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# Existence of a Weak Solution to the Maxwell-Stokes Type Equation by the Penalty Method

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**Abstract**

In this paper, we show the existence of a weak solution to the Maxwell-Stokes type equation by the penalty method introduced by Temam. Our approximate equation is nonlinear and contains so called $p$-curl system. Furthermore, we obtain the continuous dependence of the weak solution on the data.

**Keywords:** Maxwell-Stokes type equation, weak solution, penalty method, minimization problem

**2010 Mathematics Subject Classifications:** 35A05, 35H30, 35A15, 35D05

## 1. INTRODUCTION

In this paper, we show the existence of a weak solution to the Maxwell-Stokes type equation by the penalty method introduced by Temam [9] (cf. Dautray and Lions [6, vol. 7] or Girault and Raviart [7]).

More precisely, we consider the following Stokes problem in a bounded, Lipschitz-continuous domain $\Omega \subset \mathbb{R}^d$ with boundary $\Gamma$.

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \Gamma. \end{cases} \tag{1.1}$$

The penalty method replaces the Stokes problem (1.1) by

$$\begin{cases} -\Delta \boldsymbol{u}_\varepsilon - \frac{1}{\varepsilon}\nabla \operatorname{div} \boldsymbol{u}_\varepsilon = \boldsymbol{f} & \text{in } \Omega, \\ \boldsymbol{u}_\varepsilon = \boldsymbol{0} & \text{on } \Gamma, \end{cases} \tag{1.2}$$

where $\varepsilon$ is a positive parameter that will tend to zero. The pressure $\pi$ is approximated by

$\pi_\varepsilon = -\frac{1}{\varepsilon}\mathrm{div}\,\boldsymbol{u}_\varepsilon$, and $\boldsymbol{u}_\varepsilon$ approximates $\boldsymbol{u}$. Since the problem (1.2) is the Dirichlet problem for the elliptic equation, the problem (1.2) has a unique solution $\boldsymbol{u}_\varepsilon$. Temam established the convergence of $(\boldsymbol{u}_\varepsilon, \pi_\varepsilon)$ to $(\boldsymbol{u}, \pi)$, and more precisely, proved the following theorem.

**Theorem 1.1** (Temam)**.** *Let $\Omega$ be a bounded, Lipschitz-continuous domain in $\mathbb{R}^d$, and $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$. Then, as $\varepsilon \to 0$, we have*

$$\boldsymbol{u}_\varepsilon \to \boldsymbol{u} \textit{ in } \boldsymbol{H}_0^1(\Omega) \textit{ and } \pi_\varepsilon \to \pi \textit{ in } L^2(\Omega),$$

*where $(\boldsymbol{u}, \pi)$ is the solution of the homogeneous Stokes problem* (1.1).

Amrouche and Girault [1] extended this convergence to $\boldsymbol{W}^{m,p}(\Omega)$.

In the present paper, we shall show that the penalty method can be applicable to the Maxwell-Stokes problem in the case $d = 3$. In order to do so, we replace $-\Delta \boldsymbol{u}$ in the first equatin of (1.1) with a nonlinear term as in the following system.

$$\begin{cases} \mathrm{curl}\,[S_t(x, |\mathrm{curl}\,\boldsymbol{u}|^2)\mathrm{curl}\,\boldsymbol{u}] + \nabla\pi = \boldsymbol{f} & \text{in } \Omega, \\ \mathrm{div}\,\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \Gamma, \end{cases} \tag{1.3}$$

where the function $S(x,t)$ is a Carathéodory function in $\Omega \times [0, \infty)$ satisfying some structure conditions. The first equation of (1.3) contains so called $p$-curl system

$$\mathrm{curl}\,[|\mathrm{curl}\,\boldsymbol{u}|^{p-2}\mathrm{curl}\,\boldsymbol{u}] + \nabla\pi = \boldsymbol{f}$$

in a special case. The equation (1.3) is nonlinear, and when $p > 2$, it is degenerate and when $1 < p < 2$, it is singular. In a special case of $S(x,t)$ and $p = 2$, (1.3) reduces to the equation (1.1), so our result extends Theorem 1.1. Our approximate system is as follows.

$$\begin{cases} \mathrm{curl}\,[S_t(x, |\mathrm{curl}\,\boldsymbol{u}_\varepsilon|^2)\mathrm{curl}\,\boldsymbol{u}_\varepsilon] - \frac{1}{\varepsilon}\nabla[S_t(x, (\mathrm{div}\,\boldsymbol{u}_\varepsilon)^2)\mathrm{div}\,\boldsymbol{u}_\varepsilon] = \boldsymbol{f} & \text{in } \Omega, \\ \boldsymbol{u}_\varepsilon = \boldsymbol{0} & \text{on } \Gamma. \end{cases} \tag{1.4}$$

Since this equation is nonlinear, it is necessary to investigate this problem from a different point of view. Then we show that

$$\boldsymbol{u}_\varepsilon \to \boldsymbol{u} \text{ in } \boldsymbol{W}_0^{1,p}(\Omega) \text{ and } \pi_\varepsilon := -\frac{1}{\varepsilon}S_t(x, (\mathrm{div}\,\boldsymbol{u}_\varepsilon)^2)\mathrm{div}\,\boldsymbol{u}_\varepsilon \to \pi \text{ in } L^p(\Omega),$$

where $(\boldsymbol{u}, \pi)$ is a weak solution of (1.3).

The paper is organized as follows. In section 2, we give some preliminaries and the main theorem that states the existence of the weak solution to the problem (1.3). Section 3 is devoted the proof of the main theorem. In section 4, we show the continuous dependence of the solution on the data.

## 2. PRELIMINARIES AND THE MAIN THEOREM

This section consists of two subsections. In subsection 2.1, we give some preliminaries that are necessary later. In subsection 2.2, we give the notion of a weak solution for the Maxwell-Stokes problem and state the main theorem.

### 2.1 Preliminaries

Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ with a Lipshitz-continuous boundary $\Gamma$, $1 < p < \infty$ and let $p'$ be the conjugate exponent i.e., $(1/p)+(1/p') = 1$. From now on we use $L^p(\Omega)$ and $W_0^{1,p}(\Omega)$ for the standard $L^p$ and Sobolev spaces of functions. For any Banach space $B$, we denote $B \times B \times B$ by boldface character $\boldsymbol{B}$. Hereafter, we use this character to denote vector and vector-valued functions, and we denote the standard Euclidean inner product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbb{R}^3$ by $\boldsymbol{a} \cdot \boldsymbol{b}$.

We assume that a Carathéodory function $S(x,t)$ in $\Omega \times [0,\infty)$ satisfies the following structure conditions. For a.e. $x \in \Omega$, $S(x,t) \in C^2((0,\infty)) \cap C^0([0,\infty))$, and there exist $1 < p < \infty$ and positive constants $0 < \lambda \leq \Lambda < \infty$ such that for a.e. $x \in \Omega$,

$$S(x,0) = 0 \text{ and } \lambda t^{(p-2)/2} \leq S_t(x,t) \leq \Lambda t^{(p-2)/2} \text{ for } t > 0, \tag{2.1a}$$

$$\lambda t^{(p-2)/2} \leq S_t(x,t) + 2tS_{tt}(x,t) \leq \Lambda t^{(p-2)/2} \text{ for } t > 0, \tag{2.1b}$$

$$\text{If } 1 < p < 2, S_{tt}(x,t) < 0, \text{ and if } p \geq 2, S_{tt}(x,t) \geq 0 \text{ for } t > 0. \tag{2.1c}$$

We note that from (2.1a), we have

$$\frac{2}{p}\lambda t^{p/2} \leq S(x,t) \leq \frac{2}{p}\Lambda t^{p/2} \text{ for } t \geq 0. \tag{2.2}$$

**Example 2.1.** *If $S(x,t) = \nu(x)t^{p/2}$, where $\nu$ is a measurable function in $\Omega$ and satisfies $0 < \nu_* \leq \nu(x) \leq \nu^* < \infty$ for a.e. in $\Omega$ for some constants $\nu_*$ and $\nu^*$, then it follows from elementary calculations that* (2.1a)-(2.1c) *hold.* .

We give a monotonicity lemma.

