

An Introduction to Multi Metric Spaces

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Abstract

In the present paper notions of multi real point and multi metric space are presented and some basic properties of multi metric space are investigated.

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1. INTRODUCTION

Multiset (bag) is a well established notion both in mathematics and computer science ([9], [10], [19]). In mathematics, a multiset is considered to be the generalization of a set. In classical set theory, a set is a well-defined collection of distinct objects. If repeated occurrences of any object is allowed in a set, then a mathematical structure, that is known as multiset (mset, for short), is obtained ([18], [20], [21]). In various counting arguments it is convenient to distinguish between a set like $\{a, b, c\}$ and a collection like $\{a, a, a, b, c, c\}$. The latter, if viewed as a set, will be identical to the former. However, it has some of its elements purposely listed several times. We formalize it by defining a multiset as a collection of elements, each considered with certain multiplicity. For the sake of convenience a multiset is written as $\{k_1/x_1, k_2/x_2, \dots, k_n/x_n\}$ in which the element x_i occurs k_i times. We observe that each multiplicity k_i is a positive integer. A selective list of references can be given as ([1], [2],[3],[4], [5], [6], [7], [8],[11], [12],[13],[14],[15], [16], [17], [22]).

Classical set theory states that a given element can appear only once in a set, it assumes that all mathematical objects occur without repetition. Thus there is only one number four, one field of complex numbers, etc. So, the only possible relation between two mathematical objects is either they are equal or they are different.

The situation in science and in ordinary life is not like this. In the physical world it is observed that there is enormous repetition. For instance, there are many hydrogen atoms, many water molecules, many strands of DNA, etc. Coins of the same denomination and year, electrons or grains of sand appear similar, despite being obviously separate. This leads to three possible relations between any two physical objects; they are different, they are the same but separate or they coincide and are identical. For the sake of definiteness we say that two physical objects are the same or equal, if they are indistinguishable, but possibly separate, and identical if they physically coincide.

The concept of metric structure is very basic in mathematics and it has wide applications in different fields. In this paper, for the first time, we introduce a notion of multi real point and multi metric space. Starting from the definition, some basic properties of multi real points and multi metric spaces are studied.

2. PRELIMINARIES

Definition 2.1. [11] An mset M drawn from the set X is represented by a function Count M or C_M defined as $C_M : X \rightarrow N$ where N represents the set of non negative integers.

Here $C_M(x)$ is the number of occurrences of the element x in the mset M . We present the mset M drawn from the set $X = \{x_1, x_2, \dots, x_n\}$ as $M = \{m_1/x_1, m_2/x_2, \dots, m_n/x_n\}$ where m_i is the number of occurrences of the element $x_i, i = 1, 2, \dots, n$ in the mset M . However those elements which are not included in the mset M have zero count.

Example 2.2. [11] Let $X = \{a, b, c, d, e\}$ be any set. Then $M = \{2/a, 4/b, 5/d, 1/e\}$ is an mset drawn from X . Clearly, a set is a special case of an mset.

Definition 2.3. [11] Let M and N be two msets drawn from a set X . Then, the following are defined:

- (i) $M = N$ if $C_M(x) = C_N(x)$ for all $x \in X$.
- (ii) $M \subseteq N$ if $C_M(x) \leq C_N(x)$ for all $x \in X$.
- (iii) $P = M \cup N$ if $C_P(x) = \text{Max}\{C_M(x), C_N(x)\}$ for all $x \in X$.
- (iv) $P = M \cap N$ if $C_P(x) = \text{Min}\{C_M(x), C_N(x)\}$ for all $x \in X$.
- (v) $P = M \oplus N$ if $C_P(x) = C_M(x) + C_N(x)$ for all $x \in X$.

- (vi) $P = M \ominus N$ if $C_P(x) = \text{Max}\{C_M(x) - C_N(x), 0\}$ for all $x \in X$, where \oplus and \ominus represents mset addition and mset subtraction respectively.

Let M be an mset drawn from a set X . The support set of M denoted by M^* is a subset of X and $M^* = \{x \in X : C_M(x) > 0\}$, i.e., M^* is an ordinary set. M^* is also called root set.

An mset M is said to be an empty mset if for all $x \in X$, $C_M(x) = 0$. The cardinality of an mset M drawn from a set X is denoted by $\text{Card}(M)$ or $\text{Card}M$ and is given by

$$\text{Card}M = \sum_{x \in X} C_M(x).$$

Definition 2.4. [11] A domain X , is defined as a set of elements from which msets are constructed. The mset space $[X]^w$ is the set of all msets whose elements are in X such that no element in the mset occurs more than w times. The set $[X]^\infty$ is the set of all msets over a domain X such that there is no limit on the number of occurrences of an element in an mset. If $X = \{x_1, x_2, \dots, x_k\}$ then $[X]^w = \{\{m_1/x_1, m_2/x_2, \dots, m_k/x_k\} : \text{for } i = 1, 2, \dots, k; m_i \in \{0, 1, 2, \dots, w\}\}$.

