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On The Location of Zeros of Polynomials with Different Complex Coefficients

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ABSTRACT

In this paper we extend Eneström-Kakeya theorem (Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n such that $0 < a_0 \leq a_1 \leq ... \leq a_n$, then all the zeros of P(z) lie in $|z| \leq 1$) for polynomials with complex coefficients the result of [8] have been generlized by relaxing the hypothesis in different ways by considering complex coefficients.

Mathematics Subject Classification: 30C10, 30C15.[2pt]

Keywords: Zeros of polynomial, Eneström-Kakeya theorem.

1. INTRODUCTION

Location of zeros of a polynomial is a long standing classical problem [1,3-5,8,9,11-13]. It is an interesting area of research for engineers as well as mathematicians and many results on the same topic are available in literature. Here we make an attempt to extend some of the known result for real coefficients to complex coefficients. Existing results in the literature also show that there is a need to find bounds for special polynomials, for example, for those having restrictions on the coefficients,there is always a need for refinement of results in this subject. The well known result in the theory of distribution of zeros of polynomials is the following:

Theorem A_1**.** $[2,\ 7]$: *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* n *such that* $0 < a_0 \leq a_1 \leq ... \leq a_{n-1} \leq a_n$ *then all the zeros of P(z) lie in* $|z| \leq 1$.

A.Joyal, G.Labelle and Q. I. Rahman [6] obtained the following generalization, by considering the coefficients to be real, instead of being only positive.

Theorem A_2**.** *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n such that* $a_0 \leq a_1 \leq ... \leq a_{n-1} \leq a_n$ *then all the zeros of P(z) lie in* $|z| \leq \frac{1}{|a_n|}\{a_n - a_0 + |a_0|\}$.

Theorem A_3**.** [10] : *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* n *with real coefficients such that*

$$a_0 \leq a_1 \leq ... \leq a_{m-1} \leq a_m \geq a_{m+1} \geq ... \geq a_{n-1} \geq a_n.$$

Then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|a_n|}\big[2a_m + |a_0| - (a_0 + |a_n|)\big].$$

Theorem A_4**.** [8] : *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* $n \geq 2$ *with real coefficients such that* $a_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} \geq ... \leq a_4 \geq a_3 \leq a_2 \geq a_1 \leq a_0$ *if n is even*

OR

$a_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} \geq ... \geq a_4 \leq a_3 \geq a_2 \leq a_1 \geq a_0$ *if n is odd*
then all the zeros of P(z) lie in
$|z| \leq \frac{1}{|\alpha_n|}\{|a_0| + a_0 + a_n + 2([a_{n-2} + a_{n-4} + ... + a_4 + a_2] - [a_{n-1} + a_{n-3} + ...a_3 + a_1])\}$
if n is even (OR)
$|z| \leq \frac{1}{|\alpha_n|}\{a_n + 2([a_{n-2} + a_{n-4} + ... + a_3 + a_1] - [a_{n-1} + a_{n-3} + ...a_4 + a_2])\}$
if n is odd

In this paper we want to prove the following results.

Theorem 1. *Let* $P(z) = \sum_{j=0}^{n} \alpha_j z^j$ *be a polynomial with complex coefficients of degree* $n \geq 2$ *with* $Re(\alpha_j) = a_j$ *and* $Im(\alpha_j) = b_j$ *for* $j = 0, 1, 2, ..., n$ *such that*

$a_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} \geq ... \leq a_4 \geq a_3 \leq a_2 \geq a_1 \leq a_0$ *and*

$b_n \geq b_{n-1} \leq b_{n-2} \geq b_{n-3} \leq b_{n-4} \geq ... \leq b_4 \geq b_3 \leq b_2 \geq b_1 \leq b_0$ *if n is even*

OR

$a_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} \geq ... \geq a_4 \leq a_3 \geq a_2 \leq a_1 \geq a_0$ *and*

