}$$

and

$$e(x,t) = \frac{\partial u(x,t)}{\partial x}. \tag{7}$$

Instead of the classical two-temperature relation between the heat conduction φ and the thermodynamical temperature θ given by:

$$\theta = \varphi - \alpha\frac{\partial^2\varphi}{\partial x^2} \tag{8}$$

we used the following hyperbolic relation as given in[9]:

$$\frac{\partial^2\theta}{\partial t^2} = \frac{\partial^2\varphi}{\partial t^2} - \alpha\frac{\partial^2\varphi}{\partial x^2} \tag{9}$$

3. DIMENSIONLESS SYSTEM OF EQUATIONS IN LAPLACE DOMAIN

For converting the previous system of Eqs.(4)-(9) into dimensionless system we used the set of dimensionless variables as in [8] and dropping the primes for convenience, we get the following non-dimensional system of equations:

$$\frac{\partial^2\varphi(x,t)}{\partial x^2} = (\frac{\partial}{\partial t} + \frac{\tau_o^\beta}{\beta!}D_t^{\beta+1})\theta(x,t) + \xi\varepsilon_1(1+\tau^\beta D_t^\beta)e(x,t) \tag{10}$$

The dimensionless fractional-order differential equation of strain takes the form:

$$\frac{\partial^2 e(x,t)}{\partial t^2} = (1+\tau^\beta D_t^\beta)\frac{\partial^2 e(x,t)}{\partial x^2} - \omega(1+\frac{\tau_o^\beta}{\beta!}D_t^{\beta+1})\frac{\partial^2\theta(x,t)}{\partial x^2} \tag{11}$$

the constitutive equations take the following forms:

$$\sigma(x,t) = (1+\tau^\beta D_t^\beta)e(x,t) - \omega(1+\frac{\tau_o^\beta}{\beta!}D_t^{\beta+1})\theta(x,t) \tag{12}$$

and

$$e(x,t) = \frac{\partial u(x,t)}{\partial x} \tag{13}$$

The hyperbolic two-temperature non-dimensional equation becomes:

$$\frac{\partial^2\varphi}{\partial t^2} = \frac{\partial^2\theta}{\partial t^2} - \alpha\frac{\partial^2\varphi}{\partial x^2} \tag{14}$$

Applying the Laplace transform defined by:

$$L\{f(t)\} = \int_0^\infty e^{-st}f(t)dt \tag{15}$$

together with Caputo definition (3) to the system of Eqs. (10)-(14) we get the following none dimensional system of equations in Laplace domain:

$$\frac{\partial^2\overline{\varphi(x,s)}}{\partial x^2} = L_1\overline{e(x,s)} + L_2\overline{\theta(x,s)} \tag{16}$$

$$\frac{\partial^2\overline{e(x,s)}}{\partial x^2} = E_1\overline{e(x,s)} + E_2\frac{\partial^2\overline{\theta(x,s)}}{\partial x^2} \tag{17}$$

$$\overline{\sigma(x,s)} = (1+s^\beta)\overline{e(x,s)} - \omega(s^\beta\tau_1^\beta + \beta!)\overline{\theta(x,s)} \tag{18}$$

and the relation between the two types of temperature:

$$\overline{\theta(x,ps)} = \overline{\varphi(x,s)} - s^{-2}\alpha\frac{\partial^2\overline{\varphi(x,s)}}{\partial x^2}) \tag{19}$$

Combining Eqs. (17) and (18) gives:

$$\frac{\partial^2\overline{\sigma(x,s)}}{\partial x^2} = s^2\overline{e(x,s)} \tag{20}$$

where $L_1 = s^{\beta-1} + s\varepsilon_1\xi$, $L_2 = s + \frac{\tau_o^\beta s^{\beta-1}}{\beta!}$, $E_1 = \frac{s^2\beta!}{L_3}$, $E_2 = \omega\frac{s^\beta\tau_1^\beta+\beta!}{L_3}$ and $L_3 = (1 + s^\beta\tau^\beta)\beta!$
The Eqs. (16)-(20) represent the non-dimensionless governing equations of the present one - dimensional fractional-order thermoelastic model in the light of generalized fractional-order thermoelasticity with hyperbolic two-temperature equation.