Definition 2.5. [11] Let X be a support set and $[X]^w$ be the mset space defined over X . Then for any mset $M \in [X]^w$, the complement M^c of M in $[X]^w$ is an element of $[X]^w$ such that $C_{M^c}(x) = w - C_M(x)$, for all $x \in X$.

Definition 2.6. [11] Let $[X]^w$ be an mset space and $\{M_1, M_2, \dots\}$ be a collection of msets drawn from $[X]^w$. Then the following operations are possible under an arbitrary collection of msets.

- (i) The union $\bigcup_{i \in I} M_i = \{C_{\bigcup M_i}(x)/x : C_{\bigcup M_i}(x) = \text{max}\{C_{M_i}(x) : x \in X\}$.
- (ii) The intersection $\bigcap_{i \in I} M_i = \{C_{\bigcap M_i}(x)/x : C_{\bigcap M_i}(x) = \text{min}\{C_{M_i}(x) : x \in X\}$.
- (iii) The mset addition $\bigoplus_{i \in I} M_i = \{C_{\bigoplus M_i}(x)/x : C_{\bigoplus M_i}(x) = \sum_{i \in I} \{C_{M_i}(x) : x \in X\}$.
- (iv) The mset complement $M^c = Z \ominus M = \{C_{M^c}(x)/x : C_{M^c}(x) = C_Z(x) - C_M(x), x \in X\}$.

Definition 2.7. [11] Let M_1 and M_2 be two msets drawn from a set X , then the Cartesian product of M_1 and M_2 is defined as $M_1 \times M_2 = \{(m/x, n/y)/mn : x \in {}^m M_1, y \in {}^n M_2\}$.

We can define the Cartesian product of three or more nonempty msets by generalizing the definition of the Cartesian product of two msets.

Definition 2.8. [11] A sub mset R of $M \times M$ is said to be an mset relation on M if every member $(m/x, n/y)$ of R has a count, product of $C_1(x, y)$ and $C_2(x, y)$. We denote m/x related to n/y by $m/xRn/y$.

The Domain and Range of the mset relation R on M is defined as follows:

$DomR = \{x \in {}^r M : \exists y \in {}^s M \text{ such that } r/xRs/y\}$ where

$$C_{DomR}(x) = \sup\{C_1(x, y) : x \in {}^r M\}.$$

$RanR = \{y \in {}^s M : \exists x \in {}^r M \text{ such that } r/xRs/y\}$ where

$$C_{RanR}(y) = \sup\{C_2(x, y) : y \in {}^s M\}.$$

Definition 2.9. [11]

- (i) An mset relation R on an mset M is reflexive if $m/xRm/x$ for all m/x in M .
- (ii) An mset relation R on an mset M is symmetric if $m/xRn/y$ implies $n/yRm/x$.
- (iii) An mset relation R on an mset M is transitive if $m/xRn/y, n/yRk/z$ then $m/xRk/z$.

An mset relation R on an mset M is called an equivalence mset relation if it is reflexive, symmetric and transitive.

Definition 2.10. [11] An mset relation f is called an mset function if for every element m/x in $Domf$, there is exactly one n/y in $Ranf$ such that $(m/x, n/y)$ is in f with the pair occurring as the product of $C_1(x, y)$ and $C_2(x, y)$.

Definition 2.11. [11] Let $M \in [X]^w$ and $\tau \subseteq P^*(M)$. Then τ is called a multiset topology of M if τ satisfies the following properties.

1. The mset M and the empty mset \emptyset are in τ .
2. The mset union of the elements of any sub collection of τ is in τ .
3. The mset intersection of the elements of any finite sub collection of τ is in τ .

Mathematically a multiset topological space is an ordered pair (M, τ) consisting of an mset $M \in [X]^w$ and a multiset topology $\tau \subseteq P^*(M)$ on M . Note that τ is an ordinary set whose elements are msets. Multiset Topology is abbreviated as an M -topology.

3. MULTI POINTS AND MULTI REAL POINTS

Definition 3.1. Multi point: Let M be a multi set over a universal set X . Then a multi point of M is defined by a mapping $P_x^k : X \rightarrow \mathbb{N}$ such that $P_x^k(x) = k$ where $k \leq C_M(x)$.

x and k will be referred to as the **base** and the **multiplicity** of the multi point P_x^k respectively.

Collection of all multi points of an mset M is denoted by M_{pt} .

Definition 3.2. The mset generated by a collection B of multi points is denoted by $MS(B)$ and is defined by $C_{MS(B)}(x) = \text{Sup}\{k : P_x^k \in B\}$

An mset can be generated from the collection of its multi points. If M_{pt} denotes the collection of all multi points of M , then obviously

$$C_M(x) = \text{Sup}\{k : P_x^k \in M_{pt}\} \text{ and hence } M = MS(M_{pt}).$$

Definition 3.3. (i) The **elementary union** between two collections of multi points C and D is denoted by $C \sqcup D$ and is defined as

$$C \sqcup D = \{P_x^k : P_x^l \in C, P_x^m \in D \text{ and } k = \max\{l, m\}\}$$

(ii) The **elementary intersection** between two collections of multi points C and D is denoted by $C \sqcap D$ and is defined as

$$C \sqcap D = \{P_x^k : P_x^l \in C, P_x^m \in D \text{ and } k = \min\{l, m\}\}$$

(iii) For two collections of multi points C and D , C is said to be an **elementary subset** of D , denoted by $C \sqsubset D$, iff $P_x^l \in C \Rightarrow \exists m \geq l$ such that $P_x^m \in D$.

