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The form of the Third Partial Derivative Test for Functions of Three Variables

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Abstract

We consider the form of the third partial derivative test for functions of three variables. The methods used are Hessian matrix and Taylor expansion. This study was conducted for the purpose of providing an orderly tool to alleviate the computational hassle for related research of the future.

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1. INTRODUCTION

The concept of the local extremum of one-variable function can be extended to a multivariate function, but the exact method has not been established yet. In particular, the form of the third partial derivative test does not even know its shape, so this study will deal with the form. The local extremum of a multivariate function is handled vaguely by using the Hessian matrix. The concept used to find the local extremum of a multivariate function is positive definite. It is defined by

$$x^T M x > 0$$

for all non-zero $x \in R^n$, where M is an $n \times n$ symmetric real matrix. It is well-known that heat conductivity matrix is positive definite, and this term is used in machine learning. For application at extremum value, if the Hessian is positive definite(positive negative) at any critical point a, then f has a local minimum(local maximum) at a, respectively. Of course, if the Hessian has both positive and negative definite at a, then a is a saddle point for f.

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However, since this concept is not practical in extremum theory, this study will deal with a practical form. To begin with, let us see the Taylor's formula[2].

Suppose $f(x)$ has derivatives up to order $n+1$ inclusive in a neighborhood of a point a, where $f^{(n)}(a) \neq 0$ and $f^{(n+1)}(x)$ is bounded. Then Taylor's formula is given by

$$f(a+h) = f(a) + hf'(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} f^{(n)}(a)E,$$

where E denotes a quantity approaching 1 as $h \to 0$. Of course, this Taylor formula can be interpreted as an extension of the mean value theorem(or Lagrange's theorem) to n dimensions: If $f(x)$ is continuous on a finite closed interval $[a, b]$ and differentiable at every interior point of $[a, b]$, then exists a point $c \in (a, b)$ such that

$$f(b) = f(a) + f'(c)(b-a),$$

where $a \neq b$.

An algorithm for the approximate solution of two point boundary value problems of class C^2 was proposed by Dickmanns and Well[1]. As a new algorithm, an object-oriented Cartesian embedding algorithm was proposed in automated generation of high-order partial derivative models[3].

On the one hand, the proposed study is still in its infancy and aims to provide a form of third partial derivative test. This form will play a role of reducing the computational hassle for those who do related research. The form of the third partial derivative test for functions of three variables is as follows.

Suppose that $f(x, y, z)$ is continuous in an open volume containing (x_0, y_0, z_0) and has a third partial derivative. Also, let $\nabla(f)(0,0,0) = 0 = \nabla^2(f)(0,0,0)$. Then the Taylor series P_3 has the form of

$$\frac{1}{6}\left(Ax^3 + By^3 + Cz^3\right)$$
$$+\frac{1}{3}\left(Dx^2y + Ex^2z + Fxy^2 + Gy^2z + Hz^2x + Iyz^2\right) + \frac{1}{2}Jxyz,$$

where $A = f_{xxx}(0,0,0)$, $B = f_{yyy}(0,0,0)$, $C = f_{zzz}(0,0,0)$, $D = f_{xxy}(0,0,0)$, $E = f_{xxz}(0,0,0)$, $F = f_{xyy}(0,0,0)$, $G = f_{yyz}(0,0,0)$, $H = f_{zzx}(0,0,0)$, $I = f_{zzy}(0,0,0)$, and $J = f_{xyz}(0,0,0)$.

2. THE FORM OF THE THIRD PARTIAL DERIVATIVE TEST FOR FUNCTIONS OF THREE VARIABLES

We would like to consider the form of the third partial derivative test.

Lemma 2.1. *(The second partial derivative test for functions of two variables[2]) If a function $f(x, y)$ is continuous, then the extreme values of f may occur at*

i) boundary points of the domain of f

ii) interior points where $f_x = f_y = 0$

iii) points where f_x or f_y fail to exist.

If f has continuous first and second order partial derivatives on some open disc containing (a, b), and if $f_x(a, b) = f_y(a, b) = 0$, then

i) $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at $(a, b) \Rightarrow$ local maximum

ii) $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at $(a, b) \Rightarrow$ local minimum

iii) $f_{xx}f_{yy} - f_{xy}^2 < 0$ at $(a, b) \Rightarrow$ saddle point

iv) $f_{xx}f_{yy} - f_{xy}^2 = 0$ at $(a, b) \Rightarrow$ test is inconclusive.

In the above equality, $f_{xx}f_{yy} - f_{xy}^2$ is usually denoted by the discriminant

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix},$$

where D is the Hessian matrix. Its proof can be obtained simply from the second-order Taylor polynomial P_2 given by

$$\begin{aligned} P_2(x, y) &= f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ &+ \frac{1}{2}\left[f_{xx}(x_0, y_0)(x - x_0)^2 + f_{yy}(x_0, y_0)(y - y_0)^2\right. \\ &\qquad \left.+ 2f_{xy}(x_0, y_0)(x - x_0)(y - y_0)\right]. \end{aligned}$$

Example 2.2. *Find the relative values of $f(x, y) = 3x^3 + y^2 - 9x + 4y$.*

Solution. By the simple calculation, we have $f_x = 9x^2 - 9$, $f_y = 2y + 4$, $f_{xx} = 18x$, $f_{yy} = 2$, and $f_{xy} = 0$. The points that satisfy $f_x(x, y) = f_y(x, y) = 0$ are $(1, -2)$ and $(-1, -2)$. Let us put $D = f_{xx}f_{yy} - f_{xy}^2$. Since $D(1, -2) > 0$, $f(1, -2) = -10$ is a minimum value. Similarly, since $D(-1, -2) < 0$, $(-1, -2)$ is a saddle point.

