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Existence and Uniqueness Results for BVP of Nonlinear Fractional Volterra-Fredholm Integro-Differential Equation

Muhammed F. Younis[1], **Ayoob M. Abed**[2], **and Ahmed A. Hamoud**[3]

[1,2]*Department of Mathematics, Thi Qar Directorates of Education, Ministry of Education, Iraq.*
[3]*Department of Mathematics, Taiz University, Taiz, Yemen.*

Abstract

In this paper, we establish sufficient conditions for the existence and uniqueness of solutions for a class of boundary value problems (BVPs) with nonlocal conditions for nonlinear fractional Volterra-Fredholm integro-differential equations. The results are established by the application of the Arzela-Ascoli theorem, Banach and Krasnoselkii fixed point theorems.

Keywords: Volterra-Fredholm integro-differential equations; Caputo fractional derivatives; fixed point method; nonlocal conditions.

Mathematics Subject Classification (2010): 26A33, 47H10, 45J05.

1. INTRODUCTION

Many applicable models in physical, nonlinear dynamics, biological and chemical sciences can be described successfully using integro-differential equations. For example, the biological population models rely on the delayed Volterra integro-differential equations, systems of integro-differential equations characterize the evolution of nuclear reactor in a continuous medium, and many other problems in viscoelasticity, mechanics and economics as well [1, 8, 17, 19, 20]. Furthermore, converting initial and boundary value problems yields these types of equations [6, 13, 15, 21, 22].

* Corresponding author: muhammedfaleh@yahoo.com;ayob.moha75@gmail.com;ahmed.hamoud@taiz.edu.ye

Over the last decades, mathematical modeling has been supported by the field of fractional calculus, with several successful results and fractional operators shown to be an excellent tool to describe the hereditary properties of various materials and processes. Recently, this combination has gained a large amount of importance, mainly because fractional differential equations have become powerful tools for the modeling of several complex phenomena in numerous seemingly diverse and widespread fields of science and engineering; see, for instance, in [9–12, 14, 16–19].

In [7], Jaiswal and Bahuguna studied the existence and uniqueness of solutions for fractional order differential equations and nonlocal boundary condition of the form:

$$\begin{aligned} &D^{\alpha}u(t) = f(t, u(t) \\ &u(0) = h(u),\ \ u(T) = k(u),\ t \in [0, T], \end{aligned}$$

where D^{α} is the Caputo fractional derivative of order $\alpha \in (1, 2]$.

We also refer to the work in [1] discussed the existence and uniqueness of solutions for Caputo fractional Volterra-Fredholm integro-differential equation with mixed conditions in Banach space. Ahmad et al. [2] and references therein give details of recent work on the properties of solutions of sequential fractional differential equations. Ahmad et al. [3] considers solutions of fractional differential equations with non-separated type integral boundary conditions. Ahmad et al. [4] the Krasnoselskii fixed point theorem and the contraction mapping principle are used to prove the existence of solutions of the nonlinear Langevin equation with two fractional orders for a number of different intervals. Ahmad et al. [5] discusses the existence and uniqueness of solutions of nonlinear fractional differential equations with three-point integral boundary conditions. In [20], concerned with the existence and uniqueness of solutions for a boundary value problem of nonlinear fractional differential equations with three-point fractional integral boundary conditions given by

$$\begin{aligned} &D^{\alpha}u(t) = g(t, u(t) \\ &u(0) = 0,\ \ u(1) = \alpha[I^{\beta}u](b),\ t \in [0, 1], \end{aligned}$$

where D^{α} is the Caputo fractional derivative of order $\alpha,\ \alpha \in (1, 2],\ \ I^{\beta}$ is the Riemann-Liouville integral of order β.

Motivated by these results, here, we extend the previous available results of the literature to a class of fractional Volterra-Fredholm integro-differential equations.

This article is concerned with the existence and uniqueness of solutions for a boundary value problem of nonlinear fractional Volterra-Fredholm integro-differential equations given by

$$D^{\alpha}u(t) = f(t, u(t)) + \int_0^t A(t, s, u(s))ds + \int_0^T B(t, s, u(s))ds, \tag{1}$$

$$u(0) = h(u), \;\; u(T) = k(u), \; t \in J := [0, T], \tag{2}$$

where D^{α} is the Caputo fractional derivative of order $\alpha \in (1, 2]$. Let $Y = C^2(J, X)$ be a Banach space of all functions $u(t)$ having at most continuous second derivatives from a compact interval J into a Banach space X. Let $D = \{(t, s) : 0 \le s \le t \le T\}$ be subset of $\mathbb{R}^2$, and the nonlinear functions f, A, B, h and k satisfy the following hypotheses:
(H1) $f : J \times Y \longrightarrow Y, \; A, B : D \times Y \longrightarrow Y$ are continuous functions and there exists a positive constant C such that