Note 3.4. (i) Clearly $C \sqsubset C \sqcup D$, $D \sqsubset C \sqcup D$, $C \sqcap D \sqsubset C$ and $C \sqcap D \sqsubset D$.

(ii) In general the relations $C \subset C \sqcup D$, $D \subset C \sqcup D$, $C \sqcap D \subset C$ and $C \sqcap D \subset D$ do not hold. For example, in any M -metric space (M, d) , let $C, D \subset M_{pt}$ where $C = \{P_a^2, P_b^3\}$ and $D = \{P_a^3, P_b^2\}$. Then $C \sqcup D = \{P_a^3, P_b^3\}$ and $C \sqcap D = \{P_a^2, P_b^2\}$. Hence the above statement follows.

The following results can be easily proved:

Theorem 3.5. (i) For two collections of multi points C and D , $C \subset D \Rightarrow C \sqsubset D$, but the converse is not true.

(ii) For two collections of multi points C and D , $C \cup D \supset C \sqcup D$ and the equality does not hold in general.

(iii) For two collections of multi points C and D , $C \cap D \subset C \sqcap D$ and the equality does not hold in general.

(iv) For an mset M , $MS(M_{pt}) = M$.

(v) For a collection B of multi points, $[MS(B)]_{pt} \supset B$.

(vi) For two msets F and G , $F \subset G \Leftrightarrow F_{pt} \subset G_{pt}$.

(vii) For two collections of multi points C and D ,

$$C \subset D \Rightarrow MS(C) \subset MS(D).$$

(viii) For two collections of multi points C and D ,

$$C \sqsubset D \Leftrightarrow MS(C) \subset MS(D).$$

(ix) For two collections of multi points C and D ,

$$MS(C \sqcap D) = MS(C) \cap MS(D)$$

(x) For an arbitrary collection $\{B_i : i \in \Delta\}$ of multi points,

$$MS(\sqcup_{i \in \Delta} B_i) = \cup_{i \in \Delta} MS(B_i)$$

(xi) For an arbitrary collection $\{B_i : i \in \Delta\}$ of multi points,

$$MS(\cup_{i \in \Delta} B_i) = \cup_{i \in \Delta} MS(B_i)$$

Proof. (ix) Let $k = C_{MS(C) \cap MS(D)}(x) = \text{Min} \{C_{MS(C)}(x), C_{MS(D)}(x)\}$
 $\Rightarrow x \in^l MS(C), x \in^m MS(D)$ and $k = \text{Min}\{l, m\} \Rightarrow P_x^l \in C, P_x^m \in D$ and $k = \text{Min}\{l, m\}$
 $\Rightarrow P_x^k \in C \sqcap D \Rightarrow C_{MS(C \sqcap D)}(x) \geq k = C_{MS(C) \cap MS(D)}(x)$ Again clearly $C \sqcap D \sqsubset C$
and $C \sqcap D \sqsubset D \Rightarrow MS(C \sqcap D) \subset MS(C)$ and $MS(C \sqcap D) \subset MS(D)$
 $\Rightarrow MS(C \sqcap D) \subset MS(C) \cap MS(D)$

Thus from above two relations $MS(C \sqcap D) = MS(C) \cap MS(D)$.

(x) Let $k = C_{MS(\sqcup_{i \in \Delta} B_i)}(x) \Rightarrow P_x^k \in \sqcup_{i \in \Delta} B_i \Rightarrow$ For each $i \in \Delta$, $\exists P_x^{l_i} \in B_i$ such that
 $k = \text{Max}_{i \in \Delta} l_i \Rightarrow k = \text{Max}_{i \in \Delta} l_i$ and $l_i \leq C_{MS(B_i)}(x), \forall i \in \Delta \Rightarrow k = \text{Max}_{i \in \Delta} l_i \leq$
 $\text{Max}_{i \in \Delta} \{C_{MS(B_i)}(x)\} = C_{\cup_{i \in \Delta} MS(B_i)}(x) \Rightarrow C_{MS(\sqcup_{i \in \Delta} B_i)}(x) \leq C_{\cup_{i \in \Delta} MS(B_i)}(x)$
Again clearly $B_i \sqsubset \sqcup_{i \in \Delta} B_i, \forall i \in \Delta \Rightarrow MS(B_i) \subset MS(\sqcup_{i \in \Delta} B_i), \forall i \in \Delta$
 $\Rightarrow \cup_{i \in \Delta} MS(B_i) \subset MS(\sqcup_{i \in \Delta} B_i)$

Thus from the above two relations, the result holds.

