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Measures of Fuzzy Information Corresponding to Heinz Mean

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Abstract

If x and y are any two real numbers then their Heinz mean is given by $H_t(x,y) = \frac{x^t y^{1-t} + x^{1-t} y^t}{2}$., where $0 \leq t \leq 1$. In the present paper measure of fuzzy entropy and fuzzy directed divergence are obtained corresponding to the Heinz mean. $\mathrm{H_t(x,y)} = \frac{\mathrm{x^t y^{1-t} + x^{1-t} y^t}}{2}$.

Keywords – Measure of entropy, Directed Divergence, Heinz Mean, Fuzzy Set.

Mathematics Subject Classification- 94 D 05

1. INTRODUCTION

Heinz means introduced in [3], are means that interpolate in a certain way between the arithmetic and geometric mean. They are defined over R^+ as

$$H_t(x,y) = \frac{x^t y^{1-t} + x^{1-t} y^t}{2} \tag{1.1}$$

For $0 \leq t \leq 1$ One can easily show that the Heinz means are "in-between" the geometric mean and the arithmetic mean:

$$\sqrt{ab} \leq H_t(x,y) \leq \frac{a+b}{2} \tag{1.2}$$

The uncertainty associated with probability of outcomes, known as probabilistic uncertainty, is called entropy, since this is the terminology that is well entrenched in the literature. Entropies must correspond to mean values for them to be measurable. The Shannon [2] entropy corresponds to the weighted arithmetic mean, whereas the Renyi [5] entropy corresponds to the exponential mean. Shannon [2] introduced the concept of information theoretic entropy by associating uncertainty with probability distribution and found that there is unique function that can measure the uncertainty, is given by

$$H(P) = -\sum_{i=1}^{n} p_i log p_i \qquad (1.3)$$

The measure of entropy (1.3) possesses a number of interesting properties .Immediately, after Shannon's [2] gave his measure , research worker's in many fields saw the potential of the application of the application of this expression and a large number of other measures of information – theoretic entropies were derived. Renyi's [5] defined entropy of order α as:

$$H_\alpha(P) = ln\left(\frac{\sum_{i=1}^{n} p_i^{\propto}}{\sum_{i=1}^{n} p_i}\right), \propto \neq 1, \propto > 0 \qquad (1.4)$$

which includes Shannon's entropy as a limiting case as $\propto \to 1$.

A new probabilistic measures of entropy, has been deduced by N. sharma et al. [4] which is given as follows:

$$H(P) = \frac{x+y}{2} - \frac{1}{2}\sum_{i=1}^{n}\left(x^{p_i}y^{1-p_i} + x^{1-p_i}y^{p_i}\right) \qquad (1.5)$$

where x and y both are non-negative real numbers, and x,y>1.

It may be recalled that a fuzzy subset A in U (universe of discourse) is characterized by a *membership function* $\mu_A: U \to [0,1]$ which represents the *grade of membership* of $x \in U \; in \; A$ as follows

$$\mu_A(x) = 0 \; if \; x \; does \; not \; belongs \; to \; A,$$

$$and \; there \; is \; no \; uncertainty$$

$$= 1 \; if \; x \; belongs \; to \; A \; and \; there \; is \; no \; uncertainty$$

$$= 0.5 \; if \; maximum \; uncertainty$$

In fact $\mu_A(x)$ associates with each $x \in U$ a grade of membership in the set A. When $\mu_A(x)$ is valued in {0,1} it is the characteristic function of a crisp (i.e. nonfuzzy) set. Since $\mu_A(x)$ and $1 - \mu_A(x)$ gives the same degree of fuzziness, therefore, corresponding to the entropy due to Shannon [12], De Luca and Termini [13] suggested the following measure of fuzzy entropy:

$$H(A) = -\left[\sum_{i=1}^{n} \mu_A(x_i) log \mu_A(x_i) + \sum_{i=1}^{n} \big(1-\mu_A(x_i)\big) log\big(1-\mu_A(x_i)\big)\right] \qquad (1.6)$$

De Luca and Termini [1] introduced a set of properties and these properties are widely accepted as a criterion for defining any new fuzzy entropy. In fuzzy set theory, the entropy is a measure of fuzziness which expresses the amount of average ambiguity/difficulty in making a decision whether an element belongs to a set or not. So, a measure of average fuzziness in a fuzzy set should have at least the following properties to be valid fuzzy entropy:

i) $H(A) = 0$ when $\mu_A(x_i) = 0$ or 1.

ii) $H(A)$ increases as $\mu_A(x_i)$ increases from 0 to 0.5.

iii) $H(A)$ decreases as $\mu_A(x_i)$ increases from 0.5 to 1.

iv) $H(A) = H(\bar{A})$, i.e. $\mu_A(x_i) = 1 - \mu_A(x_i)$

v) $H(A)$ is a concave function of $\mu_A(x_i)$.

A measure $D(P{:}\,Q)$ of divergence or cross entropy or directed divergence is found to be very important in Mathematical, Physical and Biological sciences. This measure is probabilistic in nature and is defined as the discrepancy in the probability distribution P from another probability distribution Q. In some sense it measures the distance of P from Q. The most important and useful measure of directed divergence is obtained by Kullback and Leibler of probability distribution $P = (p_{1,}p_{2,} \ldots\ldots, p_n)$ from the probability distribution $Q = (q_{1,}q_{2,} \ldots\ldots, q_n)$ as

$$D(P{:}\,Q) = \sum_{i=1}^{n} p_i \, log \frac{p_i}{q_i} \qquad (1.7)$$

Let A and B be two standard fuzzy sets with same supporting points $x_{1,}x_{2,} \ldots\ldots, x_n$ and with fuzzy vectors $\mu_A(x_1), \mu_A(x_2), \ldots\ldots, \mu_A(x_n)$ and $\mu_B(x_1), \mu_B(x_2), \ldots\ldots, \mu_B(x_n)$. The simplest measure of fuzzy directed divergence as suggested by Bhandari and Pal (1993), is