$$\|f(t, u) - f(t, v)\| \le C\|u - v\|,$$
$$\|A(t, s, u) - A(t, s, v)\| \le C\|u - v\|,$$
$$\|B(t, s, u) - B(t, s, v)\| \le C\|u - v\|,$$

for any $t \in J, (t, s) \in D, \;\; u, v \in Y$. Moreover, let $F_{\delta} = \sup_{t \in J} \|f(t, 0)\|, \; A_{\delta} = \sup_{t,s \in D} \|A(t, s, 0)\|, \; B_{\delta} = \sup_{t,s \in D} \|B(t, s, 0)\|$, and $L = \max\{F_{\delta}, A_{\delta}, B_{\delta}, C\}$.
(H2) $h : Y \longrightarrow Y$, and $k : Y \longrightarrow Y$ are continuous functions such that

$$\|h(u) - h(v)\| \le C\|u - v\|,$$
$$\|k(u) - k(v)\| \le C\|u - v\|,$$

for any $u, v \in Y$.

2. PRELIMINARIES

In this section, we recall the necessary theory that is used throughout the work in order to obtain new results.

Definition 2.1 *[17] The left-sided Riemann-Liouville fractional integral of order $\alpha > 0$ with a lower limit a for a function $x : [a, +\infty) \longrightarrow \mathbb{R}$ is defined as*

$$I^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha-1} x(s)ds,$$

provided the right hand side is defined almost everywhere (a.e.) on $[a, +\infty)$.

Remark 2.1 *If $a = 0$, then we write $I^{\alpha}f(t) = (g_{\alpha} * f)(t)$, where*

$$g_{\alpha}(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} t^{\alpha-1}, & t > 0, \\ 0, & t \le 0, \end{cases}$$

and, as usual, $*$ *denotes the convolution of functions. Note that* $\lim_{\alpha\longrightarrow 0^+} g_\alpha(t) = \delta(t)$ *with* δ *the delta Dirac function.*

Definition 2.2 *[17] The left-sided Riemann-Liouville fractional derivative of order* $\alpha > 0,\ n-1 \leq \alpha < n,\ n \in \mathbb{N},$ *for a function* $x : [a,+\infty) \longrightarrow \mathbb{R}$, *is defined by*

$$^{L}D^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t \frac{1}{(t-s)^{\alpha-n+1}}x(s)ds,\ t > a.$$

provided the right hand side is defined a.e. on $[a,+\infty)$.

Definition 2.3 *[17] The left-sided Caputo's fractional derivative of order* $\alpha > 0,\ n-1 < \alpha < n,\ n \in \mathbb{N},$ *for a function* $x : [a,+\infty) \longrightarrow \mathbb{R}$, *is defined by*

$$\begin{aligned} D^{\alpha}x(t) &= \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{1}{(t-s)^{\alpha-n+1}}x^{(n)}(s)ds \\ &= I^{n-\alpha}x^{(n)}(t),\ t > a. \end{aligned}$$

provided the right hand side is defined a.e. on $[a,+\infty)$.

The identity

$$(I^{\alpha}D^{(\alpha)}f)(t) = f(t) + a + bt, \tag{3}$$

where $t \in J,\ a,b$ are constants and other properties of the fractional operators used in the general theory of fractional differential equations can be found in [17, 18].

3. MAIN RESULTS

In this section, we shall give an existence and uniqueness results of Eq.(1), with the condition (2).

Lemma 3.1 *Let* $1 < \alpha \leq 2,$ *and* $u \in C(J,X)$ *is called a solution of the problem* $\Longleftrightarrow$ u *satisfies*

$$\begin{aligned} D^{\alpha}u(t) &= f(t) + \int_0^t A(t,s)ds, \\ u(0) &= h(u),\ \ u(T) = k(u),\ t \in J := [0,T], \end{aligned} \tag{4}$$

$\Longleftrightarrow$ u *satisfies*

$$u(t) = \Big(\frac{T-t}{T}\Big)h(u) + \frac{t}{T}k(u) - \frac{t}{T}I^{\alpha}v(T) + I^{\alpha}v(t), \tag{5}$$

where $I^{\alpha}v(t) = f(t) + \int_0^t A(t,s)ds$ *is a fractional integrable function of order* α.