(xi) $\forall x \in X, C_{\cup_{i \in \Delta} MS(B_i)}(x) = \bigvee_{i \in \Delta} C_{MS(B_i)}(x) = \bigvee_{i \in \Delta} \bigvee_{P_x^l \in B_i} (l)$
 $= \bigvee_{P_x^l \in \cup_{i \in \Delta} B_i} (l) = C_{MS(\cup_{i \in \Delta} B_i)}(x)$ □

Definition 3.6. Let $m\mathbb{R}^+$ denotes the multi set over \mathbb{R}^+ (set of non-negative real numbers) having multiplicity of each element equal to w , $w \in \mathbb{N}$. The members of $(m\mathbb{R}^+)_{pt}$ will be called **non-negative multi real points**.

Definition 3.7. Let P_a^i and P_b^j be two multi real points of $m\mathbb{R}^+$. We define $P_a^i > P_b^j$ if $a > b$ or $P_a^i > P_b^j$ if $i > j$ when $a = b$.

Definition 3.8. (Addition of multi real points) We define $P_a^i + P_b^j = P_{a+b}^k$ where $k = \text{Max}\{i, j\}$, $P_a^i, P_b^j \in (m\mathbb{R}^+)_{pt}$.

Definition 3.9. (Multiplication of multi real points) We define multiplication of two multi real points in $m\mathbb{R}^+$ as follows:

$$P_a^i \times P_b^j = P_0^1, \text{ if either } P_a^i \text{ or } P_b^j \text{ equal to } P_0^1 \\ = P_{ab}^k, \text{ otherwise where } k = \text{Max}\{i, j\}$$

Proposition 3.10. (Properties of multiplication) Multiplication of multi real points satisfies the following properties:

- (i) Multiplication is Commutative.
- (ii) Multiplication is Associative.
- (iii) Multiplication is distributive over addition.

Proof. (i) holds from the definition of multiplication of multi real points.

(ii) If none of P_a^i, P_b^j, P_c^k is equal to P_0^1 ,

$$P_a^i \times (P_b^j \times P_c^k) = P_a^i \times P_{bc}^{\text{Max}\{j,k\}} = P_{abc}^{\text{Max}\{i,j,k\}} = (P_a^i \times P_b^j) \times P_c^k$$

If any of P_a^i, P_b^j, P_c^k is equal to P_0^1 , then clearly each of the product is P_0^1

(iii) Let $P_a^i, P_b^j, P_c^k \in (m\mathbb{R}^+)_{pt}$. To show that $P_a^i \times (P_b^j + P_c^k) = P_a^i \times P_b^j + P_a^i \times P_c^k$

If none of P_a^i, P_b^j, P_c^k is equal to P_0^1 ,

$$P_a^i \times (P_b^j + P_c^k) = P_a^i \times P_{b+c}^{\text{Max}\{j,k\}} = P_{a(b+c)}^{\text{Max}\{i, \text{Max}\{j,k\}\}} = P_{ab+ac}^{\text{Max}\{i,j,k\}}$$

$$\text{and } P_a^i \times P_b^j + P_a^i \times P_c^k = P_{ab}^{\text{Max}\{i,j\}} + P_{ac}^{\text{Max}\{i,k\}} = P_{ab+ac}^{\text{Max}\{\text{Max}\{i,j\}, \text{Max}\{i,k\}\}} = P_{ab+ac}^{\text{Max}\{i,j,k\}}$$

$$\therefore P_a^i \times (P_b^j + P_c^k) = P_a^i \times P_b^j + P_a^i \times P_c^k$$

If $P_a^i = P_0^1$ then $P_a^i \times (P_b^j + P_c^k) = P_0^1$ and $P_a^i \times P_b^j + P_a^i \times P_c^k = P_0^1 + P_0^1 = P_0^1 \forall P_b^j, P_c^k \in (m\mathbb{R}^+)_{pt}$ which gives the desired result.

If $P_b^j = P_0^1$ or $P_c^k = P_0^1$ [for definiteness say $P_b^j = P_0^1$], $P_a^i \times (P_b^j + P_c^k) = P_a^i \times (P_0^1 + P_c^k) = P_a^i \times P_c^k$ and $P_a^i \times P_b^j + P_a^i \times P_c^k = P_0^1 + P_a^i \times P_c^k = P_a^i \times P_c^k$ which also gives the desired result.

Thus the multiplication is distributive over addition. □

Note 3.11. Addition and multiplication of non negative multi real points are compatible ie $\forall P_a^l, P_b^m, P_c^n \in (m\mathbb{R}^+)_{pt}$, (i) $P_a^l \geq P_b^m \Rightarrow P_a^l + P_c^n \geq P_b^m + P_c^n$ and (ii) $P_a^l \geq P_b^m \Rightarrow P_a^l \times P_c^n \geq P_b^m \times P_c^n$

Proof. (i) $P_a^l \geq P_b^m \Rightarrow a > b$ or $a = b, l \geq m \Rightarrow a + c > b + c$ or $a + c = b + c$, $\text{Max}\{l, n\} \geq \text{Max}\{m, n\} \Rightarrow P_a^l + P_c^n \geq P_b^m + P_c^n$.

