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# Some Formulae involving Continued Fraction and Hypergeometric Function

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**Abstract**

In this paper, we aim to evaluate some formulae involving Continued fractions and generalized hypergeometric functions.

**Keywords :** Hypergeometric Function, Pochhammer symbol, Continued Fraction.

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## 1. INTRODUCTION

The following formulas are recalled (see, e.g., [12, p. 13-14, Entries 13-16]:

$$\left(\frac{tan^{-1}\,z}{z}\right)^2 = F^{1:1;2}_{1:0;1}\left(\begin{matrix}[1{:}1{,}1]{:}\,[1{:}1];\,[\tfrac{1}{2}{,}1],\,[1{:}1]; \\ [2{:}1{,}1]{:}\,-;[\tfrac{3}{2}{:}1] \qquad ;\end{matrix} - z^2, -z^2\right) \tag{1.1}$$

$$\left(\frac{tan^{-1}\,z}{z}\right)^3 = F^{2:1;1;2}_{2:0;0;1}\left(\begin{matrix}[\tfrac{3}{2}{:}1{,}1{,}1],\,[1{:}0{,}1{,}1]{:}\,[1{:}1];\,[1{:}1];\,[\tfrac{1}{2}{:}1],\,[1{:}1]\,; \\ [\tfrac{5}{2}{:}1{,}1{,}1],\,[2{:}0{,}1{,}1]{:}\,-;\,-;\tfrac{3}{2}{:}1] \qquad ;\end{matrix} - z^2, -z^2, -z^2\right) \tag{1.2}$$

$$\left(\frac{tan^{-1}\,z}{z}\right)^4 = F^{3:1;1;1;2}_{3:0;0;0;1}\left(\begin{matrix}[2{:}1{,}1{,}1{,}1],\,[\tfrac{3}{2}{:}0{,}1{,}1{,}1],\,[1{:}0{,}0{,}1{,}1]{:}\,[1{:}1];\,[1{:}1];[1{:}1];\,[1{:}1],\,[\tfrac{1}{2}{:}1]; \\ [3{:}1{,}1{,}1{,}1],\,[\tfrac{5}{2}{:}0{,}1{,}1{,}1],\,[2{:}0{,}0{,}1{,}1]{:}\,-;\,-;\,-;[\tfrac{3}{2}{:}1] \qquad ;\end{matrix} - z^2, -z^2, -z^2, -z^2\right) \tag{1.3}$$

$$\left(\frac{tan^{-1}\,z}{z}\right)^m =$$

$$F^{m-1:\overbrace{1;1;...;1;1;2}^{m-1}}_{m-1:\underbrace{0;0;...;0;0;1}_{m-1}}\left(\begin{array}{l}[\tfrac{m}{2}:\overbrace{1,1,...,1,1}^{m}],[\tfrac{m-1}{2}:0,\overbrace{1,1,...,1,1}^{m-1}],[\tfrac{m-3}{2}:0,0,\overbrace{1,1,...,1,1}^{m-2}],[\tfrac{m-4}{2}:0,0,0,\overbrace{1,1,...,1,1}^{m-3}],\\ [\tfrac{m+2}{2}:\underbrace{1,1,...,1,1}_{m}],[\tfrac{m+1}{2}:0,\underbrace{1,1,...,1,1}_{m-1}],[\tfrac{m}{2}:0,0,\underbrace{1,1,...,1,1}_{m-2}],[\tfrac{m-1}{2}:0,0,0,\underbrace{1,1,...,1,1}_{m-3}],\end{array}\right.$$

$$\begin{array}{l}...,[3:\overbrace{0,0,...,0,0}^{m-6},1,1,1,1,1,1],[\tfrac{5}{2}:\overbrace{0,0,...,0,0}^{m-5},1,1,1,1,1],[2:\overbrace{0,0,...,0,0}^{m-4},1,1,1,1],[\tfrac{3}{2}:\overbrace{0,0,...,0,0}^{m-3},1,1,1],\\ ...,[4:\underbrace{0,0,...,0,0}_{m-6},1,1,1,1,1,1],[\tfrac{7}{2}:\underbrace{0,0,...,0,0}_{m-5},1,1,1,1,1],[3:\underbrace{0,0,...,0,0}_{m-4},1,1,1,1],[\tfrac{5}{2}:\underbrace{0,0,...,0,0}_{m-3},1,1,1],\end{array}$$

$$\left.\begin{array}{l}[1:\overbrace{0,0,...,0,0}^{m-2},1,1]:\overbrace{[1:1];[1:1];...;[1:1];[1:1]}^{m-1},[1:1],[\tfrac{1}{2}:1];\\ [2:\underbrace{0,0,...,0,0}_{m-2},1,1]:\quad\underbrace{-;-;...;-;-}_{m-1};[\tfrac{3}{2}:1]\qquad ;\end{array}\underbrace{-z^2,-z^2,...,-z^2,-z^2}_{m}\right) \tag{1.4}$$