$b_n \geq b_{n-1} \leq b_{n-2} \geq b_{n-3} \leq b_{n-4} \geq ... \geq b_4 \leq b_3 \leq b_2 \leq b_1 \geq b_0$ *if n is odd*
then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|\alpha_n|}\Bigg[|a_0| + a_0 + a_n + 2([a_{n-2} + a_{n-4} + ... + a_4 + a_2] - [a_{n-1} + a_{n-3} + ...a_3 + a_1]) + |b_0| + b_0 + b_n + 2([b_{n-2} + b_{n-4} + ... + b_4 + b_2] - [b_{n-1} + b_{n-3}... + b_3 + b_1])\Bigg]$$ *if n is even*

OR

$$|z| \leq \frac{1}{|\alpha_n|}\Bigg[|a_0| - a_0 + a_n + 2([a_{n-2} + a_{n-4} + ... + a_3 + a_1] - [a_{n-1} + a_{n-3} + ...a_4 + a_2]) + |b_0| - b_0 + b_n + 2([b_{n-2} + b_{n-4} + ... + b_3 + b_1] - [b_{n-1} + b_{n-3}... + b_4 + b_2])\Bigg]$$ *if n is odd*

Corollary 1. *Let $P(z) = \sum_{j=0}^{n} \alpha_j z^j$ be a polynomial with complex coefficients of degree $n \geq 2$ with $Re(\alpha_j) = a_j \geq 0$ and $Im(\alpha_j) = b_j \geq 0$ for $j = 0, 1, 2, ..., n$ such that*

$a_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} \geq ... \leq a_4 \geq a_3 \leq a_2 \geq a_1 \leq a_0$ *and*

$b_n \geq b_{n-1} \leq b_{n-2} \geq b_{n-3} \leq b_{n-4} \geq ... \leq b_4 \geq b_3 \leq b_2 \geq b_1 \leq b_0$ *if n is even*

OR

$a_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} \geq ... \geq a_4 \leq a_3 \geq a_2 \leq a_1 \geq a_0$ *and*

$b_n \geq b_{n-1} \leq b_{n-2} \geq b_{n-3} \leq b_{n-4} \geq ... \geq b_4 \leq b_3 \leq b_2 \leq b_1 \geq b_0$ *if n is odd*
then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|\alpha_n|}\Bigg[a_n + 2([a_{n-2} + a_{n-4} + ... + a_4 + a_2 + a_0] - [a_{n-1} + a_{n-3} + ...a_3 + a_1]) + +b_n + 2([b_{n-2} + b_{n-4} + ... + b_4 + b_2 + b_0] - [b_{n-1} + b_{n-3}... + b_3 + b_1])\Bigg]$$ *if n is even*

OR

$$|z| \leq \frac{1}{|\alpha_n|}\Bigg[a_n + 2([a_{n-2} + a_{n-4} + ... + a_3 + a_1] - [a_{n-1} + a_{n-3} + ...a_4 + a_2]) + b_n + 2([b_{n-2} + b_{n-4} + ... + b_3 + b_1] - [b_{n-1} + b_{n-3}... + b_4 + b_2])\Bigg]$$ *if n is odd*

Remark 1. By taking $a_j \geq 0, b_j \geq 0$ for $j = 0, 1, 2, ..., n$ in the Theorem 1,then it reduces to Corollary 1

Theorem 2. *Let $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ be a polynomial with complex coefficients of degree $n \geq 2$ with $Re(\alpha_i) = a_i$ and $Im(\alpha_i) = b_i$ for $i = 0, 1, 2, 3, ..., n$ such that for some $k \geq 1, \delta \geq 0$,*

$ka_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} .... \leq a_4 \geq a_3 \leq a_2 \geq a_1 \leq a_0 + \delta$ *and*
$kb_n \geq b_{n-1} \leq b_{n-2} \geq b_{n-3} \leq b_{n-4} .... \leq b_4 \geq b_3 \leq b_2 \geq b_1 \leq b_0 + \delta$ *if n is even*