(ii) Same as above. □

Note 3.12. For $P_a^l, P_b^m, P_c^n \in (m\mathbb{R}^+)_{pt}$, we may have $P_a^l > P_b^m$ but $P_a^l + P_c^n = P_b^m + P_c^n$ and $P_a^l \times P_c^n = P_b^m \times P_c^n$ ie. $\forall P_a^l, P_b^m, P_c^n \in (m\mathbb{R}^+)_{pt}$, (i) $P_a^l > P_b^m \Rightarrow P_a^l + P_c^n \geq P_b^m + P_c^n$ and (ii) $P_a^l > P_b^m \Rightarrow P_a^l \times P_c^n \geq P_b^m \times P_c^n$.

To show this we consider the following example:

Let $P_a^l > P_b^m$ where $a = b, l > m$ and P_c^n is such that $n \geq \text{Max}\{l, m\} = l$. Then $P_a^l + P_c^n = P_{a+c}^n = P_{b+c}^n = P_b^m + P_c^n$ and in a similar way $P_a^l \times P_c^n = P_b^m \times P_c^n$.

On the other hand, if for $P_a^l, P_b^m, P_c^n \in (m\mathbb{R}^+)_{pt}$, the relation $P_a^l + P_c^n = P_b^m + P_c^n$ holds, then we must have $a = b$ and $n \geq \text{Max}\{l, m\}$.

4. MULTI METRIC SPACE

Definition 4.1. Multi Metric: Let $d : M_{pt} \times M_{pt} \rightarrow (m\mathbb{R}^+)_{pt}$ (M being a multi set over a Universal set X having multiplicity of any element atmost equal to w) be a mapping which satisfy the following:

$$(M1) d(P_x^l, P_y^m) \geq P_0^1, \forall P_x^l, P_y^m \in M_{pt}$$

$$(M2) d(P_x^l, P_y^m) = P_0^1 \text{ iff } P_x^l = P_y^m, \forall P_x^l, P_y^m \in M_{pt}$$

$$(M3) d(P_x^l, P_y^m) = d(P_y^m, P_x^l), \forall P_x^l, P_y^m \in M_{pt}$$

$$(M4) d(P_x^l, P_y^m) + d(P_y^m, P_z^n) \geq d(P_x^l, P_z^n), \forall P_x^l, P_y^m, P_z^n \in M_{pt}.$$

$$(M5) \text{ For } l \neq m, d(P_x^l, P_y^m) = P_0^k \Leftrightarrow x = y \text{ and } k = \text{Max}\{l, m\}.$$

Then d is said to be a multi metric on M and (M, d) is called a Multi metric (or an M -metric) space.

Example 4.2. Let M be a multi set over X having multiplicity of any element atmost equal to w . We define

$d : M_{pt} \times M_{pt} \longrightarrow (m\mathbb{R}^+)_{pt}$ such that

$$d(P_x^l, P_y^m) = P_0^1 \text{ if } P_x^l = P_y^m \\ = P_0^{Max\{l,m\}} \text{ if } x = y \text{ and } l \neq m$$

$$= P_1^j \text{ if } x \neq y \forall P_x^l, P_y^m \in M_{pt}, [1 \leq j \leq w \text{ is a fixed positive integer}]$$

Then (M1) $\forall P_x^l, P_y^m \in M_{pt}, d(P_x^l, P_y^m) \geq P_0^1$

(M2) Let $P_x^l = P_y^m \Rightarrow d(P_x^l, P_y^m) = P_0^1$ and $d(P_x^l, P_y^m) = P_0^1 \Rightarrow P_x^l = P_y^m$ since otherwise if $P_x^l \neq P_y^m$ then $d(P_x^l, P_y^m) > P_0^1$ — a contradiction.

(M3) Clearly $d(P_x^l, P_y^m) = d(P_y^m, P_x^l) \forall P_x^l, P_y^m \in M_{pt}$

(M4) Let $P_x^l, P_y^m, P_z^n \in M_{pt}$.

Case I : If $d(P_x^l, P_z^n) = P_0^1$, clearly $d(P_x^l, P_z^n) \leq d(P_x^l, P_y^m) + d(P_y^m, P_z^n)$.

Case II : If $d(P_x^l, P_z^n) = P_0^{Max\{l,n\}}$, then $x = z$ and $l \neq n$.

Subcase IIa : $y \neq x = z \Rightarrow d(P_x^l, P_y^m) = d(P_y^m, P_z^n) = P_1^j$ and obviously $d(P_x^l, P_z^n) \leq d(P_x^l, P_y^m) + d(P_y^m, P_z^n)$.