**Lemma 2.2.** (i) *There exists a constant $c_1 > 0$ such that for all $t, s \in \mathbb{R}$,*

$$(S_t(x,t^2)t - S_t(x,s^2)s)(t - s) \geq \begin{cases} c_1|t-s|^p & \text{if } p > 2, \\ c_1(|t|+|s|)^{p-2}|t-s|^2 & \text{if } 1 < p \leq 2. \end{cases}$$

(ii) *There exists a constant* $c_2 > 0$ *such that for all* $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$,

$$(S_t(x,|\boldsymbol{a}|^2)\boldsymbol{a} - S_t(x,|\boldsymbol{b}|^2)\boldsymbol{b}) \cdot (\boldsymbol{a}-\boldsymbol{b}) \geq \begin{cases} c_2|\boldsymbol{a}-\boldsymbol{b}|^p & \text{if } p > 2, \\ c_2(|\boldsymbol{a}|+|\boldsymbol{b}|)^{p-2}|\boldsymbol{a}-\boldsymbol{b}|^2 & \text{if } 1 < p \leq 2. \end{cases}$$

*Proof.* We prove (i). If we put $I(x,t) = (S_t(x,t^2)t - S_t(x,s^2)s)(t-s)$, then we can write

$$\begin{aligned} I(x,t) &= \int_0^1 \frac{d}{d\theta}\big[S_t(x,(\theta t+(1-\theta)s)^2)(\theta t+(1-\theta)s)\big]d\theta(t-s) \\ &= \int_0^1 \big\{S_t(x,(\theta t+(1-\theta)s)^2) \\ &\quad +2S_{tt}(x,(\theta t+(1-\theta)s)^2)(\theta t+(1-\theta)s)^2\big\}d\theta(t-s)^2. \end{aligned}$$

From (2.1b), we have

$$I(x,t) \geq \lambda \int_0^1 |\theta t+(1-\theta)s|^{p-2}d\theta|t-s|^2. \tag{2.3}$$

When $1 < p \leq 2$, it is trivial that

$$I(x,t) \geq \lambda(|t|+|s|)^{p-2}|t-s|^2.$$

We consider the case $p > 2$. If $|t| \geq |t-s|$, then

$$|\theta t+(1-\theta)s| = |t+(1-\theta)(s-t)| \geq |t|-(1-\theta)|s-t| \geq \theta|t-s|.$$

Thus it follows from (2.3) that

$$I(x,t) \geq \lambda \int_0^1 \theta^{p-2}d\theta|t-s|^p = \frac{\lambda}{p-1}|t-s|^p.$$

If $|t| < |t-s|$, we write

$$\int_0^1 |\theta t+(1-\theta)s|^{p-2}d\theta = \int_0^1 \frac{(|\theta t+(1-\theta)s|^2)^{p/2}}{|\theta t+(1-\theta)s|^2}d\theta.$$

Since

$$\begin{aligned} |\theta t+(1-\theta)s|^2 &= |t+(1-\theta)(s-t)|^2 \\ &\leq (|t|+(1-\theta)|s-t|)^2 \\ &\leq (|s-t|+(1-\theta)|s-t|)^2 \\ &= (2-\theta)^2|s-t|^2 \\ &\leq 4|s-t|^2 \end{aligned}$$

for $0 \le \theta \le 1$, using the Jensen inequality (cf. Jost [8, p. 122]), we have

$$\begin{aligned} I(x,t) &\ge \frac{\lambda}{4}\int_0^1 (|t+(1-\theta)(s-t)|^2)^{p/2} d\theta \\ &\ge \frac{\lambda}{4}\left(\int_0^1 (t+(1-\theta)(s-t))^2\right)^{p/2} \\ &\ge \frac{\lambda}{4}\frac{1}{3^{p/2}}(t^2+s^2+ts)^{p/2} \\ &\ge \frac{\lambda}{4}\frac{1}{3^{p/2}}\frac{1}{4^{p/2}}|t-s|^p. \end{aligned}$$

For the proof of (ii), see Aramaki [5, Lemma 3.6]. □

**Lemma 2.3.** *There exists a constants $C_1 > 0$ depending only on $\Lambda$ and $p$ such that for any $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$,*

$$|S_t(x,s,|\boldsymbol{a}|^2)\boldsymbol{a} - S_t(x,s,|\boldsymbol{b}|^2)\boldsymbol{b}| \le \begin{cases} C_1|\boldsymbol{a}-\boldsymbol{b}|^{p-1} & \text{if } 1 < p < 2, \\ C_1(|\boldsymbol{a}|+|\boldsymbol{b}|)^{p-2}|\boldsymbol{a}-\boldsymbol{b}| & \text{if } p \ge 2. \end{cases}$$

For the proof, see Aramaki [3].

## 2.2 The main theorem

We consider the following Maxwell-Stokes system.

$$\begin{cases} \mathrm{curl}\,[S_t(x,|\mathrm{curl}\,\boldsymbol{u}|^2)\mathrm{curl}\,\boldsymbol{u}] + \nabla\pi = \boldsymbol{f} & \text{in } \Omega, \\ \mathrm{div}\,\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \Gamma. \end{cases} \tag{2.4}$$

**Definition 2.4.** *Let $\boldsymbol{f} \in \boldsymbol{W}^{-1,p'}(\Omega)$ that is the dual space of $\boldsymbol{W}_0^{1,p}(\Omega)$. We say $(\boldsymbol{u},\pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^{p'}(\Omega)/\mathbb{R}$ is a weak solution of* (2.4)*, if $\boldsymbol{u}$ is divergence free and $(\boldsymbol{u},\pi)$ satisfies*

$$\int_\Omega S_t(x,|\mathrm{curl}\,\boldsymbol{u}|^2)\mathrm{curl}\,\boldsymbol{u}\cdot\mathrm{curl}\,\boldsymbol{v}dx - \int_\Omega \pi\mathrm{div}\,\boldsymbol{v}dx = \langle \boldsymbol{f},\boldsymbol{v}\rangle \tag{2.5}$$

*for all $\boldsymbol{v} \in \boldsymbol{W}_0^{1,p}(\Omega)$, where $\langle \boldsymbol{f},\boldsymbol{v}\rangle$ denotes the duality between $\boldsymbol{f} \in \boldsymbol{W}^{-1,p'}(\Omega)$ and $\boldsymbol{v} \in \boldsymbol{W}_0^{1,p}(\Omega)$.*

We are in a position to state the main theorem.

**Theorem 2.5.** *Let $\Omega \subset \mathbb{R}^3$ be a bounded, Lipcshitz-continuous domain in $\mathbb{R}^3$, and assume that a Carathéodory function $S(x,t)$ satisfies the structure conditions* (2.1a)-(2.1c). *Then for any $\boldsymbol{f} \in \boldsymbol{W}^{-1,p'}(\Omega)$, the Maxwell-Stokes system* (2.4) *has a unique weak solution $(\boldsymbol{u},\pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^{p'}(\Omega)/\mathbb{R}$, and there exists a constant $C > 0$ depending only on $p, \lambda, \Lambda$ and $\Omega$ such that*

$$\|\boldsymbol{u}\|^p_{\boldsymbol{W}_0^{1,p}(\Omega)} + \|\pi\|^{p'}_{L^{p'}(\Omega)/\mathbb{R}} \le C\|\boldsymbol{f}\|^{p'}_{\boldsymbol{W}^{-1,p'}(\Omega)}. \tag{2.6}$$

## 3. PROOF OF THEOREM 2.5

In this section, we prove Theorem 2.5 by the penalty method. In order to do so, let $0 < \varepsilon \le 1$. We consider the following system, and give the notion of its weak solution.

$$\begin{cases} \operatorname{curl}[S_t(x,|\operatorname{curl}\boldsymbol{u}|^2)\operatorname{curl}\boldsymbol{u}] - \frac{1}{\varepsilon}\nabla[S_t(x,(\operatorname{div}\boldsymbol{u})^2)\operatorname{div}\boldsymbol{u}] = \boldsymbol{f} & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \Gamma. \end{cases} \tag{3.1}$$

**Definition 3.1.** *We say that $\boldsymbol{u}_\varepsilon \in \boldsymbol{W}_0^{1,p}(\Omega)$ is a weak solution of* (3.1)*, if $\boldsymbol{u}_\varepsilon$ satisfies*

$$\int_\Omega \big\{S_t(x,|\operatorname{curl}\boldsymbol{u}_\varepsilon|^2)\operatorname{curl}\boldsymbol{u}_\varepsilon\cdot\operatorname{curl}\boldsymbol{v} + \frac{1}{\varepsilon}S_t(x,(\operatorname{div}\boldsymbol{u}_\varepsilon)^2)(\operatorname{div}\boldsymbol{u}_\varepsilon)(\operatorname{div}\boldsymbol{v})\big\}dx$$
$$= \langle\boldsymbol{f},\boldsymbol{v}\rangle \text{ for all } \boldsymbol{v} \in \boldsymbol{W}_0^{1,p}(\Omega). \tag{3.2}$$

For any fixed $0 < \varepsilon \le 1$, define a functional

$$E_\varepsilon[\boldsymbol{v}] = \frac{1}{2}\int_\Omega \big\{S(x,|\operatorname{curl}\boldsymbol{v}|^2) + \frac{1}{\varepsilon}S(x,(\operatorname{div}\boldsymbol{v})^2)\big\}dx - \langle\boldsymbol{f},\boldsymbol{v}\rangle.$$

We consider the following minimization problem: to find $\boldsymbol{u}_\varepsilon \in \boldsymbol{W}_0^{1,p}(\Omega)$ such that

$$E_\varepsilon[\boldsymbol{u}_\varepsilon] = \alpha := \inf_{\boldsymbol{v}\in\boldsymbol{W}_0^{1,p}(\Omega)} E_\varepsilon[\boldsymbol{v}]. \tag{3.3}$$

We call such a $\boldsymbol{u}_\varepsilon$ a minimizer of $\alpha$. Then we have the following proposition.

