

Uniqueness Theorems Concerning Homogeneous Differential Polynomials of L-functions and Weakly Weighted Sharing

Nirmal Kumar Datta¹ and Nintu Mandal*²

¹*Department of Physics, Suri Vidyasagar College, Suri, Birbhum-731101,
West Bengal, India.*

²*Department of Mathematics, Chandernagore College, Chandernagore,
Hooghly-712136, West Bengal, India.*

Abstract

In this paper, we prove some uniqueness results when a polynomial and a homogeneous differential polynomial of an L-function weakly share a rational function. Our results improve and generalize some earlier results due to Mandal, Datta [10].

Key words and phrases: L-function, meromorphic function, uniqueness, weakly weighted sharing, homogeneous differential polynomial.

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1. INTRODUCTION

In 1992 a model for L-functions is introduced by Selberg. The study of value distributions of L-functions is mainly concerned with the set $\{z \in \mathbb{C} : L(z) = a\}$ where $a \in \mathbb{C}$.

A meromorphic function L is said to be an L-function in the Selberg class if it satisfy the following properties.

- (i) $L(z)$ can be expressed as a Dirchlet series $L(z) = \sum_{m=1}^{\infty} a(m)/m^z$.
- (ii) $|a(m)| = O(m^\epsilon)$, for any $\epsilon > 0$.

*Corresponding Author (Nintu Mandal): nintu311209@gmail.com

(iii) There exists a nonnegative integer n such that $(z - 1)^n L(z)$ becomes an entire function of finite order.

(iv) Every L-function satisfies the functional equation

$$\lambda_L(z) = \omega \overline{\lambda_L(1 - \bar{z})},$$

where

$$\lambda_L(z) = L(z) A^z \prod_{j=1}^n \Gamma(\eta_j z + \nu_j)$$

with positive real numbers A, η_j and complex numbers ν_j, ω with $Re(\nu_j) \geq 0$ and $|\omega| = 1$.

(v) $L(z)$ satisfies $L(z) = \prod_p L_p(z)$, where $L_p(z) = \exp(\sum_{n=1}^{\infty} b(p^n)/p^{nz})$ with $b(p^n) = O(p^{n\theta})$ for some $\theta < 1/2$ and p denotes prime number.

If L satisfies (i) - (iv) then we say that L is an L-function in the extended Selberg class. In this paper, by an L-function we mean an L-function in the extended Selberg class with $a(1) = 1$. Here we use the standard notations and definitions of the value distribution theory [4].

2. PRELIMINARIES

Let ξ and ψ be a meromorphic function defined in the complex plane \mathbb{C} . If $\xi - \sigma$ and $\psi - \sigma$ have same set of zeros ignoring(counting) multiplicities, then we say that ξ and ψ share σ IM (CM). If $\frac{1}{\xi}$ and $\frac{1}{\psi}$ share 0 CM (IM), we say that ξ and ψ share ∞ CM (IM). We denote by $S(r, \xi)$ any quantity satisfying $S(r, \xi) = o(T(r, \xi))$ as $r \rightarrow \infty$, outside a possible exceptional set of finite linear measure. A meromorphic function ρ is said to be a small function of ξ if $T(r, \rho) = S(r, \xi)$. Let $\alpha \in \mathbb{C} \cup \{\infty\}$ and n be a positive integer. We denote by $E_n(\alpha; \xi)(\bar{E}_n(\alpha; \xi))$ the set of all zeros of $\xi - \alpha$ with multiplicities not exceeding n , where zeros are counted according to their multiplicities(ignoring multiplicities). We denote by $S(\xi)$ the set of all the small functions of ξ .

With the help of CM sharing Steuding [13] proved the following theorem in 2007.

Theorem 2.1. [13] *Let L and H be two L-functions with $a(1) = 1$ and $d \neq \infty$ be a complex number. If L and H share d CM, then $L \equiv H$.*

Definition 2.1. [5, 6] *Let ξ and χ be two meromorphic functions defined in the complex plane and m be an integer (≥ 0) or infinity. For $d \in \mathbb{C} \cup \{\infty\}$ we denote by $E_m(d; \xi)$*

the set of all zeros of $\xi - d$ where an zero of multiplicity t is counted t times if $t \leq m$ and $m + 1$ times if $t > m$. If $E_m(d; \xi) = E_m(d; \chi)$, we say that ξ, χ share the value d with weight m . We write ξ, χ share (d, m) to mean that ξ, χ share the value d with weight m .

Definition 2.2. [10] Let ξ be a meromorphic function defined in the complex plane and ρ be a small function of ξ . Then we denote by $E_m(\rho; \xi)$, $\overline{E}_m(\rho; \xi)$ and $E_m(\rho; \xi)$ the sets $E_m(0; \xi - \rho)$, $\overline{E}_m(0; \xi - \rho)$ and $E_m(0; \xi - \rho)$ respectively.

With the help of weighted sharing Wu and Hu [14] proved the following uniqueness result in 2015.

Theorem 2.2. [14] Let L and H be two L -functions, and let $\alpha, \beta \in \mathbb{C}$ be two distinct values. Take two positive integers m_1, m_2 with $m_1 m_2 > 1$. If $E_{m_1}(\alpha, L) = E_{m_1}(\alpha, H)$, and $E_{m_2}(\alpha, L) = E_{m_2}(\alpha, H)$, then $L \equiv H$.

In 2018 Hao and Chen [3] proved the following uniqueness theorem considering weighted sharing .

Theorem 2.3. [3] Let L be an L -function and F be a meromorphic function defined in the complex plane \mathbb{C} with finitely many poles. Let $\alpha_1, \alpha_2 \in \mathbb{C}$ be distinct and m_1, m_2 be positive integers such that $m_1 m_2 > 1$. If $E_{m_j}(\alpha_j, F) = E_{m_j}(\alpha_j, L)$, $j = 1, 2$, then $L \equiv F$.

