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On the Property (Bv)

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Abstract

In this paper, we study the property (Bv) for a bounded linear operator $T \in L(X)$ on a Banach space X, through the methods of local spectral theory. This property is equivalent to a-Browder's theorem. In particular, we shall give several conditions, by using the localized SVEP, for guarantee property (Bv) over a proper closed subspace of X, the Fredholm and upper semi-Fredholm spectrums are coincident between them, and also with other spectra, and under some topological conditions the operator T verifies property (Bv).

Keywords: Property (Bv), localized SVEP, a-Browder's theorem.

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1. INTRODUCTION AND PRELIMINARIES

1.1. Introduction

The classical a-Browder's theorem or equivalently generalized a-Browder's theorem, for operators $T \in L(X)$, defined on Banach spaces X has a lot of influence on the development of the spectral theory. This theorem admit several variants, that are stronger versions than it. Such variants have been studied by different authors,

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using methods of local spectral theory, for instance, the properties (gaz) and (V_{Π}), see [4] and [6] respectively. And somehow interest in studying a-Browder's theorem has lost momentum. Hence, in this paper, we consider an equivalent property to a-Browder's theorem, called the property (Bv), introduced in [10].

First, we set conditions through the property (Bv) so that the Fredholm and upper semi-Fredholm spectrums are coincident. These two spectra coincide with many others spectra, if the spectrum does not have isolated points. Next, we establish that there is a class of operators whose restrictions over a proper closed subspace of X, satisfy property (Bv) and finally, we present the topological definitions necessary to derive some necessary conditions involving the property (Bv). In particular, the limit of a sequence of operators that verify property (Bv) also has property (Bv), under the condition of commutativity.

1.2. Preliminaries

Let $T \in L(X)$. The various spectrums of T are defined as follows [10].

- Spectrum:
  $\sigma(T) = \{\lambda \in C : \lambda I - T \text{ is not invertible}\}$,
- Fredholm spectrum:
  $\sigma_e(T) = \{\lambda \in C : \lambda I - T \text{ is not Fredholm}\}$,
- Upper semi-Fredholm spectrum:
  $\sigma_{usf}(T) = \{\lambda \in C : \lambda I - T \text{ is not Upper semi-Fredholm}\}$,
- Lower semi-Fredholm spectrum:
  $\sigma_{lsf}(T) = \{\lambda \in C : \lambda I - T \text{ is not Lower semi-Fredholm}\}$,
- Approximate point spectrum:
  $\sigma_a(T) = \{\lambda \in C : \lambda I - T \text{ is not bounded below}\}$,
- Weyl spectrum:
  $\sigma_w(T) = \{\lambda \in C : \lambda I - T \text{ is not Weyl}\}$,
- Upper semi-Weyl spectrum:
  $\sigma_{ea}(T) = \{\lambda \in C : \lambda I - T \text{ is not Upper semi-Weyl}\}$,
- Lower semi-Weyl spectrum:
  $\sigma_{es}(T) = \{\lambda \in C : \lambda I - T \text{ is not Lower semi-Weyl}\}$,
- Browder spectrum:
  $\sigma_b(T) = \{\lambda \in C : \lambda I - T \text{ is not Browder}\}$,

- Upper semi-Browder spectrum:
  $\sigma_{ub}(T) = \{\lambda \in C : \lambda I - T$ is not Upper semi-Browder$\}$,
- B-Fredholm spectrum:
  $\sigma_{bf}(T) = \{\lambda \in C : \lambda I - T$ is not B-Fredholm$\}$,
- Upper semi B-Fredholm spectrum:
  $\sigma_{ubf}(T) = \{\lambda \in C : \lambda I - T$ is not Upper semi B-Fredholm$\}$,
- B-Weyl spectrum:
  $\sigma_{Bw}(T) = \{\lambda \in C : \lambda I - T$ is not B-Weyl$\}$,
- Upper semi B-Weyl spectrum:
  $\sigma_{uBw} = \{\lambda \in C : \lambda I - T$ is not Upper semi B-Weyl$\}$,
- Drazin invertible spectrum:
  $\sigma_d(T) = \{\lambda \in C : \lambda I - T$ is not Drazin invertible$\}$,
- Left Drazin invertible spectrum:
  $\sigma_{ld}(T) = \{\lambda \in C : \lambda I - T$ is not Left Drazin invertible$\}$.