**Proposition 3.2.** *Let $0 < \varepsilon \le 1$ and $\boldsymbol{f} \in \boldsymbol{W}^{-1,p'}(\Omega)$. Then the minimization problem* (3.3) *has a minimizer $\boldsymbol{u}_\varepsilon \in \boldsymbol{W}_0^{1,p}(\Omega)$.*

*Proof.* According to Amrouche and Seloula [2], we first note that there exists a constant $C > 0$ depending only on $p$ and $\Omega$ such that

$$\|\nabla\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \le C(\|\operatorname{curl}\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\operatorname{div}\boldsymbol{v}\|_{L^p(\Omega)})$$

for all $\boldsymbol{v} \in \boldsymbol{W}_0^{1,p}(\Omega)$, and from the Poincaré inequality, there exists a constant $c > 0$ depending only on $p$ and $\Omega$ such that

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \leq c\|\nabla \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \text{ for all } \boldsymbol{v} \in \boldsymbol{W}_0^{1,p}(\Omega).$$

Thus there exists a constant $C_2 > 0$ depending only on $p$ and $\Omega$ such that

$$\|\boldsymbol{v}\|^p_{\boldsymbol{W}_0^{1,p}(\Omega)} \leq C_2(\|\mathrm{curl}\, \boldsymbol{v}\|^p_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\, \boldsymbol{v}\|^p_{L^p(\Omega)}). \tag{3.4}$$

Since $0 < \varepsilon \leq 1$, using (2.2), the Hölder inequality and the Young inequality, for any $\delta > 0$ there exists a constant $C(\delta) > 0$ such that

$$\begin{aligned} E_\varepsilon[\boldsymbol{v}] &\geq \frac{1}{2}\int_\Omega \big\{S(x, |\mathrm{curl}\, \boldsymbol{v}|^2) + S(x, (\mathrm{div}\, \boldsymbol{v})^2)\big\}dx - \langle \boldsymbol{f}, \boldsymbol{v}\rangle \\ &\geq \frac{\lambda}{p}\int_\Omega (|\mathrm{curl}\, \boldsymbol{v}|^p + |\mathrm{div}\, \boldsymbol{v}|^p)dx - \|\boldsymbol{f}\|_{\boldsymbol{W}^{-1,p'}(\Omega)}\|\boldsymbol{v}\|_{\boldsymbol{W}_0^{1,p}(\Omega)} \\ &\geq \frac{\lambda}{pC_2}\|\boldsymbol{v}\|^p_{\boldsymbol{W}_0^{1,p}(\Omega)} - C(\delta)\|\boldsymbol{f}\|^{p'}_{\boldsymbol{W}^{-1,p'}(\Omega)} - \delta\|\boldsymbol{v}\|^p_{\boldsymbol{W}_0^{1,p}(\Omega)}. \end{aligned}$$

If we choose $\delta > 0$ so that $\delta = \lambda/2pC_2$, then we have

$$E_\varepsilon[\boldsymbol{v}] \geq \frac{\lambda}{2pC_2}\|\boldsymbol{v}\|^p_{\boldsymbol{W}_0^{1,p}(\Omega)} - C(\lambda/2pC_2)\|\boldsymbol{f}\|^{p'}_{\boldsymbol{W}^{-1,p'}(\Omega)} > -\infty. \tag{3.5}$$

Let $\{\boldsymbol{v}_j\} \subset \boldsymbol{W}_0^{1,p}(\Omega)$ be a minimizing sequence of $\alpha$, i.e.,

$$E_\varepsilon[\boldsymbol{v}_j] = \alpha + o(1) \text{ as } j \to \infty.$$

From (3.5), $\{\boldsymbol{v}_j\}$ is bounded in $\boldsymbol{W}_0^{1,p}(\Omega)$. Since $\boldsymbol{W}_0^{1,p}(\Omega)$ is a reflexive Banach space, passing to a subsequence, we may assume that

$$\boldsymbol{v}_j \to \boldsymbol{u}_\varepsilon \text{ weakly in } \boldsymbol{W}_0^{1,p}(\Omega) \text{ as } j \to \infty.$$

According to Aramaki [4], we have

$$\int_\Omega S(x, |\mathrm{curl}\, \boldsymbol{u}_\varepsilon|^2)dx = \liminf_{j\to\infty} \int_\Omega S(x, |\mathrm{curl}\, \boldsymbol{v}_j|^2)dx,$$

$$\int_\Omega S(x, (\mathrm{div}\, \boldsymbol{u}_\varepsilon)^2)dx = \liminf_{j\to\infty} \int_\Omega S(x, (\mathrm{div}\, \boldsymbol{v}_j)^2)dx$$

and clearly $\langle \boldsymbol{f}, \boldsymbol{v}_j\rangle \to \langle \boldsymbol{f}, \boldsymbol{u}_\varepsilon\rangle$ as $j \to \infty$. Therefore, we have

$$E_\varepsilon[\boldsymbol{u}_\varepsilon] \leq \liminf_{j\to\infty} E_\varepsilon[\boldsymbol{v}_j] = \alpha.$$

So $\boldsymbol{u}_\varepsilon$ is a minimizer of $\alpha$. □

For any $\boldsymbol{v} \in \boldsymbol{W}_0^{1,p}(\Omega)$ and $t \in \mathbb{R}$, we have $E_\varepsilon[\boldsymbol{u}_\varepsilon] \leq E_\varepsilon[\boldsymbol{u}_\varepsilon + t\boldsymbol{v}]$. Thus, the Euler-Lagrange equation implies

$$\frac{d}{dt}E_\varepsilon[\boldsymbol{u}_\varepsilon + t\boldsymbol{v}]\bigg|_{t=0} = \int_\Omega \big\{S_t(x, |\mathrm{curl}\,\boldsymbol{u}_\varepsilon|^2)\mathrm{curl}\,\boldsymbol{u}_\varepsilon \cdot \mathrm{curl}\,\boldsymbol{v} \\ + \frac{1}{\varepsilon}S_t(x, (\mathrm{div}\,\boldsymbol{u}_\varepsilon)^2)(\mathrm{div}\,\boldsymbol{u}_\varepsilon)(\mathrm{div}\,\boldsymbol{v})\big\}dx - \langle \boldsymbol{f}, \boldsymbol{v}\rangle = 0.$$

Thus we obtain the following proposition.

**Proposition 3.3.** *The minimizer* $\boldsymbol{u}_\varepsilon$ *in Proposition* 3.2 *is a unique weak solution of* (3.1) *in the sense of Definition* 3.3.