OR

$ka_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} .... \geq a_4 \leq a_3 \geq a_2 \leq a_1 \geq a_0 - \delta$ *and*
$kb_n \geq b_{n-1} \leq b_{n-2} \geq b_{n-3} \leq b_{n-4} .... \geq b_4 \leq b_3 \geq b_2 \leq b_1 \geq b_0 - \delta$ *if n is odd*
then all the zeros of the polynomial $P(z)$ lie in
$|z+k-1| \leq \frac{1}{|\alpha_n|}\{ka_n+|a_0|+a_0+2\delta+2[(a_{n-2}+a_{n-4+}+...+a_4+a_2)-(a_{n-1}+a_{n-3}+...+a_3+a_1)]+kb_n+|b_0|+b_0+2\delta+2[(b_{n-2}+b_{n-4+}+...+b_4+b_2)-(b_{n-1}+b_{n-3}+...+b_3+b_1)]\}$
if n is even *OR*
$|z+k-1| \leq \frac{1}{|\alpha_n|}\{ka_n+|a_0|-a_0+2\delta+2[(a_{n-2}+a_{n-4+}+...+a_3+a_1)-(a_{n-1}+a_{n-3}+...+a_4+a_2)]+kb_n+|b_0|-b_0+2\delta+2[(b_{n-2}+b_{n-4+}+...+b_3+b_1)-(b_{n-1}+b_{n-3}+...+b_4+b_2)]\}$
if n is odd

Corollary 2. *Let $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ be a polynomial with complex coefficients of degree $n \geq 2$ with $Re(\alpha_i) = a_i$ and $Im(\alpha_i) = b_i$ for $i = 0, 1, 2, 3, ..., n$ such that for some $k \geq 1$,*
$ka_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} .... \leq a_4 \geq a_3 \leq a_2 \geq a_1 \leq a_0$ *and*
$kb_n \geq b_{n-1} \leq b_{n-2} \geq b_{n-3} \leq b_{n-4} .... \leq b_4 \geq b_3 \leq b_2 \geq b_1 \leq b_0$ *if n is even*

OR

$ka_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} .... \geq a_4 \leq a_3 \geq a_2 \leq a_1 \geq a_0$ *and*
$kb_n \geq b_{n-1} \leq b_{n-2} \geq b_{n-3} \leq b_{n-4} .... \geq b_4 \leq b_3 \geq b_2 \leq b_1 \geq b_0$ *if n is odd*
then all the zeros of the polynomial $P(z)$ lie in
$|z+k-1| \leq \frac{1}{|\alpha_n|}\{ka_n+|a_0|+a_0+2[(a_{n-2}+a_{n-4+}+...+a_4+a_2)-(a_{n-1}+a_{n-3}+...+a_3+a_1)]+kb_n+|b_0|+b_0+2[(b_{n-2}+b_{n-4+}+...+b_4+b_2)-(b_{n-1}+b_{n-3}+...+b_3+b_1)]\}$
if n is even

OR

$|z+k-1| \leq \frac{1}{|\alpha_n|}\{ka_n+|a_0|-a_0+2[(a_{n-2}+a_{n-4+}+...+a_3+a_1)-(a_{n-1}+a_{n-3}+...+a_4+a_2)]+kb_n+|b_0|-b_0+2[(b_{n-2}+b_{n-4+}+...+b_3+b_1)-(b_{n-1}+b_{n-3}+...+b_4+b_2)]\}$
if n is odd

Remark 2. By taking $\delta = 0$ in the Theorem 2 , then it reduces to Corollary 2

Theorem 3. *Let $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ be a polynomial with complex coefficients degree $n \geq 2$ with $Re(\alpha_i) = a_i$ and $Im(\alpha_i) = b_i$ for $i = 0, 1, 2, ..., n$ such that*
$a_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq a_{n-4} \leq ... \geq a_4 \leq a_3 \geq a_2 \leq a_1 \geq a_0$,
$b_n \leq b_{n-1} \geq b_{n-2} \leq b_{n-3} \geq b_{n-4} \leq ... \geq b_4 \leq b_3 \geq b_2 \leq b_1 \geq b_0$ *if n is even*