Considering weighted sharing in 2020 Datta and Mandal [2] proved the following uniqueness theorem.

Theorem 2.4. [2] Let ξ be a nonconstant meromorphic function and L be a nonconstant L -function. If $E_0(0; \xi) = E_0(0; L)$, $E_1(1; \xi) = E_1(1; L)$ and $N(r; 0; \xi) + N(r; 1; \xi) = S(r; \xi)$ then either $L \equiv \xi$ or $T(r; L) = N(r; 0; L| \leq 2) + S(r; L)$ and $T(r; \xi) = N(r; 0; L| \leq 1) + S(r; L)$.

With the help of small function sharing in 2020 Mandal and Datta [10] proved the following theorem.

Theorem 2.5. [10] Let L be a nonconstant L -function and ρ be a small function of L such that $\rho \not\equiv 0, \infty$. If $\overline{E}_4(\rho; L) = \overline{E}_4(\rho; (L^m)^{(k)})$, $E_2(\rho; L) = E_2(\rho; (L^m)^{(k)})$ and

$$2N_{2+k}(r, 0; L^m) \leq (\sigma + o(1))T(r, L), \tag{2.1}$$

where $m \geq 1, k \geq 1$ are integers and $0 < \sigma < 1$, then $L \equiv (L^m)^{(k)}$.

Definition 2.3. Let ξ , ψ and χ be nonconstant meromorphic functions. We denote by $N_E(r, \chi; \xi, \psi)$ the counting function of all common zeros of $\xi - \chi$ and $\psi - \chi$ with same multiplicities. We denote by $\overline{N}_E(r, \chi; \xi, \psi)$ the corresponding reduced counting function.

Definition 2.4. Let ξ , ψ be nonconstant meromorphic functions and χ be a meromorphic function. We denote by $N_0(r, \chi; \xi, \psi)$ the counting function of all common zeros of $\xi - \chi$ and $\psi - \chi$. We denote by $\overline{N}_0(r, \chi; \xi, \psi)$ the corresponding reduced counting function.

Definition 2.5. [9] Let ξ , ψ be nonconstant meromorphic functions and $\rho \in S(\xi) \cap S(\psi)$. If

$$\overline{N}(r, \rho; \xi) + \overline{N}(r, \rho; \psi) - 2\overline{N}_E(r, \rho; \xi, \psi) = S(r, \xi) + S(r, \psi),$$

we say that ξ and ψ share ρ “CM”. If

$$\overline{N}(r, \rho; \xi) + \overline{N}(r, \rho; \psi) - 2\overline{N}_0(r, \rho; \xi, \psi) = S(r, \xi) + S(r, \psi),$$

we say that ξ and ψ share ρ “IM”.

Definition 2.6. Let ξ , ψ be nonconstant meromorphic functions and χ be a meromorphic function. We say that ξ and ψ share χ “CM” (“IM”) if $\xi - \chi$ and $\psi - \chi$ share 0 “CM” (“IM”).

Definition 2.7. [8]. Let ξ be a meromorphic function defined in the complex plane. Let n be a positive integer and $\alpha \in \mathbb{C} \cup \{\infty\}$. By $N(r, \alpha; \xi | \leq n)$ we denote the counting function of the α points of ξ with multiplicity $\leq n$ and by $\overline{N}(r, \alpha; \xi | \leq n)$ the reduced counting function. Also by $N(r, \alpha; \xi | \geq n)$ we denote the counting function of the α points of ξ with multiplicity $\geq n$ and by $\overline{N}(r, \alpha; \xi | \geq n)$ the reduced counting function. We define

$$N_n(r, \alpha; \xi) = \overline{N}(r, \alpha; \xi) + \overline{N}(r, \alpha; \xi | \geq 2) + \cdots + \overline{N}(r, \alpha; \xi | \geq n).$$

Definition 2.8. [8]. Let ξ and ψ be two meromorphic functions defined in the complex plane. Then we denote by $N(r, \psi; \xi | \leq m)$, $\overline{N}(r, \psi; \xi | \leq m)$, $N(r, \psi; \xi | \geq m)$, $\overline{N}(r, \psi; \xi | \geq m)$, $N_m(r, \psi; \xi)$ etc. the counting functions $N(r, 0; \xi - \psi | \leq m)$, $\overline{N}(r, 0; \xi - \psi | \leq m)$, $N(r, 0; \xi - \psi | \geq m)$, $\overline{N}(r, 0; \xi - \psi | \geq m)$, $N_m(r, 0; \xi - \psi)$ etc. respectively.

Definition 2.9. Let two nonconstant meromorphic functions ξ and ψ share a value α “IM” and m be a positive integer or ∞ . We denote by $\overline{N}_E(r, \alpha; \xi, \psi | \leq m)$

($\overline{N}_E(r, \alpha; \xi, \psi | \geq m)$) the counting function of the α -points of ξ and ψ with multiplicities not greater than m (not less than m) and the multiplicities with respect to ξ is equal to the multiplicities with respect to ψ , where each α -point is counted once only.

Definition 2.10. Let two nonconstant meromorphic functions ξ and ψ share a value α “IM” and m be a positive integer or ∞ . We denote by $\overline{N}_0(r, \alpha; \xi, \psi | \geq m)$ the counting function of the common α -points of ξ and ψ with multiplicities not less than m , where each α -point is counted once only.

Definition 2.11. Let two nonconstant meromorphic functions ξ, ψ share a meromorphic functions χ “IM”. By $\overline{N}_E(r, \chi; \xi, \psi | \leq m)$ and $\overline{N}_0(r, \chi; \xi, \psi | \geq m)$ we denote the counting functions $\overline{N}_E(r, 0; \xi - \chi, \psi - \chi | \leq m)$ and $\overline{N}_0(r, 0; \xi - \chi, \psi - \chi | \geq m)$ respectively.