*Proof.* It suffices to prove the uniqueness. Let $\boldsymbol{u}_\varepsilon^1, \boldsymbol{u}_\varepsilon^2 \in \boldsymbol{W}_0^{1,p}(\Omega)$ be two weak solutions of (3.1). Taking $\boldsymbol{v} = \boldsymbol{u}_\varepsilon^1 - \boldsymbol{u}_\varepsilon^2$ as a test function of (3.2), we have

$$\int_\Omega \big\{\big(S_t(x, |\mathrm{curl}\,\boldsymbol{u}_\varepsilon^1|^2)\mathrm{curl}\,\boldsymbol{u}_\varepsilon^1 - S_t(x, |\mathrm{curl}\,\boldsymbol{u}_\varepsilon^2)|^2)\mathrm{curl}\,\boldsymbol{u}_\varepsilon^2)\big) \cdot \mathrm{curl}\,(\boldsymbol{u}_\varepsilon^1 - \boldsymbol{u}_\varepsilon^2) \\ + \frac{1}{\varepsilon}\big(S_t(x, (\mathrm{div}\,\boldsymbol{u}_\varepsilon^1)^2)\mathrm{div}\,\boldsymbol{u}_\varepsilon^1 - S_t(x, (\mathrm{div}\,\boldsymbol{u}_\varepsilon^2))\mathrm{div}\,\boldsymbol{u}_\varepsilon^2\big)\mathrm{div}\,(\boldsymbol{u}_\varepsilon^1 - \boldsymbol{u}_\varepsilon^2)\big\}dx = 0.$$

It follows from Lemma 2.2 that

$$\int_\Omega (|\mathrm{curl}\,(\boldsymbol{u}_\varepsilon^1 - \boldsymbol{u}_\varepsilon^2)|^p + \frac{1}{\varepsilon}|\mathrm{div}\,(\boldsymbol{u}_\varepsilon^1 - \boldsymbol{u}_\varepsilon^2)|^p dx \leq 0 \text{ if } p \geq 2,$$

and

$$\int_\Omega \big\{(|\mathrm{curl}\,\boldsymbol{u}_\varepsilon^1| + |\mathrm{curl}\,\boldsymbol{u}_\varepsilon^2|)^{p-2}|\mathrm{curl}\,(\boldsymbol{u}_\varepsilon^1 - \boldsymbol{u}_\varepsilon^2)|^2 \\ + \frac{1}{\varepsilon}(|\mathrm{div}\,\boldsymbol{u}_\varepsilon^1| + |\mathrm{div}\,\boldsymbol{u}_\varepsilon^2|)^{p-2}|\mathrm{div}\,(\boldsymbol{u}_\varepsilon^1 - \boldsymbol{u}_\varepsilon^2)|^2\big\}dx \leq 0 \text{ if } 1 < p < 2,$$

Therefore, we have

$$\mathrm{curl}\,(\boldsymbol{u}_\varepsilon^1 - \boldsymbol{u}_\varepsilon^2) = \boldsymbol{0} \text{ and } \mathrm{div}\,(\boldsymbol{u}_\varepsilon^1 - \boldsymbol{u}_\varepsilon^2) = 0 \text{ in } \Omega.$$

By (3.4), we have $\boldsymbol{u}_\varepsilon^1 = \boldsymbol{u}_\varepsilon^2$. □

**Remark 3.4.** *From this proposition, we can see that the minimizer of the minimization problem* (3.3) *is also unique.*

*Proof of Theorem* 2.5

Let $\boldsymbol{u}_\varepsilon \in \boldsymbol{W}_0^{1,p}(\Omega)$ be a unique minimizer of (3.3). Then $\boldsymbol{u}_\varepsilon$ is a weak solution of (3.1). If we take $\boldsymbol{v} = \boldsymbol{u}_\varepsilon$ as a test function of (3.2), then

$$\int_\Omega \big\{ \big(S_t(x, |\mathrm{curl}\, \boldsymbol{u}_\varepsilon|^2)|\mathrm{curl}\, \boldsymbol{u}_\varepsilon|^2 + \frac{1}{\varepsilon} S_t(x, (\mathrm{div}\, \boldsymbol{u}_\varepsilon)^2)(\mathrm{div}\, \boldsymbol{u}_\varepsilon)^2 \big\} dx \\ = \langle \boldsymbol{f}, \boldsymbol{u}_\varepsilon \rangle. \quad (3.6)$$

Since $\frac{1}{\varepsilon} \geq 1$, we have

$$\lambda \int_\Omega (|\mathrm{curl}\, \boldsymbol{u}_\varepsilon|^p + |\mathrm{div}\, \boldsymbol{u}_\varepsilon|^p) dx \leq \|\boldsymbol{f}\|_{\boldsymbol{W}^{-1,p'}(\Omega)} \|\boldsymbol{u}_\varepsilon\|_{\boldsymbol{W}_0^{1,p}(\Omega)} \\ \leq C(\delta) \|\boldsymbol{f}\|^{p'}_{\boldsymbol{W}^{-1,p'}(\Omega)} + \delta \|\boldsymbol{u}_\varepsilon\|^p_{\boldsymbol{W}_0^{1,p}(\Omega)}$$

for any $\delta > 0$. If we choose $\delta > 0$ so that $\delta < \lambda/2C_2$, it follows from (3.4) that there exists a constant $C > 0$ depending only on $p$, $\lambda$ and $\Omega$ such that

$$\|\boldsymbol{u}_\varepsilon\|^p_{\boldsymbol{W}_0^{1,p}(\Omega)} \leq C \|\boldsymbol{f}\|^{p'}_{\boldsymbol{W}^{-1,p'}(\Omega)}. \quad (3.7)$$

Moreover, from (3.7) and (3.6), we have

$$\begin{aligned} \lambda \int_\Omega |\mathrm{div}\, \boldsymbol{u}_\varepsilon|^p dx &\leq \int_\Omega S_t(x, (\mathrm{div}\, \boldsymbol{u}_\varepsilon)^2)(\mathrm{div}\, \boldsymbol{u}_\varepsilon)^2 dx \\ &\leq \varepsilon \|\boldsymbol{f}\|_{\boldsymbol{W}^{-1,p'}(\Omega)} \|\boldsymbol{u}_\varepsilon\|_{\boldsymbol{W}_0^{1,p}(\Omega)} \\ &\leq C^{1/p} \varepsilon \|\boldsymbol{f}\|^{p'}_{\boldsymbol{W}^{-1,p'}(\Omega)}. \end{aligned} \quad (3.8)$$

Hence, we have $\mathrm{div}\, \boldsymbol{u}_\varepsilon \to 0$ strongly in $L^p(\Omega)$ as $\varepsilon \to 0$. On the other hand, from (3.7), there exists a subsequence $\{\boldsymbol{u}_{\varepsilon_j}\}$ of $\{\boldsymbol{u}_\varepsilon\}$ such that $\boldsymbol{u}_{\varepsilon_j} \to \boldsymbol{u}$ weakly in $\boldsymbol{W}_0^{1,p}(\Omega)$. This implies $\mathrm{div}\, \boldsymbol{u} = 0$ in $\Omega$. From (3.7), we have

$$\|\boldsymbol{u}\|^p_{\boldsymbol{W}_0^{1,p}(\Omega)} \leq \liminf_{\varepsilon_j \to 0} \|\boldsymbol{u}_{\varepsilon_j}\|^p_{\boldsymbol{W}_0^{1,p}(\Omega)} \leq C \|\boldsymbol{f}\|^{p'}_{\boldsymbol{W}^{-1,p'}(\Omega)}. \quad (3.9)$$

If we define $\pi_\varepsilon = -\frac{1}{\varepsilon} S_t(x, (\mathrm{div}\, \boldsymbol{u}_\varepsilon)^2)\mathrm{div}\, \boldsymbol{u}_\varepsilon$, then it follows from (3.8) and (2.1a) that

$$\|\pi_\varepsilon\|^{p'}_{L^{p'}(\Omega)} \leq \frac{1}{\varepsilon} \Lambda \int_\Omega |\mathrm{div}\, \boldsymbol{u}_\varepsilon|^p dx \leq \frac{\Lambda C^{1/p}}{\lambda} \|\boldsymbol{f}\|^{p'}_{\boldsymbol{W}^{-1,p'}(\Omega)}.$$

Passing to a subsequence, we may assume that $\pi_{\varepsilon_j} \to \pi$ weakly in $L^{p'}(\Omega)$ and

$$\|\pi\|^{p'}_{L^{p'}(\Omega)} \leq \liminf_{\varepsilon_j \to 0} \|\pi_{\varepsilon_j}\|^{p'}_{L^{p'}(\Omega)} \leq \frac{\Lambda C^{1/p}}{\lambda} \|\boldsymbol{f}\|^{p'}_{\boldsymbol{W}^{-1,p'}(\Omega)}. \quad (3.10)$$

On the other hand, since

$$\int_\Omega |S_t(x, |\mathrm{curl}\, \boldsymbol{u}_\varepsilon|^2)\mathrm{curl}\, \boldsymbol{u}_\varepsilon|^{p'} dx \leq \Lambda^{p'} \int_\Omega |\mathrm{curl}\, \boldsymbol{u}_\varepsilon|^p dx \leq \Lambda^{p'} C \|\boldsymbol{f}\|^{p'}_{\boldsymbol{W}^{-1,p}(\Omega)},$$