The subspace hyper-kernel is defined as:

$$\mathcal{N}^{\infty}(T) := \bigcup_{n=1}^{\infty} \ker T^n.$$

The subspace hyper-range is defind as:

$$T^{\infty}(X) := \bigcap_{n=1}^{\infty} T^n(X).$$

The quasi-nilpotent part of an operator T is :

$$H_0(T) := \{x \in X : \lim_{n\to\infty} \|T^n x\|^{1/n} = 0.\}$$

Also, $p(T)$ and $q(T)$ denote the ascent and descent respectively,

$$\alpha(T) = \text{dim}(\ker T) \text{ and } \beta(T) = \text{codim}(T(X)).$$

The *analytical core* of T is the set $K(T)$ of all $x \in X$ for which there is a sequence $(u_n) \subset X$ and a constant $\delta > 0$ such that:

(1) $x = u_0,$ and $Tu_{n+1} = u_n$ for every $n \in \mathbb{Z}_+$;

(2) $\|u_n\| \leq \delta^n \|x\|$ for every $n \in \mathbb{N}$.

For every subset F of $\mathbb{C}$ the *local spectral subspace of T associated with F is the set*

$$X_T(F) := \{x \in X : \sigma_T(x) \subseteq F\}.$$

The single valued extension property introduced by Finch in [9], plays a relevant role in local spectral theory. An operator $T \in L(X)$ is said to have *the single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open disc $\mathbf{D}$ with $\lambda_0 \in \mathbf{D}$, the only analytic function $f : \mathbf{D} \to X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in \mathbf{D}$ is the function $f \equiv 0$. The operator T is said to have SVEP, if it has SVEP at every point $\lambda \in \mathbb{C}$. It is easy to prove that $T \in L(X)$ has SVEP at every isolated point of $\sigma(T)$ and at each point of the resolvent set $\rho(T) := \mathbb{C} \setminus \sigma(T)$. Moreover,

$$p(\lambda I - T) < \infty \Rightarrow T \text{ has SVEP at } \lambda, \tag{1}$$

and dually

$$q(\lambda I - T) < \infty \Rightarrow T^* \text{ has SVEP at } \lambda, \tag{2}$$

see [1, Theorem 3.8]. From the definition of the localized SVEP it is easily seen that

$$\sigma_a(T) \text{ does not cluster at } \lambda \Rightarrow T \text{ has SVEP at } \lambda, \tag{3}$$

Note that $H_0(T)$ generally is not closed and by [1, Theorem 2.31], we have

$$H_0(\lambda I - T) \text{ closed} \Rightarrow T \text{ has SVEP at } \lambda. \tag{4}$$

Remark 1.1. *The converse of the implications* (1)-(3) *holds, whenever $\lambda I - T$ is a semi-Fredholm operator or a semi B-Fredholm operator, see [3].*

Let M, N be two closed linear subspaces of X and define

$$\delta(M, N) := \sup\{\text{dist}\,(u, N) : u \in M, \|u\| = 1\},$$

in the case $M \neq \{0\}$. Otherwise set $\delta(\{0\}, N) = 0$ for any subspace N.
According to [12, §2, Chapter iv], the *gap* between M and N is defined by

$$\widehat{\delta}(M, N) := \max\{\delta(M, N), \delta(N, M)\}.$$

The function $\widehat{\delta}$ is a metric on the set of all linear closed subspaces of X, the *gap metric*, and the convergence $M_n \to M$ is obviously defined by $\widehat{\delta}(M_n, M) \to 0$ as $n \to \infty$.

Remark 1.2. *Let M, N be subspaces of a Banach space X. If $\widehat{\delta}(M, N) < 1$, then dim M = dim N. See, [13, Corollary 10.10].*

2. THE PROPERTY (Bv) AND THE FREDHOLM SPECTRUM

In this section, we recall the definition of the property (Bv) and give some conditions through property (Bv) for that Fredholm spectrum and upper semi-Fredholm spectrum, are equals, also, for that they coincident with other classical spectra.

For $T \in L(X)$, we define the following sets:

$$\Delta^+(T) := \sigma(T) \setminus \sigma_{ea}(T),\ \Delta_+(T) := \sigma(T) \setminus \sigma_{usf}(T),\ \Pi^v(T) := \sigma(T) \setminus \sigma_{ub}(T).$$

In general $\Pi^v(T) \subseteq \Delta^+(T)$, but equality need not hold. Now, $\Delta^+(T) = \Pi^v(T)$ if and only if $\sigma_{ea}(T) = \sigma_{ub}(T)$ i.e., if T verifies a-Browder's theorem. Actually,

Definition 2.1. *[10] $T \in L(X)$ verifies the property (Bv), if $\Delta^+(T) = \Pi^v(T)$.*

Example 2.2. Consider the projection operator $P \in L(\ell^2(\mathbb{N}))$, defined by

$$P(x_1, x_2, \ldots) = (0, x_2, x_3, \ldots).$$

Then $\sigma(P) = \{0, 1\},\ \sigma_{ea}(P) = \sigma_{ub}(P) = \{1\},$ and $\Delta^+(P) = \Pi^v(P) = \{0\}$. Hence P verifies property (Bv).