Definition 2.12. [9] Let $\rho \in S(\xi) \cap S(\psi)$ and two nonconstant meromorphic functions ξ, ψ share ρ “IM”. If m is a positive integer or ∞ and

$$\overline{N}(r, \rho; \xi | \leq m) - \overline{N}_E(r, \rho; \xi, \psi | \leq m) = S(r, \xi)$$

$$\overline{N}(r, \rho; \psi | \leq m) - \overline{N}_E(r, \rho; \xi, \psi | \leq m) = S(r, \psi)$$

$$\overline{N}(r, \rho; \xi | \geq m + 1) - \overline{N}_0(r, \rho; \xi, \psi | \geq m + 1) = S(r, \xi)$$

$$\overline{N}(r, \rho; \psi | \geq m + 1) - \overline{N}_0(r, \rho; \xi, \psi | \geq m + 1) = S(r, \psi)$$

or $m = 0$ and

$$\overline{N}(r, \rho; \xi) - \overline{N}_0(r, \rho; \xi, \psi) = S(r, \xi)$$

$$\overline{N}(r, \rho; \psi) - \overline{N}_0(r, \rho; \xi, \psi) = S(r, \psi),$$

then we say ξ and ψ weakly share ρ with weight m . We write ξ and ψ share “ (ρ, m) ” to mean that ξ and ψ weakly share ρ with weight m .

Definition 2.13. Let two nonconstant meromorphic functions ξ, ψ share a meromorphic functions χ “IM”. Also let m be a positive integer or ∞ . We say that ξ, ψ share “ (χ, m) ” if $\xi - \chi, \psi - \chi$ share “ $(0, m)$ ”.

Definition 2.14. [1] Let ξ be a meromorphic function, t_{ij} ($i = 0, 1, 2, \dots, n$, $j = 1, 2, \dots, m$) be nonnegative integers and $\rho_j \in S(\xi)$ such that $\rho_j \neq 0$ for $j = 1, 2, \dots, m$. We define the differential polynomial $P(\xi)$ of ξ by $P(\xi) = \sum_{j=1}^m M_j(\xi)$, where $M_j(\xi) = \rho_j \prod_{i=0}^n (\xi^{(i)})^{t_{ij}}$. The numbers $\bar{d}(P) = \max_{1 \leq j \leq m} \sum_{i=0}^n t_{ij}$ and $\underline{d}(P) = \min_{1 \leq j \leq m} \sum_{i=0}^n t_{ij}$ are called degree and lower degree of $P(\xi)$ respectively. If $\bar{d}(P) = \underline{d}(P) = d$ (say), then we say that $P(\xi)$ is a homogeneous differential polynomial of degree d generated by ξ . We define Q by $Q = \max_{1 \leq j \leq m} \sum_{i=0}^n i t_{ij}$.

Definition 2.15. [7] Let ξ be a meromorphic function and k be a positive integer. We denote by $N_{\otimes}(r, 0; \xi^{(k)})$ ($\bar{N}_{\otimes}(r, 0; \xi^{(k)})$) the counting function (reduced counting function) of those zeros of $\xi^{(k)}$ which are not the zeros of $\xi(\xi - 1)$.

Definition 2.16. [11] Let two nonconstant meromorphic functions ξ and ψ share a value α "IM". We denote by $\bar{N}(r, \alpha; \xi | > \psi)$ the counting function of the α -points of ξ and ψ with multiplicities with respect to ξ is greater than the multiplicities with respect to ψ , where each α -point is counted once only.

Definition 2.17. Let two nonconstant meromorphic functions ξ and ψ share a value α "IM" and k be a positive integer. We denote by $\bar{N}(r, \alpha; \xi, \psi | \xi > \psi = k)$ the counting function of the common α -points of ξ and ψ with multiplicities with respect to ξ is greater than the multiplicities with respect to ψ and multiplicities with respect to ψ is equal to k , where each α -point is counted once only.

Now the following questions come naturally.

Question 2.1. Is it possible to consider rational function sharing in place of small function sharing in theorem 2.5?

Question 2.2. Is it possible to consider polynomial of L and homogeneous differential polynomial generated by L in place of L and $(L^m)^{(k)}$ respectively in theorem 2.5?

3. MAIN RESULTS

Let L be a nonconstant L-function and $a_i, b_j \in S(L)$, $i = 0, 1, 2, \dots, t$, $j = 0, 1, 2, \dots, s$. Henceforth we denote by $R(z)$ the function $R(z) = \frac{\sum_{i=0}^t a_i z^i}{\sum_{j=0}^s b_j z^j}$, where $a_t \neq 0$ and $b_s \neq 0$. Also we denote by $P(L)$ a homogeneous differential polynomial of degree d generated by L as defined in definition 2.14.

Using the concept of weakly weighted sharing we try to solve Questions 2.1, 2.2 and prove the following theorems.

Theorem 3.1. *Let L be a nonconstant L -function and $p(z)$ be a polynomial of degree $\lambda \geq 1$ with $p(0) = 0$. Let $P(L)$ be a homogeneous differential polynomial of degree d generated by L . If $p(L)$ and $P(L)$ share “ $(R(z), l)$ ” and one of the following holds*

(i) $l=0$ and

$$2\overline{N}(r, 0; p(L)) + N_2(r, 0; p(L)) + 2N_{2+n}(r, 0 : L) < (\lambda + o(1))T(r, L) \quad (3.1)$$

(ii) $l=1$ and

$$\frac{1}{2}\overline{N}(r, 0; p(L)) + N_2(r, 0; p(L)) + N_{2+n}(r, 0 : L) < (\lambda + o(1))T(r, L), \quad (3.2)$$

then $p(L) \equiv P(L)$.

4. LEMMAS

In this section we present some necessary lemmas.