$\{S_t(x, |\mathrm{curl}\, \boldsymbol{u}_\varepsilon|^2)\mathrm{curl}\, \boldsymbol{u}_\varepsilon\}$ is bounded in $\boldsymbol{L}^{p'}(\Omega)$. Passing to a subsequence, we may assume that

$$S_t(x, |\mathrm{curl}\, \boldsymbol{u}_{\varepsilon_j}|^2)\mathrm{curl}\, \boldsymbol{u}_{\varepsilon_j} \to \boldsymbol{w} \text{ weakly in } \boldsymbol{L}^{p'}(\Omega).$$

Since

$$\int_\Omega \big\{S_t(x, |\mathrm{curl}\, \boldsymbol{u}_{\varepsilon_j}|^2 \mathrm{curl}\, \boldsymbol{u}_{\varepsilon_j} \cdot \mathrm{curl}\, \boldsymbol{v} - \pi_{\varepsilon_j} \mathrm{div}\, \boldsymbol{v}\big\} dx = \langle \boldsymbol{f}, \boldsymbol{v}\rangle$$

for all $\boldsymbol{v} \in \boldsymbol{W}_0^{1,p}(\Omega)$, letting $\varepsilon_j \to 0$, we have

$$\int_\Omega (\boldsymbol{w} \cdot \mathrm{curl}\, \boldsymbol{v} - \pi \mathrm{div}\, \boldsymbol{v}) dx = \langle \boldsymbol{f}, \boldsymbol{v}\rangle \text{ for all } \boldsymbol{v} \in \boldsymbol{W}_0^{1,p}(\Omega). \tag{3.11}$$

In particular, since $\mathrm{div}\, \boldsymbol{u} = 0$ in $\Omega$, we have

$$\int_\Omega \boldsymbol{w} \cdot \mathrm{curl}\, \boldsymbol{u} dx = \langle \boldsymbol{f}, \boldsymbol{u}\rangle. \tag{3.12}$$

Since $\mathrm{div}\, \boldsymbol{u}_{\varepsilon_j} \to 0$ strongly in $L^p(\Omega)$, we have

$$\begin{aligned}\lim_{j\to\infty} \int_\Omega S_t(x, |\mathrm{curl}\, \boldsymbol{u}_{\varepsilon_j}|^2)|\mathrm{curl}\, \boldsymbol{u}_{\varepsilon_j}|^2 dx &= \lim_{j\to\infty} \int_\Omega \pi_{\varepsilon_j} \mathrm{div}\, \boldsymbol{u}_{\varepsilon_j} dx + \langle \boldsymbol{f}, \boldsymbol{u}_{\varepsilon_j}\rangle \\ &= \langle \boldsymbol{f}, \boldsymbol{u}\rangle = \int_\Omega \boldsymbol{w} \cdot \mathrm{curl}\, \boldsymbol{u} dx. \end{aligned} \tag{3.13}$$

By the monotonicity (Lemma 2.2), we have

$$\begin{aligned}&\int_\Omega S_t(x, |\mathrm{curl}\, \boldsymbol{u}_{\varepsilon_j}|^2)\mathrm{curl}\, \boldsymbol{u}_{\varepsilon_j} \cdot \mathrm{curl}\, (\boldsymbol{u}_{\varepsilon_j} - \boldsymbol{v}) dx \\ &\qquad - \int_\Omega S_t(x, |\mathrm{curl}\, \boldsymbol{v}|^2)\mathrm{curl}\, \boldsymbol{v} \cdot \mathrm{curl}\, (\boldsymbol{u}_{\varepsilon_j} - \boldsymbol{v}) dx \ge 0.\end{aligned}$$

Letting $j \to \infty$, we have

$$\int_\Omega (\boldsymbol{w} - S_t(x, |\mathrm{curl}\, \boldsymbol{v}|^2)\mathrm{curl}\, \boldsymbol{v}) \cdot \mathrm{curl}\, (\boldsymbol{u} - \boldsymbol{v}) dx \ge 0 \text{ for all } \boldsymbol{v} \in \boldsymbol{W}_0^{1,p}(\Omega).$$

For any $\boldsymbol{\phi} \in \boldsymbol{W}_0^{1,p}(\Omega)$, put $\boldsymbol{v} = \boldsymbol{u} - \alpha\boldsymbol{\phi}$ $(\alpha > 0)$. Then we have

$$\int_\Omega (\boldsymbol{w} - S_t(x, |\mathrm{curl}\, \boldsymbol{u} - \alpha \mathrm{curl}\, \boldsymbol{\phi}|^2)(\mathrm{curl}\, \boldsymbol{u} - \alpha \mathrm{curl}\, \boldsymbol{\phi}) \cdot \alpha \mathrm{curl}\, \boldsymbol{\phi} dx \ge 0.$$

If we divide this inequality by $\alpha$, and then let $\alpha \to 0$, we have

$$\int_\Omega (\boldsymbol{w} - S_t(x, |\mathrm{curl}\, \boldsymbol{u}|^2)\mathrm{curl}\, \boldsymbol{u}) \cdot \mathrm{curl}\, \boldsymbol{\phi} dx \ge 0$$

for all $\boldsymbol{\phi} \in \boldsymbol{W}_0^{1,p}(\Omega)$. This implies

$$\int_\Omega (\boldsymbol{w} - S_t(x, |\mathrm{curl}\, \boldsymbol{u}|^2)\mathrm{curl}\, \boldsymbol{u}) \cdot \mathrm{curl}\, \boldsymbol{\phi} dx = 0$$

for all $\boldsymbol{\phi} \in \boldsymbol{W}_0^{1,p}(\Omega)$. Thus we have

$$\int_\Omega \big(S_t(x, |\mathrm{curl}\,\boldsymbol{u}|^2)\mathrm{curl}\,\boldsymbol{u} \cdot \mathrm{curl}\,\boldsymbol{v} - \pi \mathrm{div}\,\boldsymbol{v})\big)dx = \langle \boldsymbol{f}, \boldsymbol{v}\rangle \text{ for all } \boldsymbol{v} \in \boldsymbol{W}_0^{1,p}(\Omega).$$

Therefore, $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ is a weak solution of (3.1).

Next we show the uniqueness of solution. Let $(\boldsymbol{u}_1, \pi_1), (\boldsymbol{u}_2, \pi_2) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^{p'}(\Omega)/\mathbb{R}$ be two weak solutions of (3.1). If we take $\boldsymbol{v} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ as a test function of (3.2), then we have

$$\int_\Omega S_t(x, |\mathrm{curl}\,\boldsymbol{u}_i|^2)\mathrm{curl}\,\boldsymbol{u}_i \cdot \mathrm{curl}\,(\boldsymbol{u}_1 - \boldsymbol{u}_2)dx = \langle \boldsymbol{f}, \boldsymbol{u}_1 - \boldsymbol{u}_2\rangle \text{ for } i = 1, 2$$

because of $\mathrm{div}\,\boldsymbol{u}_i = 0$ in $\Omega$. Thus we have

$$\int_\Omega \big(S_t(x, |\mathrm{curl}\,\boldsymbol{u}_1|^2)\mathrm{curl}\,\boldsymbol{u}_1 - S_t(x, |\mathrm{curl}\,\boldsymbol{u}_2|^2)\mathrm{curl}\,\boldsymbol{u}_2\big) \cdot \mathrm{curl}\,(\boldsymbol{u}_1 - \boldsymbol{u}_2)dx = 0$$

By the strict monotonicity (Lemma 2.2), we have $\mathrm{curl}\,(\boldsymbol{u}_1 - \boldsymbol{u}_2) = \boldsymbol{0}$ and $\mathrm{div}\,(\boldsymbol{u}_1 - \boldsymbol{u}_2) = 0$ in $\Omega$. By (3.4), we have $\boldsymbol{u}_1 = \boldsymbol{u}_2$. From this, we have $\nabla(\pi_1 - \pi_2) = \boldsymbol{0}$ in the distribution sense, so $\pi_1 - \pi_2$ is a constant, i.e., $\pi_1 = \pi_2$ in $L^{p'}(\Omega)/\mathbb{R}$.