Generalized hypergeometric function ${}_pF_q(a_1,...,a_p;b_1,...,b_q;z)$ is a function which can be defined in the form of a hypergeometric series, i.e., a series for which the ratio of successive terms can be written

$$\frac{c_{k+1}}{c_k}=\frac{P(k)}{Q(k)}=\frac{(k+a_1)(k+a_2)...(k+a_p)}{(k+b_1)(k+b_2)...(k+b_q)(k+1)}\,z. \tag{1.5}$$

Where $k+1$ in the denominator is present for historical reasons of notation[Koepf p.12(2.9)], and the resulting generalized hypergeometric function is written

$${}_pF_q\left[\begin{array}{ccc}a_1,a_2,\cdots,a_p & ; & \\ & & z\\ b_1,b_2,\cdots,b_q & ; & \end{array}\right]=\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_p)_k z^k}{(b_1)_k(b_2)_k\cdots(b_q)_k k!} \tag{1.6}$$

where the parameters $b_1,b_2,\cdots,b_q$ are positive integers.

The ${}_pF_q$ series converges for all finite z if $p\leq q$, converges for $\mid z\mid<1$ if $p=q+1$, diverges for all z , $z\neq 0$ if $p>q+1$[Luke p.156(3)].

The function ${}_2F_1(a,b;c;z)$ corresponding to $p=2,q=1$, is the first hypergeometric function to be studied, and so is frequently known as "the" hypergeometric equation or, more explicitly, Gauss's hypergeometric function [Gauss p.123-162]. To confuse matters even more, the term "hypergeometric function" is less commonly used to mean closed form, and "hypergeometric series" is sometimes used to mean hypergeometric function.

In mathematics, the falling factorial or Pochhammer symbol is defined as the polynomial[Steffensen p.8]

$$(\alpha)_n = \alpha(\alpha-1)(\alpha-2).......(\alpha-n+1) = \prod_{k=1}^{n}(\alpha-k+1) = \prod_{k=0}^{n-1}(\alpha-k) \quad (1.7)$$

Continued fraction, expression of a number as the sum of an integer and a quotient, the denominator of which is the sum of an integer and a quotient, and so on.
Generally,

$$z = \left[a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{a_4 + \ldots}}}}\right] \quad (1.8)$$

where $a_0, a_1, a_2,\ \& \ b_0, b_1, b_2,$ are all integers.

## 2. MAIN FORMULAE

$$\frac{1}{\left[1 + \cfrac{z^2}{3 + \cfrac{4z^2}{5 + \cfrac{9z^2}{7 + \cfrac{16z^2}{9 + \ldots}}}}\right]^2} = \frac{\left[z - \cfrac{z^3}{3 + \cfrac{9z^2}{5 + \cfrac{4z^2}{7 + \cfrac{25z^2}{9 + \cfrac{16z^2}{11 + \ldots}}}}}\right]^2}{z^2} = \frac{1}{\left[1 + z^2 + - \cfrac{2z^2}{3 - \cfrac{2z^2}{5(1+z^2) - \cfrac{12z^2}{7 - \cfrac{12z^2}{9(1+z^2) + \ldots}}}}\right]^2}$$

$$= \frac{1}{\left[1 + \cfrac{z^2}{1+z^2-2(-1+z^2) + \cfrac{9z^2}{1+z^2-4(-1+z^2) + \cfrac{25z^2}{1+z^2-6(-1+z^2) + \cfrac{49z^2}{1+\ldots+z^2-8(-1+z^2)}}}}\right]^2} =$$

$$F^{1:1;2}_{1:0;1}\left(\begin{matrix}[1{:}1,1]{:}\ [1{:}1];\ [\tfrac{1}{2}{,}1],\ [1{:}1]; \\ [2{:}1,1]{:}\ -;[\tfrac{3}{2}{:}1] \qquad\qquad ; \end{matrix} - z^2, -z^2\right) \quad (2.1)$$

$$\frac{1}{\left[1 + \cfrac{z^2}{3 + \cfrac{4z^2}{5 + \cfrac{9z^2}{7 + \cfrac{16z^2}{9 + \ldots}}}}\right]^3} = \frac{\left[z - \cfrac{z^3}{3 + \cfrac{9z^2}{5 + \cfrac{4z^2}{7 + \cfrac{25z^2}{9 + \cfrac{16z^2}{11 + \ldots}}}}}\right]^3}{z^3} = \frac{1}{\left[1 + z^2 + - \cfrac{2z^2}{3 - \cfrac{2z^2}{5(1+z^2) - \cfrac{12z^2}{7 - \cfrac{12z^2}{9(1+z^2) + \ldots}}}}\right]^3}$$