Subcase IIb: $y = x = z$

Subcase IIb(i): $m \neq l, m \neq n$. Then $d(P_x^l, P_y^m) = P_0^{Max\{l,m\}}$ and

$$d(P_y^m, P_z^n) = P_0^{Max\{m,n\}} \\ \Rightarrow d(P_x^l, P_y^m) + d(P_y^m, P_z^n) = P_0^{Max\{l,m\}} + P_0^{Max\{m,n\}} = P_0^{Max\{Max\{l,m\}, Max\{m,n\}\}} \\ = P_0^{Max\{l,m,n\}} \geq P_0^{Max\{l,n\}} = d(P_x^l, P_z^n)$$

Subcase IIb(ii) : $m = l, m \neq n$. Then $d(P_x^l, P_y^m) + d(P_y^m, P_z^n) = P_0^1 + P_0^{Max\{m,n\}} = P_0^{Max\{1, Max\{m,n\}\}} = P_0^{Max\{m,n\}} = P_0^{Max\{l,n\}} = d(P_x^l, P_z^n)$

Subcase IIb(iii): $m \neq l, m = n$. The result holds in a same manner as above.

Case III : If $d(P_x^l, P_z^n) = P_1^j \Rightarrow x \neq z \Rightarrow x \neq y$ or $y \neq z \Rightarrow d(P_x^l, P_y^m) = P_1^j$ or $d(P_y^m, P_z^n) = P_1^j \Rightarrow d(P_x^l, P_z^n) \leq d(P_x^l, P_y^m) + d(P_y^m, P_z^n)$.

(M5) Follows from the definition of d .

Thus d is an M -metric on M .

Example 4.3. Let $m\mathbb{R} \in [\mathbb{R}]^w$ (the m set space on \mathbb{R}).

Define $d : (m\mathbb{R})_{pt} \times (m\mathbb{R})_{pt} \longrightarrow (m\mathbb{R}^+)_{pt}$ such that

$$d(P_x^l, P_y^m) = P_0^1 \text{ if } P_x^l = P_y^m \\ = P_0^{Max\{l,m\}} \text{ if } x = y \text{ but } l \neq m \\ = P_{|x-y|}^{Max\{l,m\}} \text{ if } x \neq y$$

Then (M1), (M2) and (M3) are obviously satisfied.

To prove (M4) i.e. the triangle inequality, we have to prove

$$d(P_x^l, P_y^m) + d(P_y^m, P_z^n) \geq d(P_x^l, P_z^n), \forall P_x^l, P_y^m, P_z^n \in (m\mathbb{R})_{pt}$$

Case I: If $d(P_x^l, P_z^n) = P_0^1$, the result is obvious.

Case II: Let $d(P_x^l, P_z^n) > P_0^1$. Then $P_x^l \neq P_z^n$. Without loss of generality we may assume that $P_x^l \neq P_y^m$ and $P_y^m \neq P_z^n$.

Subcase IIa: $x \neq y \neq z$ Then $d(P_x^l, P_y^m) = P_{|x-y|}^{Max\{l,m\}}$, $d(P_y^m, P_z^n) = P_{|y-z|}^{Max\{m,n\}}$ and $d(P_x^l, P_z^n) = P_{|x-z|}^{Max\{l,n\}}$.

Now $d(P_x^l, P_y^m) + d(P_y^m, P_z^n) = P_{|x-y|}^{Max\{l,m\}} + P_{|y-z|}^{Max\{m,n\}} = P_{|x-y|+|y-z|}^{Max\{Max\{l,m\}, Max\{m,n\}\}}$
 $= P_{|x-y|+|y-z|}^{Max\{l,m,n\}} \geq P_{|x-z|}^{Max\{l,n\}} = d(P_x^l, P_z^n)$ [as $|x-y| + |y-z| \geq |x-z|$ and $Max\{l, m, n\} \geq Max\{l, n\}$]

Subcase IIb: $x = y$ but $y \neq z$. Then $d(P_x^l, P_y^m) = P_0^{Max\{l,m\}}$, $d(P_y^m, P_z^n) = P_{|y-z|}^{Max\{m,n\}}$ and $d(P_x^l, P_z^n) = P_{|x-z|}^{Max\{l,n\}}$

Now $d(P_x^l, P_y^m) + d(P_y^m, P_z^n) = P_0^{Max\{l,m\}} + P_{|y-z|}^{Max\{m,n\}}$
 $= P_{0+|x-z|}^{Max\{Max\{l,m\}, Max\{m,n\}\}}$ [as $x = y$] $= P_{|x-z|}^{Max\{l,m,n\}} \geq P_{|x-z|}^{Max\{l,n\}} = d(P_x^l, P_z^n)$

Subcase IIc: $x \neq y$ but $y = z$. The desired result follows in a similar way as the above case.

Subcase IIId: $x = y = z$, Then $d(P_x^l, P_y^m) = P_0^{Max\{l,m\}}$, $d(P_y^m, P_z^n) = P_0^{Max\{m,n\}}$ and $d(P_x^l, P_z^n) = P_0^{Max\{l,n\}}$.