We recall that if T or T^* has SVEP in each $\lambda \notin \sigma_{ea}(T)$, then T verifies a-Browder's theorem or equivalently property (Bv). In particular we have the next theorem.

Theorem 2.3. *If $T \in L(X)$ has SVEP at each $\lambda \in \Delta^+(T)$, then T verifies property (Bv).*

Note that $\Delta^+(T) \subseteq \Delta_+(T)$, thus we have the next corollary.

Corollary 2.4. *If $T \in L(X)$ has SVEP at each $\lambda \in \Delta_+(T)$, then T verifies property (Bv).*

The following theorems assemble some of the various conditions that allow localized SVEP, thus implying the property (Bv).

Theorem 2.5. *$T \in L(X)$ verifies property (Bv) if for each $\lambda \in \Delta^+(T)$ one of the following conditions hold:*

(i) *$\lambda I - T$ has SVEP at* 0.

(ii) *Dual of $\lambda I - T$ has SVEP at* 0.

(iii) $\mathcal{N}^\infty(\lambda I - T) \cap (\lambda I - T)^\infty(X) = \{0\}$.

(iv) $\mathcal{N}^\infty(\lambda I - T) \cap K(\lambda I - T) = \{0\}$.

(v) $\mathcal{N}^\infty(\lambda I - T) \cap X_T(\emptyset) = \{0\}$.

(vi) $H_0(\lambda I - T) \cap K(\lambda I - T) = \{0\}$.

(vii) $\ker(\lambda I - T) \cap (\lambda I - T)(X) = \{0\}$.

(viii) *The sum* $H_0(\lambda I - T) + (\lambda I - T)(X)$ *is norm dense in* X.

(ix) $\lambda \notin \operatorname{acc}(\sigma_a(T))$.

Proof. Assume that for each $\lambda \in \Delta^+(T)$, $\lambda I - T$ verifies the condition:

(i)-(ii). Then, clearly T and T^* have SVEP at each $\lambda \in \Delta^+(T)$.

(iii)-(vii). Then, we get by [1, Corollary 2.26] that T has SVEP at each $\lambda \in \Delta^+(T)$.

(viii). Then, by [1, Theorem 2.33], we get T^* has SVEP in each $\lambda \in \Delta^+(T)$.

(ix). Then $\sigma_a(T)$ does not cluster at λ. Thus T has SVEP at each $\lambda \in \Delta^+(T)$.

Hence, the conditions (i)-(ix) ensure that T or T^* has SVEP at each $\lambda \in \Delta^+(T)$. Thus, by Theorem 2.3, we have that T verifies property (Bv). ∎

Corollary 2.6. $T \in L(X)$ *verifies property* (Bv) *if for each* $\lambda \in \Delta_+(T)$, $\lambda I - T$ *verifies one of the conditions* (i)-(ix) *of Theorem 2.5.*

Proof. If for each $\lambda \in \Delta_+(T)$, $\lambda I - T$ verifies the condition (i)-(ix), then as in the proof of the Theorem 2.5, we get that T has SVEP at each $\lambda \in \Delta_+(T)$, then by Corollary 2.4 we conclude that T verifies property (Bv). ∎

We recall that $T \in L(X)$ verifies the property (gaz) if $\sigma(T) \setminus \sigma_{uBw}(T) = \sigma_a(T) \setminus \sigma_{ld}(T)$, or equivalently, $\sigma(T) = \sigma_a(T)$ and T verifies generalized a-Browder's theorem (see [4, Theorem 3.2]). Thus, the following result allows a relationship between the properties (gaz) and (Bv).

Theorem 2.7. *Let* $T \in L(X)$. *If* T^* *has SVEP at each* $\lambda \in \Delta^+(T)$, *then:*

(i) T *verifies property* (Bv).

(ii) $\sigma(T) = \sigma_a(T)$.

Proof. (i) Clearly, T verifies property (Bv).
(ii) If $\lambda \notin \sigma_a(T)$, then $\lambda \notin \sigma_{ea}(T)$ and hence T^* has SVEP at λ. Then by Remark 1.1, $q(\lambda I - T) < \infty$, whereby $\beta(\lambda I - T) \leq 0$. Consequently $\lambda \notin \sigma(T)$. Hence $\sigma(T) = \sigma_a(T)$. ∎

Corollary 2.8. *Let $T \in L(X)$. If T^* has SVEP at each $\lambda \in \Delta^+(T)$, then T verifies property (gaz).*

Corollary 2.9. *Let $T \in L(X)$. If T verifies property (Bv) and $\sigma(T) = \sigma_a(T)$, then T verifies property (gaz).*

For the remainder of the section, we suppose that $\sigma(T) = \sigma_a(T)$ and we are devoted to studying conditions for obtaining the equality $\sigma_{\mathrm{usf}}(T) = \sigma_{\mathrm{e}}(T)$.