OR

$a_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq a_{n-4} \leq ... \leq a_4 \geq a_3 \leq a_2 \geq a_1 \leq a_0$,
$b_n \leq b_{n-1} \geq b_{n-2} \leq b_{n-3} \geq b_{n-4} \leq ... \leq b_4 \geq b_3 \leq b_2 \geq b_1 \leq b_0$ *if n is odd*

then all the zeros of $P(z)$ lie in
$|z| \leq \frac{1}{|\alpha_n|}\big[|a_0| - a_0 - a_n + 2([a_{n-1} + a_{n-3} + ... + a_3 + a_1] - [a_{n-2} + a_{n-4} + ... + a_4 + a_2]) + |b_0| - b_0 - b_n + 2([b_{n-1} + b_{n-3} + ... + b_3 + b_1] - [b_{n-2} + b_{n-4} + ... + b_4 + b_2])\big]$
if n is even

OR

$|z| \leq \frac{1}{|\alpha_n|}\big[|a_0| + a_0 - a_n + 2([a_{n-2} + a_{n-4} + ... + a_3 + a_1] - [a_{n-1} + a_{n-3} + ... + a_4 + a_2]) + |b_0| + b_0 - b_n + 2([b_{n-2} + b_{n-4} + ... + b_3 + b_1] - [b_{n-1} + b_{n-3} + ... + b_4 + b_2])\big]$
if n is odd

Corollary 3. *Let $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ be a polynomial with complex coefficients degree $n \geq 2$ with $Re(\alpha_i) = a_i \geq 0$ and $Im(\alpha_i) = b_i \geq 0$ for $i = 0, 1, 2, ..., n$ such that*
$a_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq a_{n-4} \leq ... \geq a_4 \leq a_3 \geq a_2 \leq a_1 \geq a_0$,
$b_n \leq b_{n-1} \geq b_{n-2} \leq b_{n-3} \geq b_{n-4} \leq ... \geq b_4 \leq b_3 \geq b_2 \leq b_1 \geq b_0$ *if n is even*

OR

$a_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq a_{n-4} \leq ... \leq a_4 \geq a_3 \leq a_2 \geq a_1 \leq a_0$,
$b_n \leq b_{n-1} \geq b_{n-2} \leq b_{n-3} \geq b_{n-4} \leq ... \leq b_4 \geq b_3 \leq b_2 \geq b_1 \leq b_0$ *if n is odd*
then all the zeros of $P(z)$ lie in
$|z| \leq \frac{1}{|\alpha_n|}\big[2([a_{n-1} + a_{n-3} + ... + a_3 + a_1] - [a_{n-2} + a_{n-4} + ... + a_4 + a_2]) - a_n + +2([b_{n-1} + b_{n-3} + ... + b_3 + b_1] - [b_{n-2} + b_{n-4} + ... + b_4 + b_2]) - b_n\big]$ *if n is even*

OR

$|z| \leq \frac{1}{|\alpha_n|}\big[2([a_{n-2} + a_{n-4} + ... + a_3 + a_1 + a_0] - [a_{n-1} + a_{n-3} + ... + a_4 + a_2]) - a_n + +2([b_{n-2} + b_{n-4} + ... + b_3 + b_1 + b_0] - [b_{n-1} + b_{n-3} + ... + b_4 + b_2]) - b_n\big]$ *if n is odd*

Remark 3. By taking $Re(\alpha_i) = a_i \geq 0, Im(\alpha_i) = b_i \geq 0 \ for \ i = 0, 1, 2, .., n$ in the Theorem 3 ,then it reduces to Corollary 3.