Now $d(P_x^l, P_y^m) + d(P_y^m, P_z^n) = P_0^{Max\{l,m\}} + P_0^{Max\{m,n\}}$
 $= P_{0+0}^{Max\{Max\{l,m\}, Max\{m,n\}\}} = P_0^{Max\{l,m,n\}} \geq P_0^{Max\{l,n\}} = d(P_x^l, P_z^n)$

Thus in any case (M4) holds.

(M5) Follows from the definition of d .

In a similar way, another example can be given as the following:

Example 4.4. Let $M \in [X]^w$ (the mset space over X), d is an ordinary metric on X .

Define $d : M_{pt} \times M_{pt} \longrightarrow (m\mathbb{R}^+)_{pt}$ such that

$$\begin{aligned} d(P_x^l, P_y^m) &= P_0^1 \text{ if } P_x^l = P_y^m \\ &= P_0^{Max\{l,m\}} \text{ if } x = y \text{ but } l \neq m \\ &= P_{d(x,y)}^{Max\{l,m\}} \text{ if } x \neq y \end{aligned}$$

Then (M, d) is an M -metric space.

Theorem 4.5. If $d(P_a^i, P_b^j) = P_r^l$ and $d(P_a^p, P_b^q) = P_s^m$, then $r = s$,
 $P_a^i, P_b^j, P_a^p, P_b^q \in M_{pt}$ and $P_r^l, P_s^m \in (m\mathbb{R}^+)_{pt}$.

Proof. If possible let $r \neq s$. If $r < s$, then $d(P_a^p, P_b^q) \leq d(P_a^p, P_a^i) + d(P_a^i, P_b^j) + d(P_b^j, P_b^q) \Rightarrow P_s^m \leq P_0^k + P_r^l + P_0^n$ ($k = \text{Max}\{p, i\}$, $n = \text{Max}\{j, q\}$) $\Rightarrow P_s^m \leq P_r^l$

where $t = \text{Max}\{k, l, n\} \Rightarrow s \leq r$ - a contradiction. So $r \not\leq s$. Similarly $s \not\leq r$. So $r = s$. \square

5. SUBSPACE, DIAMETER & DISTANCE

Definition 5.1. Let (M, d) be an M -metric space and L be a non null sub mset of M . Then the mapping $d_L : L_{pt} \times L_{pt} \rightarrow (m\mathbb{R}^+)_{pt}$ given by $d_L(P_x^a, P_y^b) = d(P_x^a, P_y^b), \forall P_x^a, P_y^b \in L_{pt}$ is an M -metric on L . The metric is known as the **relative M -metric** induced by d on L . The M -metric space (L, d_L) is called an **M -metric subspace** or simply an **M -subspace** of the M -metric space (M, d) .

Definition 5.2. Let (M, d) be an M -metric space and L be a nonempty subset of M . Then the diameter of L , denoted by $\delta(L)$ is defined by:

$$\delta(L) = P_a^k \text{ where } a = \text{Sup}\{b : P_b^j = d(P_x^l, P_y^m), P_x^l, P_y^m \in L_{pt}\},$$

$$k = 1 \text{ if } a > b \forall P_b^j = d(P_x^l, P_y^m), P_x^l, P_y^m \in L_{pt} \text{ and}$$

$$= \text{Max}\{j : P_b^j = d(P_x^l, P_y^m), P_x^l, P_y^m \in L_{pt}\} \text{ otherwise.}$$

If supremum does not exist finitely, we call L a set of infinite diameter.

Theorem 5.3. For a sub mset L of M in an M -metric space (M, d) , $\delta(L) = P_0^1$ iff $L = \{1/a\}$ ie. L consists of a single element of the universal set X with multiplicity 1.

Proof. Let $L = \{1/a\}$. Then $L_{pt} = \{P_a^1\}$. $\therefore d(P_x^l, P_y^m) = P_0^1, \forall P_x^l, P_y^m \in L_{pt} \Rightarrow \delta(L) = P_0^1$.

Conversely, let $\delta(L) = P_0^1 \Rightarrow \text{Sup}\{b : P_b^j = d(P_x^l, P_y^m), P_x^l, P_y^m \in L_{pt}\} = 0 \Rightarrow b = 0, \forall P_b^j = d(P_x^l, P_y^m), P_x^l, P_y^m \in L_{pt} \Rightarrow d(P_x^l, P_y^m) = P_0^1, \forall P_x^l, P_y^m \in L_{pt} \Rightarrow P_x^l = P_y^m, \forall P_x^l, P_y^m \in L_{pt}$.

$\therefore L_{pt}$ consists of a single multi point ie. L consists of a single element with multiplicity 1, since otherwise if L consists of an element having multiplicity more than 1, then L_{pt} contains more than one element. \square

Note 5.4. If $L = \{m/a\}$ where $m > 1$, then $\delta(L) = P_0^m$.

Theorem 5.5. $P \subset Q \Rightarrow \delta(P) \leq \delta(Q)$.