Henceforth we denote by Ω the function defined by

$$\Omega = \left(\frac{\Phi''}{\Phi'} - \frac{2\Phi'}{\Phi - 1} \right) - \left(\frac{\Psi''}{\Psi'} - \frac{2\Psi'}{\Psi - 1} \right)$$

Lemma 4.1. [13]. *Let L be an L -function with degree q . Then $T(r, L) = \frac{q}{\pi}r \log r + O(r)$.*

Lemma 4.2. [10]. *Let L be an L -function. Then $N(r, \infty; L) = S(r, L) = O(\log r)$.*

Lemma 4.3. [16]. *Let $\xi(z) = \frac{\alpha_0 + \alpha_1 z + \dots + \alpha_t z^t}{\beta_0 + \beta_1 z + \dots + \beta_s z^s}$ be a nonconstant rational function defined in the complex plane \mathbb{C} , where $\alpha_0, \alpha_1, \dots, \alpha_t (\neq 0)$ and $\beta_0, \beta_1, \dots, \beta_s (\neq 0)$ are complex constants. Then $T(r, \xi) = \max\{t, s\} \log r + O(1)$.*

Lemma 4.4. *Let ξ be a nonconstant meromorphic function defined in the complex plane and $P(\xi)$ be a homogeneous differential polynomial of degree d generated by ξ defined as in definition 2.14. If $P(\xi) \not\equiv 0$ then for any positive integer k*

(i) $N_k(r, 0; P(\xi)) \leq N_{k+n}(r, 0; \xi) + T(r, P(\xi)) - dT(r, \xi) + S(r, \xi)$

(ii) $N_k(r, 0; P(\xi)) \leq N_{k+n}(r, 0; \xi) + Q\overline{N}(r, \infty; \xi) + S(r, \xi)$.

Proof. Using first fundamental theorem we have

$$\begin{aligned}
N_k(r, 0; P(\xi)) &\leq N(r, 0; P(\xi)) - \sum_{l=k}^{\infty} \overline{N}(r, 0; P(\xi)| \geq l) \\
&= T(r, P(\xi)) - m(r, 0; P(\xi)) - \sum_{l=k}^{\infty} \overline{N}(r, 0; P(\xi)| \geq l) + O(1) \\
&\leq T(r, P(\xi)) + m(r, \infty; \frac{P(\xi)}{\xi^d}) - m(r, 0; \xi^d) - \sum_{l=k}^{\infty} \overline{N}(r, 0; P(\xi)| \geq l) + O(1) \\
&\leq T(r, P(\xi)) - dT(r, \xi) + N(r, 0; \xi^d) - \sum_{l=k}^{\infty} \overline{N}(r, 0; P(\xi)| \geq l) + S(r, \xi) \\
&\leq T(r, P(\xi)) - dT(r, \xi) + N_{(k+n)d}(r, 0; \xi^d) + \sum_{l=(k+n+1)d}^{\infty} \overline{N}(r, 0; \xi^d| \geq l) \\
&\quad - \sum_{l=k}^{\infty} \overline{N}(r, 0; P(\xi)| \geq l) + S(r, \xi) \\
&\leq T(r, P(\xi)) - dT(r, \xi) + N_{k+n}(r, 0; \xi) + S(r, \xi). \tag{4.1}
\end{aligned}$$

This proves (i).

Now

$$\begin{aligned}
T(r, P(\xi)) &= N(r, \infty; P(\xi)) + m(r, \infty; P(\xi)) \\
&\leq N(r, \infty; P(\xi)) + m(r, \infty; \xi^d) + m(r, \infty; \frac{P(\xi)}{\xi^d}) \\
&\leq N(r, \infty; P(\xi)) + dm(r, \infty; \xi) + S(r, \xi) \\
&\leq dN(r, \infty; \xi) + Q\overline{N}(r, \infty; \xi) + dm(r, \infty; \xi) + S(r, \xi) \\
&\leq dT(r, \xi) + Q\overline{N}(r, \infty; \xi) + S(r, \xi). \tag{4.2}
\end{aligned}$$

From (4.1) and (4.2) we have

$$N_k(r, 0; P(\xi)) \leq N_{k+n}(r, 0; \xi) + Q\overline{N}(r, \infty; \xi) + S(r, \xi).$$

This proves (ii).

This completes the proof. \square

Lemma 4.5. [12] Let ξ be a nonconstant meromorphic function and let $\Phi(\xi) = \frac{\sum_{i=0}^t \alpha_i \xi^i}{\sum_{j=0}^s \beta_j \xi^j}$ be irreducible rational function in ξ with coefficients α_i and β_j , $i = 0, 1, 2, \dots, t$, $j = 0, 1, 2, \dots, s$ where $\alpha_t \neq 0$ and $\beta_s \neq 0$. Then $T(r, \Phi(\xi)) = \max\{t, s\}T(r, \xi) + S(r, \xi)$.

Lemma 4.6. Let ξ be a nonconstant meromorphic function and $\mu_i, \nu_j \in S(\xi)$, $i = 0, 1, 2, \dots, t$, $j = 0, 1, 2, \dots, s$. Also let $H(\xi) = \frac{\sum_{i=0}^t \mu_i \xi^i}{\sum_{j=0}^s \nu_j \xi^j}$, where $\mu_t \neq 0$ and $\nu_s \neq 0$. Then $T(r, H(\xi)) = \max\{t, s\}T(r, \xi) + S(r, \xi)$.