Theorem 4. *Let $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ be a polynomial with complex coefficients of degree $n \geq 2$ with $Re(\alpha_i) = a_i$ and $Im(\alpha_i) = b_i$ for $i = 0, 1, 2, ..., n$ such that for $0 \leq r, s \leq 1, \delta > 0, \eta > 0$*
$ra_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq a_{n-4} \leq ... \geq a_4 \leq a_3 \geq a_2 \leq a_1 \geq a_0 - \delta$,
$sb_n \leq b_{n-1} \geq b_{n-2} \leq b_{n-3} \geq b_{n-4} \leq ... \geq b_4 \leq b_3 \geq b_2 \leq b_1 \geq b_0 - \eta$ *if n is even*

OR

$ra_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq a_{n-4} \leq ... \leq a_4 \geq a_3 \leq a_2 \geq a_1 \leq a_0 + \delta$,
$sb_n \leq b_{n-1} \geq b_{n-2} \leq b_{n-3} \geq b_{n-4} \leq ... \leq b_4 \geq b_3 \leq b_2 \geq b_1 \leq b_0 + \eta$ *if n is odd*
then all the zeros of the polynomial $P(z)$ lie in
$|z| \leq \frac{1}{|\alpha_n|}\{2\delta + |a_n| + |a_0| - a_0 - r(|a_n| + a_n) + 2[(a_{n-1} + a_{n-3} + ... + a_3 + a_1) - (a_{n-2} + a_{n-4} + ... + a_4 + a_2)] + 2\eta + |b_n| + |b_0| - b_0 - s(|b_n| + b_n) + 2[(b_{n-1} + b_{n-3} + ... + b_3 + b_1) - (b_{n-2} + b_{n-4} + ... + b_4 + b_2)]\}$ *if n is even*

OR

$|z| \leq \frac{1}{|\alpha_n|}\{2\delta + |a_n| + |a_0| + a_0 - r(|a_n| + a_n) + 2[(a_{n-1} + a_{n-3} + ... + a_4 + a_2) - (a_{n-2} + a_{n-4} + ... + a_3 + a_1)] + 2\eta + |b_n| + |b_0| + b_0 - s(|b_n| + b_n) + 2[(b_{n-1} + b_{n-3} + ... + b_4 + b_2) - (b_{n-2} + b_{n-4} + ... + b_3 + b_1)]\}$ *if n is odd*

Corollary 4. *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree* $n \geq 2$ *with* $Re(\alpha_i) = a_i$ *and* $Im(\alpha_i) = b_i$ *for* $i = 0, 1, 2, ..., n$ *such that for* $0 \leq r, s \leq 1$
$ra_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq a_{n-4} \leq ... \geq a_4 \leq a_3 \geq a_2 \leq a_1 \geq a_0,$
$sb_n \leq b_{n-1} \geq b_{n-2} \leq b_{n-3} \geq b_{n-4} \leq ... \geq b_4 \leq b_3 \geq b_2 \leq b_1 \geq b_0$ *if n is even*

OR

$ra_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq a_{n-4} \leq ... \leq a_4 \geq a_3 \leq a_2 \geq a_1 \leq a_0,$
$sb_n \leq b_{n-1} \geq b_{n-2} \leq b_{n-3} \geq b_{n-4} \leq ... \leq b_4 \geq b_3 \leq b_2 \geq b_1 \leq b_0$ *if n is odd*
then all the zeros of the polynomial $P(z)$ *lie in*
$|z| \leq \frac{1}{|\alpha_n|}\{|a_n| + |a_0| - a_0 - r(|a_n| + a_n) + 2[(a_{n-1} + a_{n-3} + ... + a_3 + a_1) - (a_{n-2} + a_{n-4} + ... + a_4 + a_2)] + |b_n| + |b_0| - b_0 - s(|b_n| + b_n) + 2[(b_{n-1} + b_{n-3} + ... + b_3 + b_1) - (b_{n-2} + b_{n-4} + ... + b_4 + b_2)]\}$ *if n is even*

OR

$|z| \leq \frac{1}{|\alpha_n|}\{|a_n| + |a_0| + a_0 - r(|a_n| + a_n) + 2[(a_{n-1} + a_{n-3} + ... + a_4 + a_2) - (a_{n-2} + a_{n-4} + ... + a_3 + a_1)] + |b_n| + |b_0| + b_0 - s(|b_n| + b_n) + 2[(b_{n-1} + b_{n-3} + ... + b_4 + b_2) - (b_{n-2} + b_{n-4} + ... + b_3 + b_1)]\}$ *if n is odd*

Remark 4. By taking $\delta = 0,\ \eta = 0$ in the Theorem 4, then it reduces to Corollary 4.