Theorem 2.10. *If $T \in L(X)$ verifies property (Bv), then $\sigma_{\mathrm{usf}}(T) = \sigma_{\mathrm{e}}(T)$.*

Proof. Since T verifies property (Bv) and $\sigma(T) = \sigma_a(T)$, by [6, Lemma 2.1], $\sigma_{ea}(T) = \sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{b}}(T)$.
Let $\lambda \notin \sigma_{\mathrm{usf}}(T)$. We consider two cases.

Case 1. $\lambda \notin \sigma_{ea}(T)$. In this case $\lambda \notin \sigma_{\mathrm{b}}(T)$ and hence $\lambda \notin \sigma_{\mathrm{e}}(T)$.

Case 2. $\lambda \in \sigma_{ea}(T)$. In this case $ind(\lambda I - T) > 0$. Hence, $\lambda \notin \sigma_{\mathrm{lsf}}(T)$ and so $\lambda \notin \sigma_{\mathrm{usf}}(T) \cup \sigma_{\mathrm{lsf}}(T)$. Therefore $\lambda \notin \sigma_{\mathrm{e}}(T)$.

Hence $\sigma_{\mathrm{e}}(T) = \sigma_{\mathrm{usf}}(T)$. ∎

By theorems 2.7 and 2.10, we have the result:

Theorem 2.11. *Let $T \in L(X)$. If T^* has SVEP at each $\lambda \in \Delta^+(T)$, then $\sigma_{\mathrm{e}}(T) = \sigma_{\mathrm{usf}}(T)$.*

Theorem 2.12. *If $T \in L(X)$ has SVEP at each $\lambda \in \Delta^+(T)$, then T verifies property (gaz). Also, $\sigma_{usf}(T) = \sigma_e(T)$.*

Proof. By Theorem 2.3, T verifies property (Bv), or equivalently generalized a-Browder's theorem, as $\sigma(T) = \sigma_a(T)$, so T verifies property (gaz), and hence by Theorem 2.10, $\sigma_{usf}(T) = \sigma_e(T)$. ∎

The properties (gaz) and (V_Π) have been studied in-depth, through SVEP localized in [4] and [6] respectively, where we do not find any relationship with the Fredholm-type spectra, but if T verifies one of these properties, then T verifies the a-Browder's theorem or equivalently the property (Bv), and with the use of Theorem 2.12, we present a result that relates the Fredholm spectrum with other spectra.

Theorem 2.13. *If $T \in L(X)$ has SVEP at each $\lambda \in \Delta_+(T)$, then:*

(i) $\sigma_{usf}(T) = \sigma_e(T) = \sigma_w(T) = \sigma_{ea}(T) = \sigma_{ub}(T) = \sigma_b(T)$.

(ii) $\sigma_{ubf}(T) = \sigma_{bf}(T) = \sigma_{uBw}(T) = \sigma_{Bw}(T) = \sigma_{ld}(T) = \sigma_d(T)$.

Proof. (i) Since $\sigma(T) = \sigma_a(T)$ and T has SVEP at each $\lambda \in \Delta_+(T)$, employing Theorem 2.12, $\sigma_{usf}(T) = \sigma_e(T) = \sigma_{ub}(T)$. Also by [6, Lemma 2.1], we get that $\sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{b}}(T)$, whereby $\sigma_{usf}(T) = \sigma_b(T)$. But $\sigma_{usf}(T) \subseteq \sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T)$ and $\sigma_{usf}(T) \subseteq \sigma_{ea}(T) \subseteq \sigma_{ub}(T) \subseteq \sigma_b(T) = \sigma_{usf}(T)$. Hence (i) holds.

(ii) By Theorem 2.12, T verifies property (gaz) and thus by [4, Theorem 3.3], $\sigma_{ld}(T) = \sigma_d(T) = \sigma_{uBw}(T) = \sigma_{Bw}(T)$. Now let $\lambda_0 \notin \sigma_{ubf}(T)$. Then $\lambda_0 I - T$ is upper semi B-Fredholm and by [4, Theorem 2.1], there exists an open disc $\mathbb{D}(\lambda_0, \varepsilon)$ such that, $\lambda I - T$ is upper semi-Fredholm for all $\lambda \in \mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$. By hypothesis T has SVEP at every $\lambda \in \mathbb{D}(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$, so that T has SVEP at λ_0 (see, [4, Remark 2.5]) and hence $p(\lambda_0 I - T) < \infty$. So, by [2, Theorem 1.142] we have that $\lambda_0 \notin \sigma_{ld}(T)$ equivalently $\lambda_0 \notin \sigma_d(T)$. Therefore $\sigma_{ubf}(T) = \sigma_d(T)$. But $\sigma_{ubf}(T) \subseteq \sigma_{bf}(T) \subseteq \sigma_{Bw}(T) \subseteq \sigma_d(T)$. Hence $\sigma_{ubf}(T) = \sigma_{bf}(T) = \sigma_{Bw}(T) = \sigma_d(T)$. ∎

We recall that an operator $T \in L(X)$ is *Drazin invertible* if there exist an operator $S \in L(X)$ (called the *Drazin inverse* of T) and an integer $n \geq 0$ such that

$$TS = ST,\ STS = S,\ T^n ST = T^n.$$

As in the conclusion of [5], we deduce the next result.