4. THE SOLUTIONS IN THE LAPLACE DOMAIN

Eliminating $\overline{e(x,s)}$ and $\overline{\theta(x,s)}$ between the Eqs. (16) and (17), we get the following fourth order non homogeneous differential equation;

$$b\overline{\varphi(x,s)} - a\frac{\partial^2\overline{\varphi(x,s)}}{\partial x^2} + \frac{\partial^4\overline{\varphi(x,s)}}{\partial x^4} = 0, \tag{21}$$

where

$$a = \frac{s^2\left(\beta!\left(\tau^\beta s^{\beta+2} + \omega s^\beta + s^3 + s^2(\alpha + \xi\omega\epsilon 1 + 1)\right) + s^\beta\left(\tau o^\beta\left(\alpha + \tau^\beta s^\beta + 1\right) + \omega\tau 1^\beta\left(s^\beta + \xi s^2\epsilon 1\right)\right)\right)}{\beta!\left(\tau^\beta(\alpha + s)s^{\beta+2} + \alpha\omega s^\beta + s^3 + \alpha\xi s^2\omega\epsilon 1 + \alpha s^2\right) + \alpha s^\beta\left(\tau o^\beta\left(\tau^\beta s^\beta + 1\right) + \omega\tau 1^\beta\left(s^\beta + \xi s^2\epsilon 1\right)\right)}$$

$$b = \frac{s^4\left(s^2\beta! + \tau o^\beta s^\beta\right)}{\beta!\left(\tau^\beta(\alpha + s)s^{\beta+2} + \alpha\omega s^\beta + s^3 + \alpha\xi s^2\omega\epsilon 1 + \alpha s^2\right) + \alpha s^\beta\left(\tau o^\beta\left(\tau^\beta s^\beta + 1\right) + \omega\tau 1^\beta\left(s^\beta + \xi s^2\epsilon 1\right)\right)}$$

The most general solution of (21) according to the current formulation of the problem takes the form;

$$\overline{\varphi(x,s)} = \sum_{i=1}^{2} C_i e^{-k_i x}, \tag{22}$$

where C_i are coefficients depending on s whose values can be evaluated by using the given boundary conditions, $\pm k_i$ are the roots of the characteristic equations corresponding to the Eq. (21), which is;

$$N - Mk^2 + k^4 = 0.$$

After some manipulations to the system of Eqs. (16)- (19) we get the following general solutions of the physical quantities of the present model in the domain of Laplace;
The thermodynamical temperature assumes the form:

$$\overline{\theta(x,s)} = \frac{1}{s^2}\sum_{i=1}^{2} C_i\, e^{-k_i x}(s^2 - \alpha\, k_i^2) \tag{23}$$

The strain and the stress in the domain of Laplace takes the form:

$$\overline{e(x,s)} = \sum_{i=1}^{2} \frac{C_i e^{-k_i x}(-s^2 A_2 + k_i^2 A_3)}{s^2 B} \tag{24}$$

$$\overline{\sigma(x,s)} = \sum_{i=1}^{2} C_i e^{-k_i x}(L_4 + k_i^2 L_5) \tag{25}$$

where

$$L_4 = -A_1\omega - \frac{(1+s^\beta)A_2}{B}, \qquad L_5 = \frac{\alpha\ \omega\ B + (1+s^\beta)A_3}{s^2 B}$$

$$A_1 = \frac{s^\beta \tau_1^\beta + \beta!}{\beta!}, \qquad A_2 = \frac{s^\beta \tau_o^\beta + s^2\beta!}{\beta!}, \qquad A_3 = \frac{(\alpha\ s^\beta \tau_o^\beta + s^2(s+\alpha)\beta!)}{\beta!},$$

$$B = (s^\beta + s^2\varepsilon_1\xi)\beta!.$$

The Eqs.(22) -(25) represent the complete solution of the system (16)-(20) in the Laplace transform domain.

5. DETERMINATION OF THE PARAMETERS

To determine the previous parameters, the following initial conditions have been provided as well as the medium is set at rest initially and has reference temperature T_o so that the initial conditions are given by;

$$\begin{aligned} \varphi(x,0) &= 0, \quad e(x,0) = 0, \qquad \sigma(x,0) = 0, \\ \partial\varphi(x,0)/\partial t &= 0, \quad \partial e(x,0)/\partial t = 0, \quad \partial\sigma(x,0)/\partial t = 0, \end{aligned} \tag{26}$$

we assume a thermal shock loading so that the medium undergoes the following boundary conditions at the close end $x = 0$;

$$\varphi(0,t) = \varphi_0\, H(t), \quad \sigma(0,t) = 0, \tag{27}$$

while at $x = \infty$ the boundary conditions take the form:

$$\varphi(\infty,t) = 0, \quad , \quad \sigma(\infty,t) = 0, \qquad 0 < t < \infty, \tag{28}$$

where $H(t)$ is the Heaviside unit step function and $\varphi_o$is the thermal shock intensity. Applying the Laplace transform to the Eqs. (27) and (28) we obtain the following dimensionless form of the boundary conditions:

$$\begin{aligned} \overline{\varphi(0,s)} &= \frac{\varphi_o}{s}, \qquad \overline{\sigma}(0,s)=0, \\ \overline{\varphi(\infty,s)} &= 0, \qquad \overline{\sigma}(\infty,s)=0, \end{aligned} \tag{29}$$