Proof. Applying the fractional integral operator I^α to both sides of Eq. (4), and using the identity (3), we get

$$\begin{aligned} I^\alpha D^\alpha u(t) &= I^\alpha f(t) + I^\alpha \int_0^t A(t,s)ds, \\ u(t) + a + bt &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big(\int_0^s A(s,r)dr \Big) ds. \end{aligned}$$

Now, if $t = 0$, we have $a = -h(u)$, and if $t = T$, we have

$$k(u) - h(u) + bT = \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \Big(\int_0^s A(s,r)dr \Big) ds,$$

which implies that

$$b = \frac{h(u)}{T} - \frac{k(u)}{T} + \frac{1}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s)ds + \frac{1}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \Big(\int_0^s A(s,r)dr \Big) ds.$$

Therefore,

$$\begin{aligned} u(t) &= h(u) + \frac{tk(u)}{T} - \frac{th(u)}{T} - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s)ds \\ &\quad - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \Big(\int_0^s A(s,r)dr \Big) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big(\int_0^s A(s,r)dr \Big) ds \\ &= \Big(\frac{T-t}{T} \Big) h(u) + \frac{t}{T} k(u) \\ &\quad - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \Big(f(s) + \int_0^s A(s,r)dr \Big) ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big(f(s) + \int_0^s A(s,r)dr \Big) ds, \end{aligned}$$

which is Eq. (5). On the other hand, applying the fractional differential operator $D^{(\alpha)}$ to both sides of Eq. (5), it is easily to get Eq. (4).

In view of Lemma 3.1, Eq. (1) is equivalent to the integral equation

$$u(t) = \Big(\frac{T-t}{T} \Big) h(u) + \frac{t}{T} k(u) - \frac{t}{T} (I^\alpha F(u))(T) + (I^\alpha F(u))(t), \tag{6}$$

where

$$F(u) = f(s, u(s)) + \int_0^s A(s, r, u(r))dr + \int_0^T B(s, r, u(r))dr$$

is a fractional integrable (of order α) nonlinear operator. The operator F satisfies the following estimates

$$\begin{aligned} \|(I^\alpha F(u))(t)\| &\leq (I^\alpha \|F(u)-F(0)\|)(t)+(I^\alpha\|F(0)\|)(t) \\ &\leq \frac{Lt^\alpha}{\Gamma(\alpha+1)}\Big(1+\frac{t}{\alpha+1}\Big)(1+\|u\|), \end{aligned}$$

and

$$\|I^\alpha(F(u)-F(v))(t)\| \leq \frac{Lt^\alpha}{\Gamma(\alpha+1)}\Big(1+\frac{t}{\alpha+1}\Big)\|u-v\|,$$

for every $u, v \in Y,\ t \in J.$

We prove the existence of the fractional nonlinear integro-differential equation (1) by using the well-known Banach fixed point theorem. The following condition is essential to get the contraction property.

(H3) Let $0<q<1$, and r be a positive finite real number such that

$$\begin{aligned} & q \geq L\Big[1+\frac{2T^\alpha}{\Gamma(\alpha+1)}\Big(1+\frac{T}{\alpha+1}\Big)\Big] \\ & r \geq (1-q)^{-1}\Big(\|h(0)\|+\|k(0)\|++\frac{2LT^\alpha}{\Gamma(\alpha+1)}\Big(1+\frac{T}{\alpha+1}\Big)\Big). \end{aligned}$$

(H4) The functions $f: J\times Y \longrightarrow Y$, and $A, B : D \times Y \longrightarrow Y$ are jointly continuous and there exists a positive constant L such that

$$\|f(t,u)| + t\sup_{s\in J}\|A(t,s,u)\| + t\sup_{s\in J}\|B(t,s,u)\| \leq L,$$

for all $(t,u) \in J \times Y$.

Theorem 3.1 *If the hypotheses (H1)-(H3) are satisfled, then the BVP* (1)-(2) *has a unique solution on J.*

Proof. Deflne the operator $\Omega : Y \longrightarrow Y$ by

$$(\Omega u)(t) = \Big(\frac{T-t}{T}\Big)h(u)+\frac{t}{T}k(u)-\frac{t}{T}(I^\alpha F(u))(T)+(I^\alpha F(u))(t).$$

We show that Ω has a fixed point on $B_r := \{u\in Y : \|u\| \leq r\}$. This fixed point is then a solution of BVP (1)-(2).