2. PROOF OF THE THEOREMS

Proof of Theorem 1. Let $P(z) = \alpha_n z^n + \alpha_{n-1} z^{n-1} + ... + \alpha_3 z^3 + \alpha_2 z^2 + \alpha_1 z + \alpha_0$ be a polynomial with complex of degree $n \geq 2\ with\ \alpha_j = a_j + ib_j for\ \ j = 0, 1, 2, ..., n.$ Then consider the polynomial

$$\begin{aligned}
Q(z) =&(1 - z)P(z)\\
=& - \alpha_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1}\\
&+ (\alpha_{n-2} - \alpha_{n-3})z^{n-2} + ... + (\alpha_1 - \alpha_0)z + \alpha_0.\\
=& - \alpha_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1}\\
&+ (a_{n-2} - a_{n-3})z^{n-2} + ... + (a_1 - a_0)z + a_0\\
&+ i\big\{(b_n - b_{n-1})z^n + (b_{n-1} - b_{n-2})z^{n-1}\\
&+ (b_{n-2} - b_{n-3})z^{n-2} + ... + (b_1 - b_0)z + b_0\big\}.\\
If\ |z|{<}1\ then\ &\frac{1}{|z|^{n-i}}{>}1\ for\ i = 0, 1, 2, ..., n - 1.
\end{aligned}$$

$$
\begin{aligned}
Now\ |Q(z)| \geq\ & |\alpha_n||z|^{n+1} - \{|a_n - a_{n-1}||z|^n + |a_{n-1} - a_{n-2}||z|^{n-1} \\
& + |a_{n-2} - a_{n-3}||z|^{n-2} + ... + |a_3 - a_2||z|^3 \\
& + |a_2 - a_1||z|^2 + |a_1 - a_0||z| + |a_0| \\
& + |b_n - b_{n-1}||z|^n + |b_{n-1} - b_{n-2}||z|^{n-1} \\
& + |b_{n-2} - b_{n-3}||z|^{n-2} + ... \\
& + |b_3 - b_2||z|^3 + |b_2 - b_1||z|^2 + |b_1 - b_0||z| + |b_0|\} \\
\geq & |\alpha_n||z|^n \Bigg[|z| - \frac{1}{|\alpha_n|}\{|a_n - a_{n-1}| \\
& + \frac{|a_{n-1} - a_{n-2}|}{|z|} + \frac{|a_{n-2} - a_{n-3}|}{|z|^2} + ... + \frac{|a_3 - a_2|}{|z|^{n-3}} \\
& + \frac{|a_2 - a_1|}{|z|^{n-2}} + \frac{|a_1 - a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n} + |b_n - b_{n-1}| \\
& + \frac{|b_{n-1} - b_{n-2}|}{|z|^1} + \frac{|b_{n-2} - b_{n-3}|}{|z|^2} + ... \\
& + \frac{|b_3 - b_2|}{|z|^{n-3}} + \frac{|b_2 - b_1|}{|z|^{n-2}} + \frac{|b_1 - b_0|}{|z|^n} + \frac{|b_0|}{|z|^n}\} \Bigg] \\
\geq & |\alpha_n||z|^n \Bigg[|z| - \frac{1}{|\alpha_n|}\{|a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| \\
& + |a_{n-2} - a_{n-3}| + ... + |a_3 - a_2| + |a_2 - a_1| + |a_1 - a_0| \\
& + |a_0| + |b_n - b_{n-1}| + |b_{n-1} - b_{n-2}| + |b_{n-2} - b_{n-3}| + ... + |b_3 - b_2| \\
& + |b_2 - b_1| + |b_1 - b_0| + |b_0|\} \Bigg] \\
\geq & |\alpha_n||z|^n \Bigg[|z| - \frac{1}{|\alpha_n|}\{(a_n - a_{n-1}) \\
& + (a_{n-1} - a_{n-2}) + (a_{n-2} - a_{n-3}) + ... + (a_2 - a_3) + (a_2 - a_1) \\
& + (a_0 - a_1) + |a_0| + (b_n - b_{n-1}) + (b_{n-1} - b_{n-2}) \\
& + (b_{n-2} - b_{n-3}) + ... + (b_2 - b_3) + (b_2 - b_1) \\
& + (b_0 - b_1) + |b_0|\} \Bigg]\ if\ n\ is\ even\ , by\ hypothesis \\
= & |\alpha_n||z|^n \Bigg[|z| - \frac{1}{|\alpha_n|} \\
& \{|a_0| + a_0 + a_n + 2([a_{n-2} + a_{n-4} + ... + a_4 + a_2] \\
& - [a_{n-1} + a_{n-3} + ...a_3 + a_1]) \\
& + |b_0| + b_0 + b_n + 2([b_{n-2} + b_{n-4} + ... + b_4 + b_2] \\
& - [b_{n-1} + b_{n-3} ... + b_3 + b_1])\} \Bigg] > 0
\end{aligned}
$$