Similarly the dimensionless initial conditions in the domain of Laplace can be obtained. By applying these conditions to (22)-(25), the constants C_i can be obtained as given below;

$$C_1 = \frac{\varphi_o[s^2(s^2 - \ \alpha\, k_2^2)\, L_6\, \tau_1^\beta\, \omega \ + \ (s^2\, L_7 - k_2^2\, L_8\, \beta!)]}{(k_1^2 - k_2^2)(\alpha\, s^\beta\, L_6\, \tau_1^\beta\, \omega \ + \ L_8\, \beta!)} \tag{30}$$

$$C_2 = \frac{\varphi_o[s^2(s^2 - \ \alpha\, k_1^2)\, L_6\, \tau_1^\beta\, \omega \ + \ (s^2\, L_7 - k_1^2\, L_8\, \beta!)]}{(k_1^2 - k_2^2)(\alpha\, s^\beta\, L_6\, \tau_1^\beta\, \omega \ + \ L_8\, \beta!)} \tag{31}$$

where

$$\begin{aligned} L_6 &= (1+s^\beta)\tau_o^\beta + (s^\beta + s^2\varepsilon_1\xi)\tau_1^\beta\omega \\ L_7 &= s^{2+\beta} + \omega\, s^\beta + s^2(1+\varepsilon_1\xi\omega) \\ L_8 &= (s^3 + s^{3+\beta} + \alpha\, s^{2+\beta} + \ \alpha\, \omega s^\beta + \alpha\, s^2(1+\varepsilon_1\xi\omega)\beta!) \end{aligned}$$

After substituting with the constants given by the Eq. (30)-(31) into the Eqs. (22)-(25), we obtain the complete solution in the Laplace domain of the non-dimensional field functions; temperature, stress and strain respectively.

6. NUMERICAL FORM OF THE INVERSION OF THE LAPLACE TRANSFORM

The physical quantities $\varphi(x,t)$, $\theta(x,t)$, $\sigma(x,t)$ and $e(x,t)$ can be obtained by inverting the system of Eqs.(22)-(25) back to the time domain. Therefore, we use a numerical equation based on the expansion of Fourier. In this technique any function $\overline{f}\,(s)$ is inverted back to the original function $f(t)$ in the time domain as given below;

$$f(t) = \frac{\exp(ct)}{t_1}[\frac{1}{2}\overline{f}\,(c) + \Re(\sum_{1}^{N}\overline{f}\,(c+\frac{ik\pi}{t_1}\exp(\frac{ik\pi}{t_1})], \qquad 0<t_1<2t, \tag{32}$$

where $\Re$ is the real part, i is imaginary number unit and N is a sufficiently large integer representing the number of terms in the truncated Fourier series chosen such that;

$$\exp(ct)\Re\left[\overline{f}\,(c+\frac{iN\pi}{t_1})\exp(\frac{iN\pi t}{t_1})\right] \le \epsilon_1, \tag{33}$$

where ϵ_1 is a small positive number that represents the degree of accuracy required. The parameter c is a positive free parameter that must be greater than the real part of all the singularities of $\overline{f}(s)$. The optimal choice of c was obtained according to the criteria described in Honig and Hirdes [23]. Details about the analysis of the formula (32) can be found in [24].

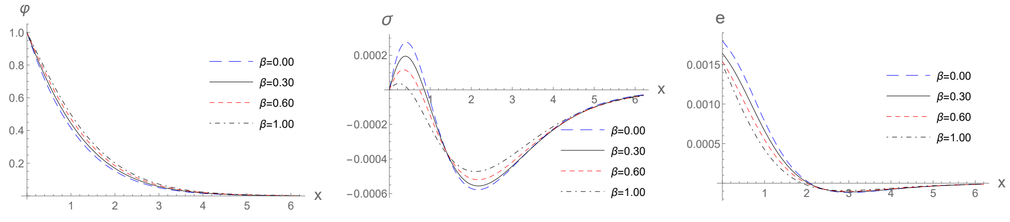
7. DISCUSSION OF THE RESULTS

For numerical computations, we used the physical constants of the Copper material used in [21]. We investigate the distributions of the field functions φ, σ and e with the variation of the values of the parameters β, τ_1 and t. The results are collected in groups of figures; each group presents the effect of one of the mentioned parameters on the physical quantities.