Proof. Since $\mu_i, \nu_j \in S(\xi)$, $i = 0, 1, 2, \dots, t$, $j = 0, 1, 2, \dots, s$, therefore $T(r, \mu_i) = S(r, \xi)$, $i = 0, 1, \dots, t$ and $T(r, \nu_j) = S(r, \xi)$, $j = 0, 1, \dots, s$. Hence the result follows by lemma 4.5 □

Lemma 4.7. *Let L be a nonconstant L-function. Then $T(r, R(z)) = S(r, L)$.*

Proof. By lemma 4.1, lemma 4.3 and lemma 4.6 we get the required result. □

Lemma 4.8. [1] *Let $P(\xi)$ be a homogeneous differential polynomial of degree d generated by a nonconstant meromorphic function ξ as defined in definition 2.14. Then*

$$N(r, 0; \frac{P(\xi)}{f^d}) \leq Q(\overline{N}(r, 0; \xi) + \overline{N}(r, \infty; \xi)) + S(r, \xi).$$

Lemma 4.9. [9] *Let l be a nonnegative integer and two nonconstant meromorphic functions Φ and Ψ share “(1, l)”. If $\Omega \not\equiv 0$ and $2 \leq l \leq \infty$, then*

$$T(r, \Phi) \leq N_2(r, \infty; \Phi) + N_2(r, 0; \Phi) + N_2(r, \infty; \Psi) + N_2(r, 0; \Psi) + S(r, \Phi) + S(r, \Psi)$$

and

$$T(r, \Psi) \leq N_2(r, \infty; \Phi) + N_2(r, 0; \Phi) + N_2(r, \infty; \Psi) + N_2(r, 0; \Psi) + S(r, \Phi) + S(r, \Psi).$$

Lemma 4.10. [15] *Let two nonconstant meromorphic functions Φ and Ψ share “(1, 0)”, then*

$$(i) \overline{N}(r, 1; \Phi | > \Psi) + 2\overline{N}(r, 1; \Psi | > \Phi) + N(r, 1; \Phi, \Psi | \geq 2) - \overline{N}(r, 1, \Phi, \Psi | \Phi > \Psi = 1) - \overline{N}(r, 1, \Psi, \Phi | \Psi > \Phi = 1) \leq N(r, 1; \Psi) - \overline{N}(r, 1; \Psi).$$

$$(ii) \overline{N}(r, 1; \Phi | > \Psi) \leq \overline{N}(r, \infty; \Phi) + \overline{N}(r, 0; \Phi) + S(r, \Phi).$$

$$(iii) \overline{N}(r, 1; \Phi, \Psi | \Phi > \Psi = 1) \leq \overline{N}(r, 0; \Phi) + \overline{N}(r, \infty; \Phi) - \overline{N}_{\otimes}(r, 0; \Phi^{(1)}) + S(r, \Phi).$$

$$(iv) \overline{N}(r, 1; \Psi, \Phi | \Psi > \Phi = 1) \leq \overline{N}(r, 0; \Psi) + \overline{N}(r, \infty; \Psi) - \overline{N}_{\otimes}(r, 0; \Psi^{(1)}) + S(r, \Psi).$$

Lemma 4.11. [15] *Let two nonconstant meromorphic functions Φ and Ψ share “(1, 1)”, then*

$$(i) 2\overline{N}(r, 1; \Phi | > \Psi) + 2\overline{N}(r, 1; \Psi | > \Phi) + N(r, 1; \Phi, \Psi | \geq 2) - \overline{N}(r, 1, \Phi, \Psi | \Phi > \Psi = 2) \leq N(r, 1; \Psi) - \overline{N}(r, 1; \Psi).$$

$$(ii) \bar{N}(r, 1; \Phi, \Psi | \Phi > \Psi = 2) \leq \frac{1}{2} \bar{N}(r, 0; \Phi) + \frac{1}{2} \bar{N}(r, \infty; \Phi) - \frac{1}{2} \bar{N}_{\otimes}(r, 0; \Phi^{(1)}) + S(r, \Phi).$$

Lemma 4.12. *Let L be a nonconstant L -function and $p(z)$ be a polynomial of degree $\lambda \geq 1$ with $p(0) = 0$. Let $P(L)$ be a homogeneous differential polynomial of degree d generated by L . Let $\Phi(z) = \frac{p(L(z))}{R(z)}$ and $\Psi(z) = \frac{P(L(z))}{R(z)}$. If $p(L)$ and $P(L)$ share “ $(R(z), 0)$ ” and $\Omega \neq 0$, then*

$$2\bar{N}(r, 0; p(L)) + N_2(r, 0; p(L)) + 2N_{2+n}(r, 0 : L) \geq (\lambda + o(1))T(r, L).$$

Proof. Clearly Φ and Ψ share “ $(1, 0)$ ” except for the zeros and poles of $R(z)$.

By lemma 4.7 we have $R(z)$ is a small function of L .

Hence

$$\begin{aligned} N(r, \infty; \Omega) &\leq \bar{N}(r, \infty; \Phi) + \bar{N}(r, 0; \Phi | \geq 2) + \bar{N}(r, 0; \Psi | \geq 2) \\ &+ \bar{N}(r, 1; \Phi | \geq \Psi) + \bar{N}(r, 1; \Psi | \geq \Phi) + \bar{N}_{\otimes}(r, 0; \Phi^{(1)}) \\ &+ \bar{N}_{\otimes}(r, 0; \Psi^{(1)}) + S(r, \Phi) + S(r, \Psi) \end{aligned} \quad (4.3)$$

and

$$\bar{N}_E(r, 1; \Phi, \Psi | \leq 1) \leq N(r, \infty; \Omega) + S(r, \Phi). \quad (4.4)$$