Theorem 2.14. *Let $T \in L(X)$ be Drazin invertible with Drazin inverse S. If T verifies property (Bv), then S verifies property (Bv).*

Thus, the property (Bv) is transferred from T to their Drazin inverse S. In this way, the properties that are consequential to property (Bv) for T are also valid for the operator S, without the need to know to S. These happen in the case that the operator is algebraic.

Example 2.15. The class of algebraic operators check Theorem 2.13, in particular verify property (Bv), since if T is an algebraic operator so $\sigma_a(T) = \sigma(T) =$

$\{\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n\}$, thus, T have SVEP. Note that by [11, Corollary 2.10] result $\sigma_d(T) = \emptyset$. Nilpotent operators are special cases of algebraic operator. An extensive class of nilpotent operators is the class of the analytically quasi-$\mathcal{THN}$ operators which are quasi-nilpotent over $L(H)$, where H is a Hilbert space (see [1, Theorem 6.188]). Also, idempotent operators are algebraic, likewise operators for which some power has finite-dimensional range.

Example 2.16. Let V denote the Volterra operator on the Banach space $X := C[0,1]$ defined by

$$(Vf)(t) := \int_0^t f(s)\mathrm{d}s \quad \text{for all } f \in C[0,1] \quad \text{and } t \in [0,1].$$

Then V is injective and quasinilpotent. Note that:

$$\{0\} = \sigma_a(V) = \sigma(V) = \sigma_{usf}(V) = \sigma_e(V) = \sigma_w(V) = \sigma_{ea}(V) = \sigma_{ub}(V) = \sigma_b(V) = \sigma_d(V) = \sigma_{ubf}(V) = \sigma_{bf}(V) = \sigma_{uBw}(V) = \sigma_{Bw}(V) = \sigma_{ld}(V).$$

Example 2.17. Let $T \in L(\ell^2(\mathbb{N}))$ be defined by

$$T(x_1, x_2, x_3, \cdots) = \left(\frac{x_2}{2}, \frac{x_3}{3}, \cdots\right), \text{where } x = (x_n) \in \ell^2(\mathbb{N}).$$

Note that:

$$\{0\} = \sigma_a(T) = \sigma(T) = \sigma_{usf}(T) = \sigma_e(T) = \sigma_w(T) = \sigma_{ea}(T) = \sigma_{ub}(T) = \sigma_b(T) = \sigma_d(T) = \sigma_{ubf}(T) = \sigma_{bf}(T) = \sigma_{uBw}(T) = \sigma_{Bw}(T) = \sigma_{ld}(T).$$

The following result establishes the conditions by which various spectra coincide.

Corollary 2.18. *Let $T \in L(X)$. If T has SVEP at each $\lambda \in \Delta_+(T)$ and $iso\sigma_a(T) = \emptyset$, then:*

$$\sigma_{usf}(T) = \sigma_e(T) = \sigma_w(T) = \sigma_{ea}(T) = \sigma_{ub}(T) = \sigma_b(T) = \sigma(T) = \sigma_a(T) = \sigma_d(T) = \sigma_{ubf}(T) = \sigma_{bf}(T) = \sigma_{uBw}(T) = \sigma_{Bw}(T) = \sigma_{ld}(T).$$

Proof. By hypothesis, $\mathrm{iso}\sigma_a(T) = \emptyset$ whereby $\sigma(T) = \sigma_b(T)$ and $\sigma(T) = \sigma_d(T)$. Hence, the result follows by Theorem 2.13. ∎

3. THE PROPERTY (Bv) AND PROPER SUBSPACES

Let W a proper closed subspace of X. Define :

$$\mathcal{P}(X,W) = \{T \in L(X) : T(W) \subseteq W \text{ and } T^n(X) \subseteq W, \text{ for some integer } n \geq 1\}.$$

For $T \in \mathcal{P}(X,W)$, T_W denotes the restriction of T over the T-invariant subspace W of X. Note that T is not surjective.

For V the Volterra operator defined in Example 2.16, result that

$$V^\infty(X) = \{f \in C^\infty[0,1] : f^{(n)}(0) = 0,\ n \in \mathbb{Z}_+\},$$

is a subspace not closed.