Figure (1) illustrates the effects of the fractional-order parameter β on the field functions. Fig.1 (a) illustrates the effects of the variation of the fractional order parameter β on the distribution of the field functions. It is noticed that there is a direct variation between β and the amplitude of the heat conduction φ. An asymptotic stability in the heat conduction curve can also be noticed. Figure 1(b) represents the stress with different values of β. It is noticed that the amplitude changed its sign from positive to negative at $x \simeq 1.35$. The absolute vale of the magnitude of the stress distribution is in inversely proportional to the increasing value of the fractional order parameter. Figure 1(c) represents the variation of the strain distribution with the variance of β. It shows that the strain resembles the behaviour of the stress up to the point $x \simeq 2.0$ but for $x \simeq 2$ the effect of the fractional order on the distribution curve of the strain disappear. The strain attains its equilibrium point faster than the other than the physical quantities.

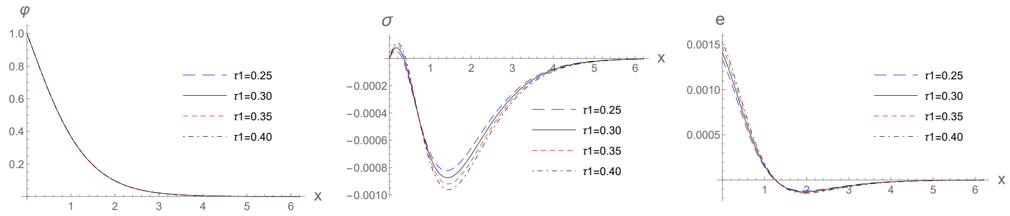
Figure (2) represents the effect of the second relaxation time τ_1 on the functions φ, σ and e. It is seen that there is no changes on the distribution curve of the heat conduction φ with different values of τ_1. Figure 2(b) illustrates the behaviour of the stress with the variance of the second relaxation time τ_1. A very slight change in the amplitude of the stress near the point where we apply the thermal shock is seen up to the point $x \simeq 1.0$. Changes of the amplitude of the stress have been noticed at $x > 1.0$. The stress distribution has an asymptotic stability. Unlike the behaviour of the stress, the strain distribution curve changes its absolute value of the amplitude near the point of the heat source and the effect of the second relaxation time starts to disappear at $x = 1.0$. Strain distribution curve has an asymptotic stability like the stress.

Figure (3) shows the variation of the field functions with the variation of time t. The variation of the heat conduction φ with the variation of time t resemble its behaviour with the variation of the fractional order parameter β. In fig.3(b) the stress distribution curves show an inverse variation with change in time. It is also noticed that the peak positions of the stress curve move away from the point of the heat source. The stress distribution curve has an asymptotic stability during the variation of time.



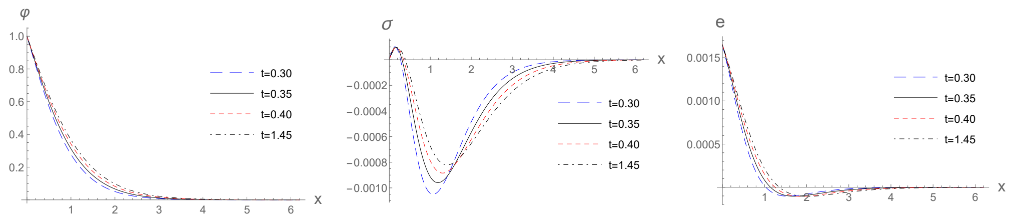
(a) **(b)** **(c)**

FIGURE 1: Effect of fractional-order parameter β on φ, σ and e at $t = 0.05$, $\tau_o = 0.2$, $\tau 1 = 0.25$
(a)Distribution of hyperbolic conductive temperature φ; (b) Distribution of Stress σ; (c) Distribution of Strain



(a) **(b)** **(c)**

FIGURE 2: Effect of the Second Relaxation Time τ_1 on φ, σ and e at $t = 0.5$, $\tau_o = 0.01$, $\beta = 0.6$
(a) Distribution of hyperbolic conductive temperature φ; (b) Distribution of Stress σ; (c) Distribution of Strain



(a) **(b)** **(c)**

FIGURE 3: Variation of Physical field functions with Time t on φ, σ and e at $t = 0.5$, $\tau_o = 0.01$, $\beta = 0.6$
(a) Distribution of hyperbolic conductive temperature φ; (b) Distribution of Stress σ; (c) Distribution of Strain

8. CONCLUSION

We noticed that the distributions of the field function φ, σ and e has an asymptotic stability with the variation of all parameters β, τ_1 and time t. We also noticed that the second relaxation time τ_1 has no effect on the heat conduction φ. The amplitude of the distribution curve of the stress changes its sign at the same point with the variation of all parameters. The figures of the heat conduction and the stress are coincident with the boundary conditions.

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