$$= \frac{1}{\left[1 + \cfrac{z^2}{1+z^2-2(-1+z^2) + \cfrac{9z^2}{1+z^2-4(-1+z^2) + \cfrac{25z^2}{1+z^2-6(-1+z^2) + \cfrac{49z^2}{1+\ldots+z^2-8(-1+z^2)}}}}\right]^3} =$$

$$= F^{2:1;1;2}_{2:0;0;1}\left(\begin{matrix}[\frac{3}{2}:1,1,1],[1:0,1,1]:[1:1];[1:1];[\frac{1}{2}:1],[1:1]\,; \\ [\frac{5}{2}:1,1,1],[2:0,1,1]:-;-;[\frac{3}{2}:1]\qquad ;\end{matrix} -z^2,-z^2,-z^2\right) \tag{2.2}$$

$$\frac{1}{\left[1+\cfrac{z^2}{3+\cfrac{4z^2}{5+\cfrac{9z^2}{7+\cfrac{16z^2}{9+\ldots}}}}\right]^4} = \frac{\left[z-\cfrac{z^3}{3+\cfrac{9z^2}{5+\cfrac{4z^2}{7+\cfrac{25z^2}{9+\cfrac{16z^2}{11+\ldots}}}}}\right]^4}{z^4} = \frac{1}{\left[1+z^2+-\cfrac{2z^2}{3-\cfrac{2z^2}{5(1+z^2)-\cfrac{12z^2}{7-\cfrac{12z^2}{9(1+z^2)+\ldots}}}}\right]^4}$$

$$= \frac{1}{\left[1+\cfrac{z^2}{1+z^2-2(-1+z^2)+\cfrac{9z^2}{1+z^2-4(-1+z^2)+\cfrac{25z^2}{1+z^2-6(-1+z^2)+\cfrac{49z^2}{1+\ldots+z^2-8(-1+z^2)}}}}\right]^4} =$$

$$= F^{3:1;1;1;2}_{3:0;0;0;1}\left(\begin{matrix}[2:1,1,1,1],[\frac{3}{2}:0,1,1,1],[1:0,0,1,1]:[1:1];[1:1];[1:1];[1:1],[\frac{1}{2}:1]; \\ [3:1,1,1,1],[\frac{5}{2}:0,1,1,1],[2:0,0,1,1]:-;-;-;[\frac{3}{2}:1]\qquad ;\end{matrix} -z^2,-z^2,-z^2,-z^2\right) \tag{2.3}$$

$$\frac{1}{\left[1+\cfrac{z^2}{3+\cfrac{4z^2}{5+\cfrac{9z^2}{7+\cfrac{16z^2}{9+\ldots}}}}\right]^m} = \frac{\left[z-\cfrac{z^3}{3+\cfrac{9z^2}{5+\cfrac{4z^2}{7+\cfrac{25z^2}{9+\cfrac{16z^2}{11+\ldots}}}}}\right]^m}{z^m} = \frac{1}{\left[1+z^2+-\cfrac{2z^2}{3-\cfrac{2z^2}{5(1+z^2)-\cfrac{12z^2}{7-\cfrac{12z^2}{9(1+z^2)+\ldots}}}}\right]^m}$$

$$= \frac{1}{\left[1+\cfrac{z^2}{1+z^2-2(-1+z^2)+\cfrac{9z^2}{1+z^2-4(-1+z^2)+\cfrac{25z^2}{1+z^2-6(-1+z^2)+\cfrac{49z^2}{1+\ldots+z^2-8(-1+z^2)}}}}\right]^m} =$$

$$= F^{m-1:\overbrace{1;1;\ldots;1;1;2}^{m-1}}_{m-1:\underbrace{0;0;\ldots;0;0;1}_{m-1}}\left(\begin{matrix}[\frac{m}{2}:\overbrace{1,1,\ldots,1,1}^{m}],[\frac{m-1}{2}:0,\overbrace{1,1,\ldots,1,1}^{m-1}],[\frac{m-3}{2}:0,0,\overbrace{1,1,\ldots,1,1}^{m-2}],[\frac{m-4}{2}:0,0,0,\overbrace{1,1,\ldots,1,1}^{m-3}], \\ [\frac{m+2}{2}:\underbrace{1,1,\ldots,1,1}_{m}],[\frac{m+1}{2}:0,\underbrace{1,1,\ldots,1,1}_{m-1}],[\frac{m}{2}:0,0,\underbrace{1,1,\ldots,1,1}_{m-2}],[\frac{m-1}{2}:0,0,0,\underbrace{1,1,\ldots,1,1}_{m-3}],\end{matrix}\right.$$