Finally we show the estimate (2.6). Taking $\boldsymbol{v} = \boldsymbol{u}$ as a test function of (2.5), since $\mathrm{div}\,\boldsymbol{u} = 0$ in $\Omega$, we have

$$\begin{aligned} \lambda \int_\Omega |\mathrm{curl}\,\boldsymbol{u}|^p dx \le \int_\Omega S_t(x, |\mathrm{curl}\,\boldsymbol{u}|^2)|\mathrm{curl}\,\boldsymbol{u}|^2 dx \\ = \langle \boldsymbol{f}, \boldsymbol{u}\rangle \le \|\boldsymbol{f}\|_{\boldsymbol{W}^{-1,p'}(\Omega)} \|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\Omega)}. \end{aligned}$$

By the same arguments as above, we have

$$\|\boldsymbol{u}\|^p_{\boldsymbol{W}_0^{1,p}(\Omega)} \le C_1 \|\boldsymbol{f}\|^{p'}_{\boldsymbol{W}^{-1,p'}(\Omega)}. \tag{3.14}$$

If $c_\pi = \frac{1}{|\Omega|}\int_\Omega \pi dx$, we have

$$\int_\Omega (\pi - c_\pi)\mathrm{div}\,\boldsymbol{v} dx = \int_\Omega \pi \mathrm{div}\,\boldsymbol{v} dx \text{ for all } \boldsymbol{v} \in \boldsymbol{W}_0^{1,p}(\Omega)$$

because of the fact $\int_\Omega \mathrm{div}\,\boldsymbol{v} dx = 0$ from the divergence theorem. Therefore we may assume that $\pi \in L_0^{p'}(\Omega) = \{\varphi \in L^{p'}(\Omega); \int_\Omega \varphi dx = 0\}$. For any $\phi \in L^p(\Omega)$, $\phi - c_\phi \in L_0^p(\Omega)$. According to Amrouche and Girault [1, Corollary 3.1], the operator $\mathrm{div} : \boldsymbol{W}_0^{1,p}(\Omega)/V^{1,p} \to L_0^p(\Omega)$ is isomorphism onto, where $V^{1,p} = \{\boldsymbol{v} \in \boldsymbol{W}_0^{1,p}(\Omega); \mathrm{div}\,\boldsymbol{v} = 0 \text{ in } \Omega\}$. Hence there exists $[\boldsymbol{w}] \in \boldsymbol{W}_0^{1,p}(\Omega)/V^{1,p}$ such that $\mathrm{div}\,\boldsymbol{w} = \phi - c_\phi$ and

$$\|[\boldsymbol{w}]\|_{\boldsymbol{W}_0^{1,p}(\Omega)/V^{1,p}} \le C\|\phi - c_\phi\|_{L^p(\Omega)} \le C_1 \|\phi\|_{L^p(\Omega)}.$$

We claim that $\|[\boldsymbol{w}]\|_{\boldsymbol{W}_0^{1,p}(\Omega)} = \inf_{\boldsymbol{v}\in V^{1,p}} \|\boldsymbol{w}+\boldsymbol{v}\|_{\boldsymbol{W}_0^{1,p}(\Omega)}$ is achieved. In fact, $\{\boldsymbol{v}_j\} \subset V^{1,p}$ be a minimizing sequence of $\|[\boldsymbol{w}]\|_{\boldsymbol{W}_0^{1,p}(\Omega)/V^{1,p}}$. Then

$$\|\boldsymbol{w}+\boldsymbol{v}_j\|_{\boldsymbol{W}_0^{1,p}(\Omega)} = \|[\boldsymbol{w}]\|_{\boldsymbol{W}_0^{1,p}(\Omega)/V^{1,p}} + o(1) \text{ as } j\to\infty.$$

Then $\{\boldsymbol{v}_j\}$ is bounded in $\boldsymbol{W}_0^{1,p}(\Omega)$. Passing to a subsequence, we may assume that $\boldsymbol{v}_j \to \boldsymbol{v}_0$ weakly in $\boldsymbol{W}_0^{1,p}(\Omega)$, so $\mathrm{div}\,\boldsymbol{v}_0 = 0$ in $\Omega$, i.e,, $\boldsymbol{v}_0 \in V^{1,p}$. Moreover, we have

$$\|\boldsymbol{w}+\boldsymbol{v}_0\|_{\boldsymbol{W}_0^{1,p}(\Omega)} \le \liminf_{j\to\infty} \|\boldsymbol{w}+\boldsymbol{v}_j\|_{\boldsymbol{W}_0^{1,p}(\Omega)} = \|[\boldsymbol{w}]\|_{\boldsymbol{W}_0^{1,p}(\Omega)/V^{1,p}}.$$

Thus we have $\mathrm{div}\,(\boldsymbol{w}+\boldsymbol{v}_0) = \mathrm{div}\,\boldsymbol{w} = \phi - c_\phi$. Hence we can assume that $\mathrm{div}\,\boldsymbol{w} = \phi - c_\phi$ and

$$\|\boldsymbol{w}\|_{\boldsymbol{W}_0^{1,p}(\Omega)} \le C\|\phi\|_{L^p(\Omega)}. \tag{3.15}$$

Taking $\boldsymbol{v} = \boldsymbol{w}$ as a test function of (2.5), since $\pi \in L_0^{p'}(\Omega)$ and satisfies $\int_\Omega \pi c_\phi dx = 0$, we have

$$\int_\Omega \pi\phi dx = \int_\Omega S_t(x, |\mathrm{curl}\,\boldsymbol{u}|^2)\mathrm{curl}\,\boldsymbol{u}\cdot \mathrm{curl}\,\boldsymbol{w} dx - \langle \boldsymbol{f}, \boldsymbol{w}\rangle.$$

Therefore, by the Hölder inequality and (3.14), we have

$$\begin{aligned}\left|\int_\Omega \pi\phi dx\right| &\le \Lambda \int_\Omega |\mathrm{curl}\,\boldsymbol{u}|^{p-1}|\mathrm{curl}\,\boldsymbol{w}|dx + \|\boldsymbol{f}\|_{\boldsymbol{W}^{-1,p'}(\Omega)}\|\boldsymbol{w}\|_{\boldsymbol{W}_0^{1,p}(\Omega)}\\ &\le \Lambda\|\mathrm{curl}\,\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}\|\mathrm{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{f}\|_{\boldsymbol{W}^{-1,p'}(\Omega)}\|\boldsymbol{w}\|_{\boldsymbol{W}_0^{1,p}(\Omega)}\\ &\le C\Lambda(\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{W}_0^{1,p}(\Omega)} + \|\boldsymbol{f}\|_{\boldsymbol{W}^{-1.p'}(\Omega)})\|\phi\|_{L^p(\Omega)}\\ &\le C\|\boldsymbol{f}\|_{\boldsymbol{W}^{-1,p'}(\Omega)}\|\phi\|_{L^p(\Omega)}.\end{aligned}$$

for all $\phi \in L^p(\Omega)$. Hence we have

$$\|\pi\|_{L^{p'}(\Omega)} \le C\|\boldsymbol{f}\|_{\boldsymbol{W}^{-1,p'}(\Omega)}. \tag{3.16}$$

Summing up (3.14) and (3.16), we get the estimate (2.6). This completes the proof of Theorem 2.5.

## 4. CONTINUOUS DEPENDENCE OF A WEAK SOLUTION ON THE DATA

In this section, we consult the continuous dependence of a weak solution of (2.4) on the data. In order to do so, for every $n = 0, 1, \ldots$, let $S^{(n)}(x,t)$ satisfy (2.1a)-(2.1c) with the same constants $\lambda$ and $\Lambda$, and let $\boldsymbol{f}_n \in \boldsymbol{W}^{-1,p'}(\Omega)$. For $n = 0, 1, \ldots$, assume that $(\boldsymbol{u}_n, \pi_n) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^{p'}(\Omega)/\mathbb{R}$ is weak solution of (2.4), i.e.,

$$\begin{cases} \mathrm{curl}\,[S_t^{(n)}(x, |\mathrm{curl}\,\boldsymbol{u}_n|^2)\mathrm{curl}\,\boldsymbol{u}_n] + \nabla\pi_n = \boldsymbol{f}_n & \text{in } \Omega,\\ \mathrm{div}\,\boldsymbol{u}_n = 0 & \text{in } \Omega,\\ \boldsymbol{u}_n = \boldsymbol{0} & \text{on } \Gamma.\end{cases} \tag{4.1}$$

More precisely, $(\boldsymbol{u}_n, \pi_n)$ satisfies

$$\int_\Omega S_t^{(n)}(x, |\mathrm{curl}\,\boldsymbol{u}_n|^2)\mathrm{curl}\,\boldsymbol{u}_n \cdot \mathrm{curl}\,\boldsymbol{v} dx - \int_\Omega \pi_n \mathrm{div}\,\boldsymbol{v} dx \\ = \langle \boldsymbol{f}_n, \boldsymbol{v}\rangle \text{ for all } \boldsymbol{v} \in \boldsymbol{W}_0^{1,p}(\Omega)\ (n = 0, 1, \ldots). \quad (4.2)$$

Then we have the following theorem on the continuous dependence on the data.