Consider extending lemma 2.1 to the case of the following two cases.

Case 1. The second partial derivative test of a three-variable function

The value of discriminant can be obtained by using the Hessian matrix

$$D = \begin{vmatrix} f_{xx} & f_{xy} & f_{zx} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{yz} & f_{zz} \end{vmatrix}.$$

Case 2. The form of the third partial derivative test of a three-variable function

We consider this in the next theorem.

Theorem 2.3. *Suppose that $f(x, y, z)$ is continuous in an open volume containing (x_0, y_0, z_0) and has a third partial derivative. Also, let $\nabla(f)(0, 0, 0) = 0 = \nabla^2(f)(0, 0, 0)$. Then the Taylor series P_3 has the form of*

$$\frac{1}{6}(Ax^3 + By^3 + Cz^3)$$

$$+\frac{1}{3}(Dx^2y + Ex^2z + Fxy^2 + Gy^2z + Hz^2x + Iyz^2) + \frac{1}{2}Jxyz,$$

where $A = f_{xxx}(0, 0, 0)$, $B = f_{yyy}(0, 0, 0)$, $C = f_{zzz}(0, 0, 0)$, $D = f_{xxy}(0, 0, 0)$, $E = f_{xxz}(0, 0, 0)$, $F = f_{xyy}(0, 0, 0)$, $G = f_{yyz}(0, 0, 0)$, $H = f_{zzx}(0, 0, 0)$, $I = f_{zzy}(0, 0, 0)$, and $J = f_{xyz}(0, 0, 0)$.

Proof. We consider the third-order Taylor polynomial $P_3(x, y, z)$. Then P_3 is represented as

$$\begin{aligned}
P_3(x, y, z) &= f(x_0, y_0, z_0) + [f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) \\
&\quad + f_z(x_0, y_0, z_0)(z - z_0)] + \frac{1}{2}\,[f_{xx}(x_0, y_0, z_0)(x - x_0)^2 \\
&\quad + f_{yy}(x_0, y_0, z_0)(y - y_0)^2 + f_{zz}(x_0, y_0, z_0)(z - z_0)^2 \\
&\quad + 2f_{xy}(x_0, y_0, z_0)(x - x_0)(y - y_0) + 2f_{yz}(x_0, y_0, z_0)(y - y_0)(z - z_0)] \\
&\quad + 2f_{zx}(x_0, y_0, z_0)(x - x_0)(z - z_0)] + \frac{1}{6}\,[f_{xxx}(x_0, y_0, z_0)(x - x_0)^3 \\
&\quad + f_{yyy}(x_0, y_0, z_0)(y - y_0)^3 + f_{zzz}(x_0, y_0, z_0)(z - z_0)^3] \\
&\quad + \frac{1}{3}\,[f_{xxy}(x_0, y_0, z_0)(x - x_0)^2(y - y_0) + f_{xxz}(x_0, y_0, z_0)(x - x_0)^2(z - z_0) \\
&\quad + f_{xyy}(x_0, y_0, z_0)(x - x_0)(y - y_0)^2 + f_{yyz}(x_0, y_0, z_0)(y - y_0)^2(z - z_0) \\
&\quad + f_{zzx}(x_0, y_0, z_0)(z - z_0)^2(x - x_0) + f_{zzy}(x_0, y_0, z_0)(y - y_0)(z - z_0)^2] \\
&\quad + \frac{1}{2}\,[f_{xyz}(x_0, y_0, z_0)(x - x_0)(y - y_0)(z - z_0)].
\end{aligned}$$

We assume that $f(0, 0, 0) = 0$ and $x_0 = y_0 = z_0 = 0$. Since $\nabla(f)(0, 0, 0) = 0 = \nabla^2(f)(0, 0, 0)$, we have $f_x(0, 0, 0) = f_y(0, 0, 0) = f_z(0, 0, 0) = 0$ and

$$f_{xx}(0, 0, 0) = f_{xy}(0, 0, 0) = f_{zx}(0, 0, 0)$$

$$= f_{yy}(0,0,0) = f_{yz}(0,0,0) = f_{zz}(0,0,0) = 0.$$

Simplification gives

$$P_3(x,y,z) = \frac{1}{6}\left[f_{xxx}(0,0,0)x^3 + f_{yyy}(0,0,0)y^3 + f_{zzz}(0,0,0)z^3\right]$$

$$+\frac{1}{3}\left[f_{xxy}(0,0,0)x^2y + f_{xxz}(0,0,0)x^2z + f_{xyy}(0,0,0)xy^2 + f_{yyz}(0,0,0)y^2z\right.$$

$$\left.+f_{zzx}(0,0,0)z^2x + f_{zzy}(0,0,0)yz^2\right] + \frac{1}{2}\left[f_{xyz}(0,0,0)xyz\right]. \qquad (*)$$

All coefficients here are constants, so let us put $A = f_{xxx}(0,0,0)$, $B = f_{yyy}(0,0,0)$, $C = f_{zzz}(0,0,0)$, $D = f_{xxy}(0,0,0)$, $E = f_{xxz}(0,0,0)$, $F = f_{xyy}(0,0,0)$, $G = f_{yyz}(0,0,0)$, $H = f_{zzx}(0,0,0)$, $I = f_{zzy}(0,0,0)$, and $J = f_{xyz}(0,0,0)$. Then (*) becomes

$$P_3(x,y,z) = \frac{1}{6}(Ax^3 + By^3 + Cz^3)$$

$$+\frac{1}{3}(Dx^2y + Ex^2z + Fxy^2 + Gy^2z + Hz^2x + Iyz^2) + \frac{1}{2}Jxyz. \qquad (**)$$

It is believed that a simple shape will be derived by obtaining the relational expression between each alphabet.

Conflict of interest. The authors declare no conflicts of interest.

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