Example 3.1. Let $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ the operator defined in Example 2.17. It is easily seen that

$$\|T^k\| = \frac{1}{(k+1)!} \quad \text{for every } k = 0,1,\dots$$

So the operator T is quasi-nilpotent or equivalently $H_0(T) = \ell^2(\mathbb{N})$, thus T has SVEP at 0. But note that $p(T) = \infty$, whereby T is not semi-Fredholm operator.

Now, following [1, Theorem 1.42], it is possible obtain a closed subspace $T^\infty(X) = K(T)$, when $T \in L(X)$ is a semi-Fredholm operator.

Theorem 3.2. *Let $T \in L(X)$ be a semi-Fredholm operator with ascent or descent not finite. If for each $\lambda \in \Delta^+(T)$, $\lambda I - T$ verifies one of the conditions* (i)-(ix) *of Theorem 2.5, then there exists a proper closed subspace W of X such that T_W verifies property (Bv).*

Proof. Since T is an upper semi-Fredholm operator so for all $n \geq 1$, T^n is an upper semi-Fredholm operator, whereby $T^n(X)$ is closed. Also T has ascent or descent not finite so T is not surjective. Hence $W = T^\infty(X) = K(T)$ is a proper closed subspace of X. Clearly $T \in \mathcal{P}(X,W)$. Also by [7, Theorem 4.1] and Theorem 2.5, we get $\sigma_{ea}(T_W) = \sigma_{ea}(T) = \sigma_{\mathrm{ub}}(T) = \sigma_{ub}(T_W)$. Hence T_W verifies property (Bv). ∎

Corollary 3.3. *Let W be a proper closed subspace of X and $T \in \mathcal{P}(X,W)$ such that $q(T) = \infty,$ or $p(T) = \infty$. Then,*

(i) $\Pi^v(T) = \Pi^v(T_W)$.

(ii) $\Delta^+(T) = \Delta^+(T_W)$.

Consequently, T verifies property (Bv) if and only if T_W verifies property (Bv).

Proof. Since, $q(T) = \infty$ or $p(T) = \infty$, [7, Theorem 4.1] result that $\sigma(T) = \sigma(T_W)$ and also as in Theorem 3.2, $\sigma_{ea}(T) = \sigma_{ea}(T_W)$ and $\sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{ub}}(T_W)$. Therefore (i) and (ii) hold. ∎

Example 3.4. Let $R \in \ell^2(\mathbb{N})$ be the unilateral right shift given by

$$R(x_1, x_2, \dots) := (0, x_1, x_2, \cdots) \quad \text{for all } (x_n) \in \ell^2(\mathbb{N}).$$

R is an upper semi-Fredholm operator and $W = R^\infty(\ell^2(\mathbb{N}))$ is a proper closed subspace of $\ell^2(\mathbb{N})$. Thus, $R \in \mathcal{P}(\ell^2(\mathbb{N}), W)$. On the other hand $q(R) = \infty$ and $\sigma_{ea}(R) = \sigma_{ub}(R) = \Gamma$, where Γ is the unit circle. Hence R verifies property (Bv). Therefore, by Corollary 3.3, R_W verifies property (Bv).

Theorem 3.5. *Let W be a proper closed subspace of X and $T \in \mathcal{P}(X, W)$ such that $0 \in \sigma_{ea}(T_W)$. Then T verifies property (Bv) if and only if T_W verifies property (Bv).*

Proof. Since $0 \in \sigma_{ea}(T_W) \subseteq \sigma_{ea}(T)$, if $\lambda \notin \sigma_{ea}(T_W)$ or $\lambda \notin \sigma_{ea}(T)$, then $\lambda \neq 0$. Note that by [7], the range of $\lambda I - T$ is closed in X if and only if the range of $\lambda I - T_W$ is closed in W, also $\alpha(\lambda I - T) = \alpha(\lambda I - T_W)$, $\beta(\lambda I - T) = \beta(\lambda I - T_W)$ and $p(\lambda I - T) = p(\lambda I - T_W)$. Hence we obtain that $\sigma_{ea}(T) = \sigma_{ea}(T_W)$ and $\sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{ub}}(T_W)$. Therefore, T verifies property (Bv) if and only if T_W verifies property (Bv). ∎

4. THE PROPERTY (Bv) AND TOPOLOGICAL NOTIONS

In metric space $\mathbb{C}$, we denote by $Cl(A)$, $\mathrm{int}\,(A)$ and $\partial(A)$, the closure, interior and boundary respectively of $A \subseteq \mathbb{C}$. With this notation, we introduce some topological notions enough for an operator to verify the property (Bv). We also give the conditions for the limit of a sequence of operators verifying property (Bv) to verify property (Bv). Note that if T verifies property (Bv), then it is not necessarily $\mathrm{int}\,(\Delta^+(T)) = \emptyset$.