$$if\ |z|>\{|a_0|+a_0+a_n+2([a_{n-2}+a_{n-4}+....+a_4+a_2]$$
$$-[a_{n-1}+a_{n-3}+....a_3+a_1])+|b_0|+b_0+b_n$$
$$+2([b_{n-2}+b_{n-4}+...+b_4+b_2]-[b_{n-1}+b_{n-3}...+b_3+b_1])\}$$

$This\ shows\ that\ if\ |z|>1\ then\ |Q(z)|>0\ whenver$

$$|z|>\frac{1}{|\alpha_n|}\{|a_0|+a_0+a_n+2([a_{n-2}+a_{n-4}+....+a_4+a_2]$$
$$-[a_{n-1}+a_{n-3}+....a_3+a_1])+|b_0|+b_0+b_n$$
$$+2([b_{n-2}+b_{n-4}+...+b_4+b_2]-[b_{n-1}+b_{n-3}...+b_3+b_1])\}$$

$Hence\ all\ the\ zeros\ of\ Q(z)\ with\ |z|>1\ lie\ in$

$$|z|<\frac{1}{|\alpha_n|}\{|a_0|+a_0+a_n+2([a_{n-2}+a_{n-4}+....+a_4+a_2]$$
$$-[a_{n-1}+a_{n-3}+....a_3+a_1])+|b_0|+b_0+b_n$$
$$+2([b_{n-2}+b_{n-4}+...+b_4+b_2]-[b_{n-1}+b_{n-3}...+b_3+b_1])\}\ if\ n\ is\ even$$

But the zeros of Q(z) whose modules is less than or equal to 1 already satisfy the above inquality. Since all the zeros of Q(z) lie in the circle defined by the above ineauality , we conclude that the proof of the Theorem 1 is complete, if n is even. Similarly we can also prove for odd degree polynomials. For this we can rearrange the terms of the given polynomial and compute as above. That is if n is odd then all the zeros of P(z) lie in

$$|z|\le\frac{1}{|\alpha_n|}\Big[|a_0|-a_0+a_n+2([a_{n-2}+a_{n-4}+....+a_3+a_1]$$
$$-[a_{n-1}+a_{n-3}+....a_4+a_2])$$
$$+|b_0|-b_0+b_n+2([b_{n-2}+b_{n-4}+...+b_3+b_1]$$
$$-[b_{n-1}+b_{n-3}...+b_4+b_2])\Big].$$

This completes the proof of the Theorem 1.

Proof of Theorem 2. Proof of theorem 2 is similar to the proof of theorem 1.

Proof of Theorem 3. Proof of theorem 3 is similar to the proof of theorem 1.

Proof of Theorem 4. Proof of theorem 4 is similar to the proof of theorem 1.

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