**Theorem 4.1.** *We assume that for every $n = 0, 1, \ldots$, a Carathéodory function $S^{(n)}(x,t)$ satisfies* (2.1a)-(2.1c) *with the same constants $\lambda$ and $\Lambda$, and $\boldsymbol{f}_n \in \boldsymbol{W}^{-1,p'}(\Omega)$. Let $(\boldsymbol{u}_n, \pi_n) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^{p'}(\Omega)/\mathbb{R}$ be a unique weak solution of* (2.4). *If $S_t^{(n)}(x,t) \to S_t^{(0)}(x,t)$ a.e. in $\Omega \times [0,\infty)$ and $\boldsymbol{f}_n \to \boldsymbol{f}_0$ in $\boldsymbol{W}^{-1,p'}(\Omega)$ as $n \to \infty$, then $\boldsymbol{u}_n \to \boldsymbol{u}_0$ in $\boldsymbol{W}_0^{1,p}(\Omega)$ and $\pi_n \to \pi_0$ in $L^{p'}(\Omega)/\mathbb{R}$ as $n \to \infty$.*

*In particular case where $S^{(n)}(x,t) = S^{(0)}(x,t)$ for all $n = 1, \ldots$, there exists a constant $C > 0$ depending only on $p, \lambda, \Lambda, \Omega$ and $\|\boldsymbol{f}_0\|_{\boldsymbol{W}^{-1,p'}(\Omega)}$ such that*

$$\|\boldsymbol{u}_n - \boldsymbol{u}_0\|^{p\vee p'}_{\boldsymbol{W}_0^{1,p}(\Omega)} + \|\pi_n - \pi_0\|^{p\vee p'}_{L^p(\Omega)/\mathbb{R}} \\ \le C(\|\boldsymbol{f}_n - \boldsymbol{f}_0\|^{p'}_{\boldsymbol{W}^{-1,p'}(\Omega)} + \|\boldsymbol{f}_n - \boldsymbol{f}_0\|^{p}_{\boldsymbol{W}^{-1,p'}(\Omega)}),$$

*where $p \vee p' = \max\{p, p'\}$.*

*Proof.* Taking $\boldsymbol{v} = \boldsymbol{u}_n - \boldsymbol{u}_0$ as a test function of (4.2), since $\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_0) = 0$ in $\Omega$, we have

$$\int_\Omega \big(S_t^{(n)}(x, |\mathrm{curl}\,\boldsymbol{u}_n|^2)\mathrm{curl}\,\boldsymbol{u}_n - S_t^{(n)}(x, |\mathrm{curl}\,\boldsymbol{u}_0|^2)\mathrm{curl}\,\boldsymbol{u}_0\big) \cdot \mathrm{curl}\,(\boldsymbol{u}_n - \boldsymbol{u}_0) dx \\ = \langle \boldsymbol{f}_n - \boldsymbol{f}_0, \boldsymbol{u}_n - \boldsymbol{u}_0\rangle \\ - \int_\Omega \big(S_t^{(n)}(x, |\mathrm{curl}\,\boldsymbol{u}_0|^2)\mathrm{curl}\,\boldsymbol{u}_0 - S_t^{(0)}(x, |\mathrm{curl}\,\boldsymbol{u}_0|^2)\mathrm{curl}\,\boldsymbol{u}_0\big) \cdot \mathrm{curl}\,(\boldsymbol{u}_n - \boldsymbol{u}_0) dx. \quad (4.3)$$

Hereafter, for the brevity of notations, we put

$$F_n(k) = \|\boldsymbol{f}_n - \boldsymbol{f}_0\|^k_{\boldsymbol{W}^{-1,p'}(\Omega)} \\ + \|S_t^{(n)}(x, |\mathrm{curl}\,\boldsymbol{u}_0|^2)\mathrm{curl}\,\boldsymbol{u}_0 - S_t^{(0)}(x, |\mathrm{curl}\,\boldsymbol{u}_0|^2)\mathrm{curl}\,\boldsymbol{u}_0\|^k_{\boldsymbol{L}^{p'}(\Omega)},$$

and we denote constants depending only on $p, \lambda, \Lambda, \Omega$ and the constants in Lemma 2.2, 2.3 by $C, C_1, C_2, \ldots$ which may vary from line to line.

We estimate (4.3).

When $p \geq 2$, it follows from Lemma 2.2 (ii) and the Young inequality that we have

$$\begin{aligned}
&c_2 \int_\Omega |\mathrm{curl}\,(\boldsymbol{u}_n - \boldsymbol{u}_0)|^p dx \\
&\leq \|\boldsymbol{f}_n - \boldsymbol{f}_0\|_{\boldsymbol{W}^{-1,p'}(\Omega)} \|\boldsymbol{u}_n - \boldsymbol{u}_0\|_{\boldsymbol{W}_0^{1,p}(\Omega)} \\
&+\|S_t^{(n)}(x, |\mathrm{curl}\,\boldsymbol{u}_0|^2)\mathrm{curl}\,\boldsymbol{u}_0 - S_t^{(0)}(x, |\mathrm{curl}\,\boldsymbol{u}_0|^2)\mathrm{curl}\,\boldsymbol{u}_0\|_{\boldsymbol{L}^{p'}(\Omega)} \\
&\times\|\mathrm{curl}\,(\boldsymbol{u}_n - \boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)} \\
&\leq C(\varepsilon)F_n(p') + \varepsilon\|\boldsymbol{u}_n - \boldsymbol{u}_0\|^p_{\boldsymbol{W}_0^{1,p}(\Omega)}
\end{aligned}$$

for any $\varepsilon > 0$. On the other hand, from (3.4),

$$c_2\|\boldsymbol{u}_n - \boldsymbol{u}_0\|^p_{\boldsymbol{W}_0^{1,p}(\Omega)} \leq c_2C_2\|\mathrm{curl}\,(\boldsymbol{u}_n - \boldsymbol{u}_0)\|^p_{\boldsymbol{L}^p(\Omega)}.$$

If we choose $\varepsilon > 0$ so that $C_2\varepsilon = c_2/2$, we have

$$\|\boldsymbol{u}_n - \boldsymbol{u}_0\|^p_{\boldsymbol{W}_0^{1,p}(\Omega)} \leq C_2F_n(p'). \tag{4.4}$$

When $1 < p < 2$, using Lemma 2.2 (ii) to (4.3), we have

$$\int_\Omega (|\mathrm{curl}\,\boldsymbol{u}_n| + |\mathrm{curl}\,\boldsymbol{u}_0|)^{p-2}|\mathrm{curl}\,(\boldsymbol{u}_n - \boldsymbol{u}_0)|^2 dx \leq CF_n(1)\|\boldsymbol{u}_n - \boldsymbol{u}_0\|_{\boldsymbol{W}_0^{1,p}(\Omega)}.$$

Here if we use the reverse Hölder inequality (cf. Sobolev [10, p. 8]) with $0 < s = p/2 < 1$ and $s' = p/(p-2)$, then there exists a constant $c > 0$ such that

$$\begin{aligned}
&\int_\Omega (|\mathrm{curl}\,\boldsymbol{u}_n| + |\mathrm{curl}\,\boldsymbol{u}_0|)^{p-2}|\mathrm{curl}\,(\boldsymbol{u}_n - \boldsymbol{u}_0)|^2 dx \\
&\qquad \geq c(\|\mathrm{curl}\,\boldsymbol{u}_n\|^p_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{curl}\,\boldsymbol{u}_0\|^p_{\boldsymbol{L}^p(\Omega)})^{(p-2)/2}\|\mathrm{curl}\,(\boldsymbol{u}_n - \boldsymbol{u}_0)\|^2_{\boldsymbol{L}^p(\Omega)}.
\end{aligned}$$

Hence, it follows from (3.9) that

$$\begin{aligned}
&\|\boldsymbol{u}_n - \boldsymbol{u}_0\|^2_{\boldsymbol{W}_0^{1,p}(\Omega)} \\
&\qquad \leq C(\|\boldsymbol{f}_n\|^{p'}_{\boldsymbol{W}^{-1,p'}(\Omega)} + \|\boldsymbol{f}_0\|^{p'}_{\boldsymbol{W}^{-1,p'}(\Omega)})^{(2-p)/2}F_n(1)\|\boldsymbol{u}_n - \boldsymbol{u}_0\|_{\boldsymbol{W}_0^{1,p}(\Omega)}.
\end{aligned}$$