Using (4.3), (4.4) and lemma 4.10 we have

$$\begin{aligned} \bar{N}(r, 1; \Phi) + \bar{N}(r, 1; \Psi) &\leq \bar{N}_E(r, 1; \Phi, \Psi | \leq 1) + \bar{N}(r, 1; \Phi | > \Psi) + \bar{N}(r, 1; \Psi | > \Phi) \\ &+ \bar{N}_E(r, 1; \Phi, \Psi | \geq 2) + \bar{N}(r, 1; \Psi) + S(r, \Phi) + S(r, \Psi) \\ &\leq \bar{N}_E(r, 1; \Phi, \Psi | \leq 1) + N(r, 1; \Psi) - \bar{N}(r, 1; \Psi | > \Phi) \\ &+ \bar{N}(r, 1; \Phi, \Psi | \Phi > \Psi = 1) + \bar{N}(r, 1; \Psi, \Phi | \Psi > \Phi = 1) \\ &+ S(r, \Phi) + S(r, \Psi) \\ &\leq \bar{N}(r, \infty; \Phi) + \bar{N}(r, 0; \Phi | \geq 2) + \bar{N}(r, 0; \Psi | \geq 2) \\ &+ \bar{N}(r, 1; \Phi | \geq \Psi) + \bar{N}_{\otimes}(r, 0; \Phi^{(1)}) \\ &+ \bar{N}_{\otimes}(r, 0; \Psi^{(1)}) + N(r, 1; \Psi) + \bar{N}(r, 1; \Phi, \Psi | \Phi > \Psi = 1) \\ &+ \bar{N}(r, 1; \Psi, \Phi | \Psi > \Phi = 1) + S(r, \Phi) + S(r, \Psi). \end{aligned} \quad (4.5)$$

Using Nevanlinna second fundamental theorem we get from (4.5), lemma 4.2, lemma

4.4, lemma 4.7 and lemma 4.10

$$\begin{aligned}
 T(r, \Phi) + T(r, \Psi) &\leq \bar{N}(r, \infty; \Phi) + \bar{N}(r, \infty; \Psi) + \bar{N}(r, 0; \Phi) + \bar{N}(r, 0; \Psi) + \bar{N}(r, 1; \Phi) \\
 &+ \bar{N}(r, 1; \Psi) - \bar{N}_{\otimes}(r, 0; \Phi^{(1)}) - \bar{N}_{\otimes}(r, 0; \Psi^{(1)}) + S(r, \Phi) + S(r, \Psi) \\
 &\leq N(r, 1; \Psi) + 4\bar{N}(r, \infty; \Phi) + 2\bar{N}(r, \infty; \Psi) + 3\bar{N}(r, 0; \Phi) \\
 &+ 2\bar{N}(r, 0; \Psi) + \bar{N}(r, 0; \Phi \geq 2) + \bar{N}(r, 0; \Psi \geq 2) + S(r, \Phi) + S(r, \Psi) \\
 &\leq N(r, 1; \Psi) + 4\bar{N}(r, \infty; L) + 2\bar{N}(r, \infty; L) + 3\bar{N}(r, 0; \Phi) \\
 &+ 2\bar{N}(r, 0; \Psi) + \bar{N}(r, 0; \Phi \geq 2) + \bar{N}(r, 0; \Psi \geq 2) + S(r, L) \\
 &\leq T(r, \Psi) + 2\bar{N}(r, 0; p(L)) + N_2(r, 0; p(L)) + \bar{N}(r, 0; P(L)) \\
 &+ N_2(r, 0; P(L)) + S(r, L) \\
 &\leq T(r, \Psi) + 2\bar{N}(r, 0; p(L)) + N_2(r, 0; p(L)) + 2N_{2+n}(r, 0; L) \\
 &+ 2Q\bar{N}(r, \infty; L) + S(r, L) \\
 &\leq T(r, \Psi) + 2\bar{N}(r, 0; p(L)) + N_2(r, 0; p(L)) \\
 &+ 2N_{2+n}(r, 0; L) + S(r, L).
 \end{aligned} \tag{4.6}$$

Using lemma 4.7 we have from (4.6)

$$2\bar{N}(r, 0; p(L)) + N_2(r, 0; p(L)) + 2N_{2+n}(r, 0; L) \geq (\lambda + o(1))T(r, L).$$

This completes the proof of the lemma. □

Lemma 4.13. *Let L be a nonconstant L -function and $p(z)$ be a polynomial of degree $\lambda \geq 1$ with $p(0) = 0$. Let $P(L)$ be a homogeneous differential polynomial of degree d generated by L . Let $\Phi(z) = \frac{p(L(z))}{R(z)}$ and $\Psi(z) = \frac{P(L(z))}{R(z)}$. If $p(L)$ and $P(L)$ share “ $(R(z), 1)$ ” and $\Omega \not\equiv 0$, then*

$$\frac{1}{2}\bar{N}(r, 0; p(L)) + N_2(r, 0; p(L)) + N_{2+n}(r, 0; L) \geq (\lambda + o(1))T(r, L).$$

Proof. Clearly Φ and Ψ share “ $(1, 1)$ ” except for the zeros and poles of $R(z)$.

By lemma 4.7 we have $R(z)$ is a small function of L .

Hence using Lemma 4.2 and Lemma 4.7 we get

$$\begin{aligned}
 N(r, \infty; \Omega) &\leq \bar{N}(r, \infty; \Phi) + \bar{N}(r, 0; \Phi \geq 2) + \bar{N}(r, 0; \Psi \geq 2) \\
 &+ \bar{N}_{\otimes}(r, 0; \Phi^{(1)}) + \bar{N}_{\otimes}(r, 0; \Psi^{(1)}) + S(r, \Phi) + S(r, \Psi) \\
 &\leq \bar{N}(r, \infty; L) + \bar{N}(r, 0; \Phi \geq 2) + \bar{N}(r, 0; \Psi \geq 2) \\
 &+ \bar{N}_{\otimes}(r, 0; \Phi^{(1)}) + \bar{N}_{\otimes}(r, 0; \Psi^{(1)}) + S(r, L) \\
 &\leq \bar{N}(r, 0; \Phi \geq 2) + \bar{N}(r, 0; \Psi \geq 2) + \bar{N}_{\otimes}(r, 0; \Phi^{(1)}) \\
 &+ \bar{N}_{\otimes}(r, 0; \Psi^{(1)}) + S(r, L)
 \end{aligned} \tag{4.7}$$

and

$$N(r, 1; \Phi | = 1) \leq N(r, 0; \Omega) + S(r, \Phi) \leq N(r, \infty; \Omega) + S(r, \Phi). \quad (4.8)$$