Firstly, we show that $\Omega B_r \subset B_r$. Let $u \in B_r$, then

$$\begin{aligned}
\|\Omega u(t)\| &\leq \Big\|\Big(\frac{T-t}{T}\Big)h(u)\Big\| + \Big\|\frac{t}{T}k(u)\Big\| + \Big\|\frac{t}{T}(I^\alpha F(u))(T)\Big\| + \Big\|(I^\alpha F(u))(t)\Big\| \\
&\leq \Big(\frac{T-t}{T}\Big)\|h(u)\| + \frac{t}{T}\|k(u)\| + \frac{t}{T}\|(I^\alpha F(u))(T)\| + \|(I^\alpha F(u))(t)\| \\
&\leq \Big(\frac{T-t}{T}\Big)\|h(0)\| + C\Big(\frac{T-t}{T}\Big)\|u\| + C\frac{t}{T}\|u\| + \frac{t}{T}\|k(0)\| \\
&\quad + \frac{Lt^\alpha}{\Gamma(\alpha+1)}\Big(1+\frac{t}{\alpha+1}\Big)(1+\|u\|) \\
&\quad + \frac{tLT^\alpha}{T\Gamma(\alpha+1)}\Big(1+\frac{T}{\alpha+1}\Big)(1+\|u\|) \\
&\leq \|h(0)\| + \|k(0)\| + \frac{2LT^\alpha}{\Gamma(\alpha+1)}\Big(1+\frac{T}{\alpha+1}\Big) \\
&\quad + L\Big(1+\frac{2T^\alpha}{\Gamma(\alpha+1)}\Big(1+\frac{T}{\alpha+1}\Big)\Big)\|u\| \\
&\leq (1-q)r + qr \\
&= r.
\end{aligned}$$

Hence, the operator Ω maps B_r into itself. Next, we prove that Ω is a contraction mapping on B_r. Let $u, v \in B_r$, then

$$\begin{aligned}
\|\Omega u(t) - \Omega v(t)\| &= \Big\|\Big(\frac{T-t}{T}\Big)h(u) + \frac{t}{T}k(u) - \frac{t}{T}(I^\alpha F(u))(T) + (I^\alpha F(u))(t) \\
&\quad -\Big(\frac{T-t}{T}\Big)h(v) - \frac{t}{T}k(v) + \frac{t}{T}(I^\alpha F(v))(T) - (I^\alpha F(v))(t)\Big\| \\
&\leq \Big(\frac{T-t}{T}\Big)\|h(u)-h(v)\| + \frac{t}{T}\|k(u)-k(v)\| \\
&\quad + \frac{t}{T}(I^\alpha\|F(v)-F(u)\|)(T) + (I^\alpha\|F(u)-F(v)\|)(t) \\
&\leq L\Big(\frac{T-t}{T}\Big)\|u-v\| + L\frac{t}{T}\|u-v\| + \frac{LT^\alpha}{\Gamma(\alpha+1)}\Big(1+\frac{T}{\alpha+1}\Big)\|u-v\| \\
&\quad + \frac{Lt^\alpha}{\Gamma(\alpha+1)}\Big(1+\frac{t}{\alpha+1}\Big)\|u-v\| \\
&\leq L\Big(1+\frac{2T^\alpha}{\Gamma(\alpha+1)}\Big(1+\frac{T}{\alpha+1}\Big)\Big)\|u-v\| \\
&\leq q\|u-v\|.
\end{aligned}$$

Hence, the operator Ω has a unique fixed point which is a solution to the BVP (1)-(2).

Theorem 3.2 *If the hypotheses (H2) and (H4) are satisfied, and if $C < 1$, then the BVP* (1)-(2) *has a solution on J.*

Proof. Let $r \geq (1-C)^{-1}\Big(\|h(0)\| + \|k(0)\| + \frac{2LT^\alpha}{\Gamma(\alpha+1)}\Big)\Delta$ Define the operators Ψ_1 and Ψ_2 on the compact set $B_r = \{u \in Y : \|u\| \leq r\} \subset Y$ by

$$\begin{aligned}\Psi_1 u(t) &= \Big(\frac{T-t}{T}\Big)h(u) + \frac{t}{T}k(u),\\ \Psi_2 u(t) &= (I^\alpha F(u))(t) - \frac{t}{T}(I^\alpha F(u))(T).\end{aligned}$$