$$\begin{matrix}\ldots,[3:\overbrace{0,0,\ldots,0,0}^{m-6},1,1,1,1,1,1,1],[\frac{5}{2}:\overbrace{0,0,\ldots,0,0}^{m-5},1,1,1,1,1,1],[2:\overbrace{0,0,\ldots,0,0}^{m-4},1,1,1,1,1],[\frac{3}{2}:\overbrace{0,0,\ldots,0,0}^{m-3},1,1,1], \\ \ldots,[4:\underbrace{0,0,\ldots,0,0}_{m-6},1,1,1,1,1,1,1],[\frac{7}{2}:\underbrace{0,0,\ldots,0,0}_{m-5},1,1,1,1,1,1],[3:\underbrace{0,0,\ldots,0,0}_{m-4},1,1,1,1,1],[\frac{5}{2}:\underbrace{0,0,\ldots,0,0}_{m-3},1,1,1],\end{matrix}$$

$$\left.\begin{matrix} [1:\overbrace{0,0,...,0,0}^{m-2},1,1]: \overbrace{[1:1];[1:1];...;[1:1];[1:1]}^{m-1},[1:1],[\frac{1}{2}:1]; \\ [2:\underbrace{0,0,...,0,0}_{m-2},1,1]: \underbrace{-;-;...;-;-}_{m-1};[\frac{3}{2}:1] \qquad ; \end{matrix} \underbrace{-z^2,-z^2,...,-z^2,-z^2}_{m}\right) \tag{2.4}$$

## 3. DERIVATION OF THE FORMULAE

Derivation of (2.1) We have

$$\left(\frac{tan^{-1}z}{z}\right)^2 = \frac{1}{\left[1+\frac{z^2}{3+\frac{4z^2}{5+\frac{9z^2}{7+\frac{16z^2}{9+...}}}}\right]^2} = \frac{\left[z-\frac{z^3}{3+\frac{9z^2}{5+\frac{4z^2}{7+\frac{25z^2}{9+\frac{16z^2}{11+...}}}}}\right]^2}{z^2} = \frac{1}{\left[1+z^2+-\frac{2z^2}{3-\frac{2z^2}{5(1+z^2)-\frac{12z^2}{7-\frac{12z^2}{9(1+z^2)+...}}}}\right]^2}$$

$$= \frac{1}{\left[1+\frac{z^2}{1+z^2-2(-1+z^2)+\frac{9z^2}{1+z^2-4(-1+z^2)+\frac{25z^2}{1+z^2-6(-1+z^2)+\frac{49z^2}{1+...+z^2-8(-1+z^2)}}}}\right]^2} \tag{3.1}$$

Again from(1.1),

$$\left(\frac{tan^{-1}z}{z}\right)^2 = F^{1:1;2}_{1:0;1}\left(\begin{matrix}[1:1,1]:[1:1];[\frac{1}{2},1],[1:1]; \\ [2:1,1]:-;[\frac{3}{2}:1] \qquad ;\end{matrix} -z^2,-z^2\right) \tag{3.2}$$

Now from (3.1) and (3.2), we get

$$\frac{1}{\left[1+\frac{z^2}{3+\frac{4z^2}{5+\frac{9z^2}{7+\frac{16z^2}{9+...}}}}\right]^2} = \frac{\left[z-\frac{z^3}{3+\frac{9z^2}{5+\frac{4z^2}{7+\frac{25z^2}{9+\frac{16z^2}{11+...}}}}}\right]^2}{z^2} = \frac{1}{\left[1+z^2+-\frac{2z^2}{3-\frac{2z^2}{5(1+z^2)-\frac{12z^2}{7-\frac{12z^2}{9(1+z^2)+...}}}}\right]^2}$$

$$= \frac{1}{\left[1+\frac{z^2}{1+z^2-2(-1+z^2)+\frac{9z^2}{1+z^2-4(-1+z^2)+\frac{25z^2}{1+z^2-6(-1+z^2)+\frac{49z^2}{1+...+z^2-8(-1+z^2)}}}}\right]^2} =$$

$$F^{1:1;2}_{1:0;1}\left(\begin{matrix}[1:1,1]:[1:1];[\frac{1}{2},1],[1:1]; \\ [2:1,1]:-;[\frac{3}{2}:1] \qquad ;\end{matrix} -z^2,-z^2\right)$$

Hence the result(2.1) is proved. In this way remaining formulae can be proved.

**Conflicts of Interest:** The author declares that there is no conflicts of interest regarding the publication of this article.

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