$$D(A{:}\,B) = \sum_{i=1}^{n} \mu_A(x_i) \, log \frac{\mu_A(x_i)}{\mu_B(x_i)} + \sum_{i=1}^{n} \big(1-\mu_A(x_i)\big) \, log \frac{\big(1-\mu_A(x_i)\big)}{\big(1-\mu_B(x_i)\big)} \quad (1.8)$$

satisfying the conditions:

i) $D(A{:}\,B) \geq 0$

ii) $D(A:B) = 0\ iff\ A = B$

iii) $D(A:B) = D(B:A)$

iv) $D(A:B)$ is a convex function of $\mu_A(x_i)$

2. NEW MEASURE OF FUZZY ENTROPY

Corresponding to (1.8) we propose new measure of intuitionistic fuzzy entropy as

$$H(A) = (x+y) - \frac{1}{2}\sum_{i=1}^{n}\left(x^{\mu_A(x_i)}y^{1-\mu_A(x_i)} + x^{1-\mu_A(x_i)}y^{\mu_A(x_i)}\right) - \frac{1}{2}\sum_{i=1}^{n}\left(x^{1-\mu_A(x_i)}y^{\mu_A(x_i)} + x^{\mu_A(x_i)}y^{1-\mu_A(x_i)}\right)$$

Or

$$H(A) = (x+y) - \sum_{i=1}^{n}\left(x^{\mu_A(x_i)}y^{1-\mu_A(x_i)} + x^{1-\mu_A(x_i)}y^{\mu_A(x_i)}\right) \quad (2.1)$$

x,y	$\mu_A(x_i)$	H(A)
	0	0
	0.1	0.04
x=2	0.2	0.06
y=3	0.3	0.08
	0.4	0.096
	0.5	0.101
	0.6	0.09
	0.7	0.08
	0.8	0.06
	0.9	0.04
	1	0

Table 2.1

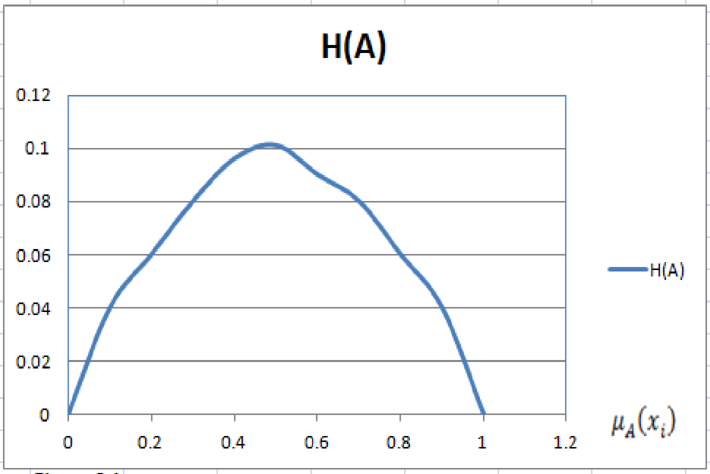

Figure 2.1

From table 2.1 and figure 2.1 it is clear that

i) $H(A) = 0$ when $\mu_A(x_i) = 0\ or\ 1$.

ii) $H(A)$ increases as $\mu_A(x_i)$ increases from 0 to 0.5.

iii) $H(A)$ decreases as $\mu_A(x_i)$ increases from 0.5 to 1.

iv) $H(A) = H(\bar{A})$, i.e. $\mu_A(x_i) = 1 - \mu_A(x_i)$

v) To verify $H(A)$ is a concave function of $\mu_A(x_i)$ let $s = \mu_A(x_i)$ then
$H(A) = (x + y) - (x^s y^{1-s} + x^{1-s} y^s)$ we have
$\frac{dH}{ds} = (-x^s y^{1-s} + x^{1-s} y^s)(logx - logy)$
and $\frac{d^2H}{ds^2} = -(logx - logy)^2 (x^s y^{1-s} + x^{1-s} y^s)$ clearly $\frac{d^2H}{ds^2} < 0$. So $H(A)$ is a concave function of $\mu_A(x_i)$.

Hence $H(A)$ satisfies all the properties of fuzzy entropy so it is a valid measure of fuzzy entropy.

3. NEW MEASURE OF FUZZY DIRECTED DIVERGENCE

Corresponding to (1.8) we propose new measure of fuzzy directed divergence as

$$D(A:B) = (x + y) - \sum_{i=1}^{n} \left(x^{\left(\frac{\mu_A(x_i)}{\mu_B(x_i)}\right)} y^{1-\left(\frac{\mu_A(x_i)}{\mu_B(x_i)}\right)} + x^{1-\left(\frac{\mu_A(x_i)}{\mu_B(x_i)}\right)} y^{\left(\frac{\mu_A(x_i)}{\mu_B(x_i)}\right)} \right) \quad (3.1)$$

Clearly

i) $D(A:B) \geq 0$

ii) $D(A:B) = 0 \; iff \; A = B$

iii) $D(A:B) = D(B:A)$

iv) $D(A:B)$ is a convex function of $\mu_A(x_i)$ as $\frac{\partial^2 D(A:B)}{\partial \mu_A(x_i)^2} \geq 0$

Thus $D(A:B)$ is a valid measure of fuzzy directed divergence.

4. CONCLUSION

This work introduces a Fuzzy Entropy and Fuzzy Divergence measure in the setting of fuzzy set theory. Basic properties of proposed measures have been examined.

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