Using (4.7), (4.8) and Lemma 4.2, Lemma 4.7 and Lemma 4.11 we have

$$\begin{aligned} \overline{N}(r, 1; \Phi) + \overline{N}(r, 1; \Psi) &\leq \overline{N}_E(r, 1; \Phi, \Psi | \leq 1) + \overline{N}(r, 1; \Phi | > \Psi) + \overline{N}(r, 1; \Psi | > \Phi) \\ &+ \overline{N}_E(r, 1; \Phi, \Psi | \geq 2) + \overline{N}(r, 1; \Psi) + S(r, \Phi) + S(r, \Psi) \\ &\leq \overline{N}_E(r, 1; \Phi, \Psi | \leq 1) + N(r, 1; \Psi) - \overline{N}(r, 1; \Phi | > \Psi) \\ &- \overline{N}(r, 1; \Psi | > \Phi) + \overline{N}(r, 1, \Phi, \Psi | \Phi > \Psi = 2) + S(r, \Phi) + S(r, \Psi) \\ &\leq \overline{N}_E(r, 1; \Phi, \Psi | \leq 1) + N(r, 1; \Psi) - \overline{N}(r, 1; \Phi | > \Psi) \\ &- \overline{N}(r, 1; \Psi | > \Phi) + \frac{1}{2}\overline{N}(r, 0, \Phi) + \frac{1}{2}\overline{N}(r, \infty, \Phi) \\ &- \frac{1}{2}\overline{N}_\otimes(r, 0; \Phi^{(1)}) + S(r, \Phi) + S(r, \Psi) \\ &\leq \overline{N}(r, \infty; \Phi) + \overline{N}(r, 0; \Phi | \geq 2) + \overline{N}(r, 0; \Psi | \geq 2) - \overline{N}(r, 1; \Phi | \geq \Psi) \\ &- \overline{N}(r, 1; \Psi | \geq \Phi) + \overline{N}_\otimes(r, 0; \Phi^{(1)}) + \overline{N}_\otimes(r, 0; \Psi^{(1)}) + N(r, 1; \Psi) \\ &+ \frac{1}{2}\overline{N}(r, 0, \Phi) + \frac{1}{2}\overline{N}(r, \infty, \Psi) - \frac{1}{2}\overline{N}_\otimes(r, 0; \Phi^{(1)}) + S(r, \Phi) + S(r, \Psi) \\ &\leq \overline{N}(r, 0; \Phi | \geq 2) + \overline{N}(r, 0; \Psi | \geq 2) - \overline{N}(r, 1; \Phi | \geq \Psi) \\ &- \overline{N}(r, 1; \Psi | \geq \Phi) + \overline{N}_\otimes(r, 0; \Phi^{(1)}) + \overline{N}_\otimes(r, 0; \Psi^{(1)}) \\ &+ N(r, 1; \Psi) + \frac{1}{2}\overline{N}(r, 0, \Phi) - \frac{1}{2}\overline{N}_\otimes(r, 0; \Phi^{(1)}) + S(r, L). \end{aligned} \quad (4.9)$$

Using Nevanlinna second fundamental theorem we get from (4.9), lemma 4.2, lemma 4.4, lemma 4.7 and lemma 4.11

$$\begin{aligned} T(r, \Phi) + T(r, \Psi) &\leq \overline{N}(r, \infty; \Phi) + \overline{N}(r, \infty; \Psi) + \overline{N}(r, 0; \Phi) \\ &+ \overline{N}(r, 0; \Psi) + \overline{N}(r, 1; \Phi) + \overline{N}(r, 1; \Psi) \\ &- \overline{N}_\otimes(r, 0; \Phi^{(1)}) - \overline{N}_\otimes(r, 0; \Psi^{(1)}) + S(r, \Phi) + S(r, \Psi) \\ &\leq N(r, 1; \Psi) + \overline{N}(r, \infty; \Phi) + \overline{N}(r, \infty; \Psi) + \frac{3}{2}\overline{N}(r, 0; \Phi) \\ &+ \overline{N}(r, 0; \Psi) + \overline{N}(r, 0; \Phi | \geq 2) + \overline{N}(r, 0; \Psi | \geq 2) + S(r, L) \\ &\leq T(r, \Psi) + \overline{N}(r, \infty; L) + \overline{N}(r, \infty; L) + \frac{3}{2}\overline{N}(r, 0; p(L)) \\ &+ \overline{N}(r, 0; P(L)) + \overline{N}(r, 0; p(L) | \geq 2) + \overline{N}(r, 0; P(L) | \geq 2) + S(r, L) \\ &\leq T(r, \Psi) + \frac{1}{2}\overline{N}(r, 0; p(L)) + N_2(r, 0; p(L)) + N_2(r, 0; P(L)) + S(r, L) \\ &\leq T(r, \Psi) + \frac{1}{2}\overline{N}(r, 0; p(L)) + N_2(r, 0; p(L)) + N_{2+n}(r, 0; L) \\ &+ Q\overline{N}(r, \infty; L) + S(r, L) \\ &\leq T(r, \Psi) + \frac{1}{2}\overline{N}(r, 0; p(L)) + N_2(r, 0; p(L)) \\ &+ N_{2+n}(r, 0; L) + S(r, L). \end{aligned} \quad (4.10)$$

Using lemma 4.7 we have from (4.10)

$$\frac{1}{2}\overline{N}(r, 0; p(L)) + N_2(r, 0; p(L)) + N_{2+n}(r, 0; L) \geq (\lambda + o(1))T(r, L).$$

This completes the proof of the lemma. □

5. PROOF OF THE MAIN RESULT

Proof of Theorem 3.1

Let $\Phi(z) = \frac{p(L(z))}{R(z)}$ and $\Psi(z) = \frac{P(L(z))}{R(z)}$.