Example 4.1. Let R be the unilateral right shift operator defined in Example 3.4 and let $P \in L(\ell^2(\mathbb{N}))$ be the projection operator defined in Example 2.2. Define an operator T on $X = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by $T = R \oplus P$. Then $\sigma(T) = \mathbf{D}(0,1)$ and $\sigma_{ub}(T) = \sigma_{ea}(T) = \Gamma$, where $\mathbf{D}(0,1)$ is the closed unit disc. Hence T verifies property (Bv), but $\mathrm{int}\,(\Delta^+(T)) \neq \emptyset$.

Theorem 4.2. *Let $T \in L(X)$ such that $\mathrm{int}\,(\Delta^+(T)) = \emptyset$. Then:*

(i) $\Delta^+(T) = \Pi^v(T)$. *Consequently T verifies property (Bv).*

(ii) $\sigma(T) = \sigma_a(T)$.

(iii) $\sigma_{usf}(T) = \sigma_e(T)$.

Proof. Suppose that $\text{int}\,(\Delta^+(T)) = \emptyset$.

(i) If $\lambda_0 \in \Delta^+(T)$, there exists an open disc $\mathbb{D}(\lambda_0, \epsilon)$ such that $\lambda \notin \sigma_{ea}(T)$ for all $\lambda \in \mathbb{D}(\lambda_0, \epsilon)$, because the set of upper Weyl operators is open. It must happen that $\lambda_0 \in \partial(\sigma(T))$. Otherwise there exists an open disc $\mathbb{D}(\lambda_0, \epsilon_1)$, such that $\mathbb{D}(\lambda_0, \epsilon_1) \subseteq \sigma(T)$. Then λ_0 is an interior point of $\Delta^+(T)$, a contradiction. Hence $\lambda_0 \in \partial(\sigma(T))$ and so T verifies the SVEP at λ_0, as $\lambda_0 \notin \sigma_{ea}(T)$ then $p(\lambda_0 I - T) < \infty$. Thus $\lambda_0 \in \Pi^v(T)$. We conclude that $\Delta^+(T) = \Pi^v(T)$.

(ii) Let $\lambda_0 \notin \sigma_a(T)$. Then $\lambda_0 \in \sigma(T)$ result that $\lambda_0 \in \Delta^+(T)$. So if we proceed as in the previous part (i), we get that $\lambda_0 \in \partial(\sigma(T))$, which is a contradiction, since $\partial(\sigma(T)) \subseteq \sigma_a(T)$. Hence $\sigma(T) = \sigma_a(T)$.

(iii) It follows from Theorem 2.10. ■

From parts (i), (ii) of Theorem 4.2 and Corollary 2.9, we get the following result.

Corollary 4.3. *Let $T \in L(X)$ such that $\text{int}\,(\Delta^+(T)) = \emptyset$, then T verifies property (gaz).*

If $\text{acc}\,(\sigma(T)) = \emptyset$, or $\Delta^+(T) \subseteq \text{iso}\sigma_a(T)$, or $\Delta^+(T) \subseteq \partial\sigma_a(T)$, then $\text{int}\,(\Delta^+(T)) = \emptyset$, and hence by Theorem 4.2, we get the following results.

Corollary 4.4. *Let $T \in L(X)$. If $\text{acc}\,(\sigma(T)) = \emptyset$, or $\Delta^+(T) \subseteq \text{iso}\sigma_a(T)$, or $\Delta^+(T) \subseteq \partial\sigma_a(T)$, then T verify the statements* (i)-(iii) *of Theorem 4.2. Consequently T verifies property (gaz).*

Example 4.5. Let $\{a_n\}$ be any convergent sequence of scalars, say $a_n \to a$, define $T_n := a_n I$ and $T = aI$. Then $T_n \to T$. Clearly every T_n verifies property (Bv) and so also T.

Next, we inquire about the conditions for property (Bv) to remain under limit convergence.

Theorem 4.6. *Let $T \in L(X)$ and T_n be sequence of operators in $L(X)$ such that $\lim_{n\to+\infty} \| T_n - T \| = 0$. Then $\Delta^+(T) \subseteq \Delta^+(T_n)$, for sufficiently large n.*

Proof. The set of upper semi-Weyl operators is open. Let $\lambda_0 \in \Delta^+(T)$, so by convergence of T_n to T, $\exists\, N_0 \in \mathbb{N}$ such that $\lambda_0 \in \Delta^+(T_n), \forall n \geq N_0$. Truly if $\lambda_0 \notin \sigma_{ea}(T)$, then $\lambda I - T$ has closed range. Thus the hypothesis implies that

$\widehat{\delta}(\ker(T_n), \ker(T)) \to 0$ as $n \to \infty$, and, $\widehat{\delta}(\mathrm{R}(T_n), \mathrm{R}(T)) \to 0$ as $n \to \infty$, (see [13, Theorem 10.17]). Hence, if $\lambda_0 \in \sigma(T)$ we have by Remark 1.2 that $\lambda_0 \in \sigma(T_n)$, for sufficiently large n. Hence $\Delta^+(T) \subseteq \Delta^+(T_n)$, for sufficiently large n. ∎

Theorem 4.7. *Let* T_n *be sequence of operators in* $L(X)$ *verifying property* (Bv) *and that commutes with* $T \in L(X)$. *If* $\lim_{n\to+\infty} \| T_n - T \| = 0$, *then* T *verifies property* (Bv).