We observe that

$$\begin{aligned}(I^\alpha F(u))(t) &\leq I^\alpha\|f(t,u(t))\| + I^\alpha\Big(\int_0^t \|A(t,s,u(s))\|ds + \int_0^T \|B(t,s,u(s))\|ds\Big)\\ &\leq \frac{LT^\alpha}{\Gamma(\alpha+1)},\end{aligned}$$

hence,

$$\|\Psi_2 u(t)\| \leq \frac{2LT^\alpha}{\Gamma(\alpha+1)}, \tag{7}$$

and

$$\begin{aligned}\|\Psi_1 u(t) + \Psi_2 v(t)\| &\leq \|\Psi_1 u(t)\| + \|\Psi_2 v(t)\|\\ &= \|\Big(\frac{T-t}{T}\Big)h(u) + \frac{t}{T}k(u)\|\\ &\quad + \|(I^\alpha F(v))(t) - \frac{t}{T}(I^\alpha F(v))(T)\|\\ &\leq \frac{(T-t)}{T}\|h(0)\| + \frac{t}{T}\|k(0)\| + \frac{2LT^\alpha}{\Gamma(\alpha+1)}\\ &\quad + \frac{Ct}{T}\|u\| + \frac{C(T-t)}{T}\|u\|.\end{aligned}$$

Therefore, if $u, v \in B_r$, then $\Psi_1 u + \Psi_2 v \in B_r$. On the other hand, it is easily to show that the operator Ψ_1 is a contraction. Indeed, since

$$\begin{aligned}\|\Psi_1 u(t) - \Psi_1 v(t)\| &\leq \Big(\frac{T-t}{T}\Big)\|h(u) - h(v)\| + \frac{t}{T}\|k(u) - k(v)\|\\ &\leq C\Big(\frac{T-t}{T}\Big)\|u-v\| + C\frac{t}{T}\|u-v\|\\ &= C\|u-v\|.\end{aligned}$$

By the hypothesis (H4), the operator Ψ_2 is continuous and by the inequality (7), it is uniformly bounded on B_r. For the equicontinuity of $\Psi_2 v(t)$, let $t_1, t_2 \in J$, and $v \in B_r$, we have

$$\begin{aligned}
&\|(I^\alpha F(v))(t_1) - (I^\alpha F(v))(t_2)\| \\
=\ & \frac{1}{\Gamma(\alpha)}\| \int_0^{t_1} (t_1 - s)^{\alpha-1} f(s, v(s))ds + \int_0^{t_1} (t_1 - s)^{\alpha-1}\Big(\int_0^s A(s, r, u(r))dr \\
&+ \int_0^T \|B(s, r, u(r))dr\Big) ds - \int_0^{t_2} (t_2 - s)^{\alpha-1} f(s, v(s))ds - \int_0^{t_2} (t_2 - s)^{\alpha-1} \\
&\times\Big(\int_0^s A(s, r, u(r))dr + \int_0^T \|B(s, r, u(r))dr\Big) ds\| \\
\leq\ & \frac{L}{\Gamma(\alpha)} \int_0^{t_1} |(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}|ds + \frac{L}{\Gamma(\alpha)} \int_{t_1}^{t_2} |(t_2 - s)^{\alpha-1}|ds \\
\leq\ & \frac{L}{\Gamma(\alpha + 1)}(2|t_2 - t_1|^\alpha + |t_2^\alpha - t_1^\alpha|)
\end{aligned}$$

hence by (H4) one can get

$$\begin{aligned}
\|\Psi_2 v(t_1) - \Psi_2 v(t_2)\| &\leq \|(I^\alpha F(v))(t_1) - (I^\alpha F(v))(t_2)\| \\
&\quad + \|\frac{t_2}{T}(I^\alpha F(v))(T) - \frac{t_1}{T}(I^\alpha F(v))(T)\| \\
&\leq \frac{L}{\Gamma(\alpha + 1)}(2|t_2 - t_1|^\alpha + |t_2^\alpha - t_1^\alpha|) + |t_2 - t_1|\frac{LT^\alpha}{\Gamma(\alpha + 2)} \\
&= \frac{L}{\Gamma(\alpha + 1)}(2|t_2 - t_1|^\alpha + |t_2^\alpha - t_1^\alpha| + T^{\alpha-1}|t_2 - t_1|) \\
&\quad \longrightarrow 0 \ \text{ as } \ t_2 \longrightarrow t_1,
\end{aligned}$$

which gives the equicontinuity of $\Psi_2 v(t)$, so $\Psi_2(B_r)$ is relatively compact. By the Arzela-Ascoli theorem, Ψ_2 is compact. Hence by the Krasnoselkii theorem there exists a solution to the problem (1)-(2).

4. CONCLUDING REMARKS

In this paper, we applied Banach and Krasnoselkii's fixed point theorems to investigate the existence and uniqueness of solution of a class of fractional Volterra-Fredholm integro-differential equations. The obtained existence and uniqueness results were subject to an appropriate set of sufficient conditions. As a future direction of research, it would be desirable to consider the study of Ψ-Hilfer fractional nonlocal nonlinear stochastic systems involving almost sectorial operators and impulsive effects, generalizing the current work. Another open line of research consists of the development of numerical methods to approximate solutions.

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