Since we may assume that

$$\|\boldsymbol{f}_n\|^{p'}_{\boldsymbol{W}^{-1,p'}(\Omega)} \leq C(\|\boldsymbol{f}_0\|^{p'}_{\boldsymbol{W}^{-1,p'}(\Omega)} + 1)$$

under the hypothesis, we can write

$$\|\boldsymbol{u}_n - \boldsymbol{u}_0\|_{\boldsymbol{W}_0^{1,p}(\Omega)} \leq C_2F_n(1). \tag{4.5}$$

We may assume that $\pi_n - \pi_0 \in L_0^{p'}(\Omega)$, so for any $\phi \in L^p(\Omega)$,

$$\int_\Omega (\pi_n - \pi_0)(\phi - c_\phi)dx = \int_\Omega (\pi_n - \pi_0)\phi dx.$$

There exists $\boldsymbol{w} \in \boldsymbol{W}_0^{1,p}(\Omega)$ such that $\operatorname{div} \boldsymbol{w} = \phi - c_\phi$ (cf. [1, Corollary 3.1]), and

$$\|\boldsymbol{w}\|_{\boldsymbol{W}_0^{1,p}(\Omega)} \leq C\|\phi\|_{L^p(\Omega)}. \tag{4.6}$$

Taking $\boldsymbol{v} = \boldsymbol{w}$ as a test function of (4.2), we have

$$\int_\Omega \big(S_t^{(n)}(x, |\operatorname{curl} \boldsymbol{u}_n|^2)\operatorname{curl} \boldsymbol{u}_n - S_t^{(0)}(x, |\operatorname{curl} \boldsymbol{u}_0|^2)\operatorname{curl} \boldsymbol{u}_0\big) \cdot \operatorname{curl} \boldsymbol{w} dx$$
$$- \int_\Omega (\pi_n - \pi_0)\phi dx = \langle \boldsymbol{f}_n - \boldsymbol{f}_0, \boldsymbol{w}\rangle.$$

We write this equality in the following form.

$$\int_\Omega (\pi_n - \pi_0)\phi dx = I_1 + I_2 - \langle \boldsymbol{f}_n - \boldsymbol{f}_0, \boldsymbol{w}\rangle. \tag{4.7}$$

where

$$\begin{aligned} I_1 &= \int_\Omega \big(S_t^{(n)}(x, |\operatorname{curl} \boldsymbol{u}_n|^2)\operatorname{curl} \boldsymbol{u}_n - S_t^{(n)}(x, |\operatorname{curl} \boldsymbol{u}_0|^2)\operatorname{curl} \boldsymbol{u}_0\big) \cdot \operatorname{curl} \boldsymbol{w} dx, \\ I_2 &= \int_\Omega \big(S_t^{(n)}(x, |\operatorname{curl} \boldsymbol{u}_0|^2)\operatorname{curl} \boldsymbol{u}_0 - S_t^{(0)}(x, |\operatorname{curl} \boldsymbol{u}_0|^2)\operatorname{curl} \boldsymbol{u}_0\big) \cdot \operatorname{curl} \boldsymbol{w} dx. \end{aligned}$$

From (4.6) and the Hölder inequality, we have

$$|I_2| \leq C\|S_t^{(n)}(x, |\operatorname{curl} \boldsymbol{u}_0|^2)\operatorname{curl} \boldsymbol{u}_0 - S_t^{(0)}(x, |\operatorname{curl} \boldsymbol{u}_0|^2)\operatorname{curl} \boldsymbol{u}_0\|_{\boldsymbol{L}^{p'}(\Omega)}$$
$$\times \|\boldsymbol{w}\|_{\boldsymbol{W}_0^{1,p}(\Omega)}.$$

We now estimate $I_1$. When $1 < p < 2$, using Lemma 2.3, (4.5) and (4.6), we have

$$\begin{aligned} |I_1| &\leq C_4 \int_\Omega |\operatorname{curl}(\boldsymbol{u}_n - \boldsymbol{u}_0)|^{p-1}|\operatorname{curl} \boldsymbol{w}|dx \\ &\leq C_4\|\boldsymbol{u}_n - \boldsymbol{u}_0\|_{\boldsymbol{W}_0^{1,p}(\Omega)}^{p-1}\|\boldsymbol{w}\|_{\boldsymbol{W}_0^{1,p}(\Omega)} \\ &\leq C_5 F_n(p-1)\|\phi\|_{L^p(\Omega)}. \end{aligned}$$

When $p \geq 2$, using Lemma 2.3, (4.4), (4.6), (2.6) and the Hölder inequality, we have

$$\begin{aligned} |I_1| &\leq C_1 \int_\Omega (|\operatorname{curl} \boldsymbol{u}_n| + |\operatorname{curl} \boldsymbol{u}_0)|)^{p-2}|\operatorname{curl}(\boldsymbol{u}_n - \boldsymbol{u}_0)||\operatorname{curl} \boldsymbol{w}|dx \\ &\leq C_2(\|\operatorname{curl} \boldsymbol{u}_n\|_{\boldsymbol{L}^p(\Omega)} + \|\operatorname{curl} \boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)})^{p-2} \\ &\qquad \times\|\operatorname{curl}(\boldsymbol{u}_n - \boldsymbol{u}_0)\|_{\boldsymbol{L}^p(\Omega)}\|\operatorname{curl} \boldsymbol{w}\|_{\boldsymbol{L}^p(\Omega)} \\ &\leq C_3(\|\boldsymbol{f}_n\|_{\boldsymbol{W}^{-1,p'}(\Omega)}^{p'-1} + \|\boldsymbol{f}_0\|_{\boldsymbol{W}^{-1,p'}(\Omega)}^{p'-1})^{p-2} F_n(p'-1)\|\phi\|_{L^p(\Omega)}. \end{aligned}$$

Therefore, we have

$$\|\pi_n - \pi\|_{L^{p'}(\Omega)} \le \begin{cases} C_1 F_n(1) + C_2 F_n(p-1) & \text{if } 1 < p < 2, \\ C_1 F_n(1) + C_2 F_n(p'-1) & \text{if } p \ge 2. \end{cases} \tag{4.8}$$

Hence we have

$$\|\boldsymbol{u}_n - \boldsymbol{u}_0\|^{p\vee p'}_{\boldsymbol{W}^{1,p}_0(\Omega)} + \|\pi_n - \pi_0\|^{p\vee p'}_{L^{p'}(\Omega)} \le C(F_n(p) + F_n(p')).$$

Finally, we claim that $F_n(k) \to 0$ as $n \to \infty$, if $\boldsymbol{f}_n \to \boldsymbol{f}_0$ in $\boldsymbol{W}^{-1,p'}(\Omega)$ and $S^{(n)}_t(x,t) \to S^{(0)}_t(x,t)$ a.e. $(x,t) \in \Omega \times [0,\infty)$.

In fact, from (2.1a), we have

$$\begin{aligned} &|S^{(n)}_t(x, |\mathrm{curl}\,\boldsymbol{u}_0|^2)\mathrm{curl}\,\boldsymbol{u}_0 - S^{(0)}_t(x, |\mathrm{curl}\,\boldsymbol{u}_0|^2)\mathrm{curl}\,\boldsymbol{u}_0|^{p'} \\ &\quad \le (2\Lambda)^{p'} |\mathrm{curl}\,\boldsymbol{u}_0|^p \in L^1(\Omega) \end{aligned}$$

and $S^{(n)}_t(x, |\mathrm{curl}\,\boldsymbol{u}_0|^2)\mathrm{curl}\,\boldsymbol{u}_0 - S^0_t(x, |\mathrm{curl}\,\boldsymbol{u}_0|^2)\mathrm{curl}\,\boldsymbol{u}_0 \to 0$ a.e. in $\Omega$, so it follows from the Lebesgue dominated theorem that

$$\|S^{(n)}_t(x, |\mathrm{curl}\,\boldsymbol{u}_0|^2)\mathrm{curl}\,\boldsymbol{u}_0 - S^{(0)}_t(x, |\mathrm{curl}\,\boldsymbol{u}_0|^2)\mathrm{curl}\,\boldsymbol{u}_0\|_{\boldsymbol{L}^{p'}(\Omega)} \to 0$$

as $n \to \infty$. In the particular case, since $F_n(k) = \|\boldsymbol{f}_n - \boldsymbol{f}_0\|^k_{\boldsymbol{W}^{-1,p'}(\Omega)}$, the estimate is clear. □

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