Proof. By Theorem 4.6, if $\lambda_0 \in \Delta^+(T)$, then $\exists N_0 \in \mathbb{N}$ such that $\lambda_0 \in \Delta^+(T_n)$ $\forall n \geq N_0$. In this way $\forall n \geq N_0$ we have that $\lambda_0 \notin \sigma_{ub}(T_n)$ and $p(\lambda_0 I - T_n) < \infty$. On another hand, since $\lim_{n\to+\infty} \| T_n - T \| = 0$, so as T_n and T commutes result that $\lim_{n\to+\infty} \| (\lambda_0 I - T_n)^k - (\lambda_0 I - T)^k \| = 0$, where k is a natural number. Hence by Remark 1.2 we deduce that, $\forall n \geq N_1$, $\alpha\left((\lambda_0 I - T_n)^k\right) = \alpha\left((\lambda_0 I - T)^k\right)$ and so $\ker(\lambda_0 I - T_n)^k = \ker(\lambda_0 I - T)^k$. Now, with $k := p(\lambda_0 I - T_{N_1}) < \infty$, result that $p(\lambda_0 I - T) < \infty$. Therefore, we get T has SVEP at $\lambda_0 \notin \sigma_{ea}(T)$, thus, we conclude that T verifies property (Bv). ∎

Let X and Y be an infinite dimensional Banach spaces. Consider $S \in L(Y)$ and $T \in L(X)$. If $\lambda \notin (\sigma_{ea}(T) \cup \sigma_{ea}(S))$, then $\lambda I - T$ and $\lambda I - S$ are upper semi-Weyl operators. Hence $\lambda \notin \sigma_{usf}(T)$ and $\lambda \notin \sigma_{usf}(S)$. Hence $\lambda \notin \sigma_{usf}(T \bigoplus S)$, but $\alpha(\lambda(I \bigoplus I) - T \bigoplus S) = \alpha(\lambda I - T) + \alpha(\lambda I - S)$, and $\beta(\lambda(I \bigoplus I) - T \bigoplus S) = \beta(\lambda I - T) + \beta(\lambda I - S)$, whereby $\lambda \notin \sigma_{ea}(T \bigoplus S)$. Hence, $\sigma_{ea}(T \bigoplus S) \subseteq \sigma_{ea}(T) \cup \sigma_{ea}(S)$. However, if $T \in B(X)$ and $S \in B(Y)$ satisfy the property (Bv), then, $\sigma_{ea}(T \bigoplus S) = \sigma_{ea}(T) \cup \sigma_{ea}(S)$ if and only if $T \bigoplus S$ verifies properties (Bv). See [8, Theorem 3.11].

Theorem 4.8. *Let* $T \in L(X)$ *and* $S \in L(Y)$ *be such that* $\sigma_{ea}(T) \subseteq \sigma_{es}(T)$ *and* $\sigma_{ea}(S) \subseteq \sigma_{es}(S)$. *If* $\mathrm{int}\,(\Delta^+(T) \cup \Delta^+(S)) = \emptyset$, *then* $T \bigoplus S$ *verifies property* (Bv).

Proof. If $\lambda \in \Delta^+(T \bigoplus S)$, then $\lambda \in \sigma(T \bigoplus S)$ and $\lambda \notin \sigma_{ea}(T \bigoplus S)$. Thus, $\lambda \in \sigma(T)$, $\lambda \in \sigma(S)$, but $\alpha(\lambda(I \bigoplus I) - T \bigoplus S) - \beta(\lambda(I \bigoplus I) - T \bigoplus S) \leq 0$. Without loss of generality, we assume that $\lambda \notin \sigma_{ea}(T)$ and $\alpha(\lambda I - S) - \beta(\lambda I - S) \geq 0$, whereby $\lambda \notin \sigma_{es}(S)$ and as $\sigma_{ea}(S) \subseteq \sigma_{es}(S)$, $\lambda \notin \sigma_{ea}(S)$. Thus, we deduce that $\lambda \in (\Delta^+(T) \cup \Delta^+(S))$ and hence $\Delta^+(T \bigoplus S) \subseteq (\Delta^+(T) \cup \Delta^+(S))$. By hypothesis, we get that $\mathrm{int}\,(\Delta^+(T \bigoplus S)) = \emptyset$. Hence by Theorem 4.2 , $T \bigoplus S$ verifies property (Bv).

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