By lemma 4.7 we have $T(r, R(z)) = S(r, L)$. Now we have to consider the following two cases

Case 1 Let $\Omega \neq 0$.

Then by Lemma 4.12 and Lemma 4.13 we arrive at a contradiction.

Case 2 Let $\Omega \equiv 0$.

Hence

$$\left(\frac{\Phi''}{\Phi'} - \frac{2\Phi'}{\Phi - 1}\right) - \left(\frac{\Psi''}{\Psi'} - \frac{2\Psi'}{\Psi - 1}\right) = 0. \tag{5.1}$$

Integrating (5.1) we get

$$\Psi = \frac{(D + 1)\Phi + (C - D - 1)}{D\Phi + (C - D)}, \tag{5.2}$$

where $C \neq 0$ and D are constants.

Now we have to consider the following three subcases.

Subcase 2.1 Let $D = 0$. Then from (5.2) we have

$$\Psi = \frac{\Phi + C - 1}{C}. \tag{5.3}$$

If $C \neq 1$, then from (5.3) we get

$$\overline{N}(r, 1 - C, \Phi) = \overline{N}(r, 0; \Psi). \tag{5.4}$$

Using lemma 4.2, lemma 4.4, (5.4) we get by Nevanlinna second fundamental theorem

$$\begin{aligned} T(r, p(L)) &= T(r, \Phi) + S(r, L) \\ &\leq \overline{N}(r, \infty; \Phi) + \overline{N}(r, 0; \Phi) + \overline{N}(r, 1 - C; \Phi) + S(r, L) \\ &\leq \overline{N}(r, \infty; L) + \overline{N}(r, 0; \Phi) + \overline{N}(r, 0; \Psi) + S(r, L) \\ &\leq \overline{N}(r, 0; P(L)) + \overline{N}(r, 0; p(L)) + S(r, L) \\ &\leq N_{1+n}(r, 0; L) + Q\overline{N}(r, \infty; L) + \overline{N}(r, 0; p(L)) + S(r, L) \\ &\leq N_{2+n}(r, 0; L) + N_2(r, 0; p(L)) + S(r, L). \end{aligned} \tag{5.5}$$

From (5.5) we get $N_2(r, 0; p(L)) + N_{2+n}(r, 0; L) \geq (\lambda + o(1))T(r, L)$, which contradicts (3.2). Hence $C = 1$ and therefore $p(L) \equiv P(L)$.

Subcase 2.2 Let $D = -1$. Then from (5.2) we have

$$\Psi = \frac{C}{C + 1 - \Phi}. \tag{5.6}$$

If $C \neq -1$, then using lemma 4.2 we get from (5.6)

$$\bar{N}(r, 1 + C, \Phi) = \bar{N}(r, \infty; \Psi) = \bar{N}(r, \infty; L) + S(r, L) = S(r, L). \tag{5.7}$$

Using lemma 4.2, lemma 4.4, (5.7) we get by Nevanlinna second fundamental theorem

$$\begin{aligned} T(r, p(L)) &= T(r, \Phi) + S(r, L) \\ &\leq \bar{N}(r, \infty; \Phi) + \bar{N}(r, 0; \Phi) + \bar{N}(r, 1 + C; \Phi) + S(r, L) \\ &\leq \bar{N}(r, \infty; L) + \bar{N}(r, 0; \Phi) + \bar{N}(r, \infty; \Phi) + S(r, L) \\ &\leq \bar{N}(r, 0; p(L)) + \bar{N}(r, \infty; L) + S(r, L) \\ &\leq \bar{N}(r, 0; p(L)) + S(r, L). \end{aligned} \tag{5.8}$$

From (5.8) we get $\bar{N}(r, 0; p(L)) \geq (\lambda + o(1))T(r, L)$, which contradicts (3.2).

If $C = -1$, then

$$\Phi\Psi = 1. \tag{5.9}$$

From (5.9) we have

$$p(L)P(L) = R^2(z). \tag{5.10}$$

From (5.10) we have

$$\bar{N}(r, \infty; L) + \bar{N}(r, 0; L) = S(r, L). \tag{5.11}$$

Using lemma 4.8 and (5.11) we get $N(r, \infty; \frac{P(L)}{L^d}) = S(r, L)$ and hence

$$T(r, \frac{P(L)}{L^d}) = N(r, \infty; \frac{P(L)}{L^d}) + m(r, \infty; \frac{P(L)}{L^d}) = S(r, L). \tag{5.12}$$

Using lemma 4.6 and (5.12) we have

$$\begin{aligned} (d + \lambda)T(r, L) &\leq T(r, \frac{R^2(z)}{d^{\lambda+d}}) + O(1) \\ &\leq T(r, (1 + \frac{\rho_{\lambda-1}}{L} + \dots + \frac{\rho_1}{L^{\lambda-1}}) \frac{P(L)}{L^d}) + O(1) \\ &\leq (\lambda - 1)T(r, L) + T(r, \frac{P(L)}{L^d}) + S(r, L) \\ &\leq (\lambda - 1)T(r, L) + S(r, L). \end{aligned} \tag{5.13}$$

From (5.13) we get $T(r, L) = S(r, L)$, which is a contradiction.

Subcase 2.3 Let $D \neq 0, -1$.

If $C - D \neq 1$, then from (5.2) we get $\overline{N}(r, \frac{-C+D+1}{D+1}; \Phi) = \overline{N}(r, 0; \Psi)$. Now proceeding as in Subcase 2.1 we arrive at a contradiction.

If $C - D = 1$, then by (5.2) we get $\overline{N}(r, \frac{-1}{D}; \Phi) = \overline{N}(r, \infty; \Psi)$. Now proceeding as in Subcase 2.2 we arrive at a contradiction.

This completes the proof.

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