Proof. The proof is straight forward. \square

Definition 5.6. Let (M, d) be an M -metric space, P_u^c be a fixed multi point of M and L be a non-null sub mset of M . Then the **distance of P_u^c from the multi set L** is denoted by $\delta(P_u^c, L)$ and is defined by

$\delta(P_u^c, L) = P_a^k$ where $a = \text{Inf} \{b : P_b^j = d(P_u^c, P_x^l), P_x^l \in L_{pt}\}$,

$k = w$ if $a < b \forall P_b^j = d(P_u^c, P_x^l), P_x^l \in L_{pt}$ and

$k = \text{Min} \{j : P_a^j = d(P_u^c, P_x^l), P_x^l \in L_{pt}\}$ if $a = b$.

Note 5.7. In case P_u^c is a multi point of L , $\delta(P_u^c, L) = P_0^1$ since if $\delta(P_u^c, L) = P_a^i$ then $0 \leq a = \text{Inf} \{b : P_b^j = d(P_u^c, P_x^l), P_x^l \in L_{pt}\} \leq 0$ as $P_0^1 = d(P_u^c, P_u^c)$ and $P_u^c \in L_{pt} \Rightarrow a = 0$

Also $i = \text{Min} \{j : P_0^j = d(P_u^c, P_x^l), P_x^l \in L_{pt}\} = 1$ as $P_0^1 = d(P_u^c, P_u^c), P_u^c \in L_{pt} \therefore \delta(P_u^c, L) = P_0^1$.

On the other hand if $P_u^c \notin L_{pt}$, then $\delta(P_u^c, L) \geq P_0^2$ since $P_u^c \notin L_{pt} \Rightarrow d(P_u^c, P_x^l) > P_0^1 \forall P_x^l \in L_{pt}$ and hence $\delta(P_u^c, L) > P_0^1$.

Thus $\delta(P_u^c, L) = P_0^1$ iff $P_u^c \in L_{pt}$.

Definition 5.8. Let A and B be two sub msets of M in an M -metric space (M, d) . Then the **distance between** A and B , denoted by $\delta(A, B)$, is defined by

$\delta(A, B) = P_a^k$ where $a = \text{Inf} \{b : P_b^j = d(P_x^l, P_y^m), P_x^l \in A_{pt}, P_y^m \in B_{pt}\}$ and

$k = w$ if $a < b \forall P_b^j = d(P_x^l, P_y^m), P_x^l \in A_{pt}, P_y^m \in B_{pt}$,

$k = \text{Min} \{j : P_a^j = d(P_x^l, P_y^m), P_x^l \in A_{pt}, P_y^m \in B_{pt}\}$ otherwise.

Theorem 5.9. Let A and B be two sub msets of M in an M -metric space (M, d) . Then (i) $\delta(A, B) = \delta(B, A)$ (ii) $A \cap B \neq \emptyset \Leftrightarrow \delta(A, B) = P_0^1$.

Proof. Let $A \cap B \neq \emptyset \Rightarrow A_{pt} \cap B_{pt} \neq \emptyset \therefore \exists P_u^c \in A_{pt} \cap B_{pt}$ Let $\delta(A, B) = P_a^k$. Then $0 \leq \text{Inf} \{b : P_b^j = d(P_x^l, P_y^m), P_x^l \in A_{pt}, P_y^m \in B_{pt}\} \leq 0$ as $P_0^1 = d(P_u^c, P_u^c), P_u^c \in A_{pt}, P_u^c \in B_{pt} \Rightarrow a = 0$.

Also as $a = 0$ and $P_0^1 = P_a^1 = d(P_u^c, P_u^c), P_u^c \in A_{pt}, P_u^c \in B_{pt}$ So $k = \text{Min} \{j : P_0^j = d(P_x^l, P_y^m), P_x^l \in A_{pt}, P_y^m \in B_{pt}\} = 1$ as $P_0^1 = d(P_u^c, P_u^c), P_u^c \in A_{pt}, P_u^c \in B_{pt} \therefore \delta(A, B) = P_0^1$.

$\therefore A \cap B \neq \emptyset \Rightarrow \delta(A, B) = P_0^1$.

Conversely let $A \cap B = \emptyset$ and $\delta(A, B) = P_a^k$.

If $a > 0$, then obviously $\delta(A, B) > P_0^1$.

If $a = 0$ and $a = 0 < b \forall b$ such that $P_b^j = d(P_x^l, P_y^m), P_x^l \in A_{pt}, P_y^m \in B_{pt}$, then $k = w > 1$.

If $a = 0 = b$ for some $P_b^j = d(P_x^l, P_y^m), P_x^l \in A_{pt}, P_y^m \in B_{pt}$, then obviously $j > 1$ as $A \cap B = \emptyset \Rightarrow A_{pt} \cap B_{pt} = \emptyset$ and consequently $k > 1$.

Thus in any case $A \cap B = \emptyset \Rightarrow \delta(A, B) = P_a^k > P_0^1$. \square

6. CONCLUSIONS

Functional analysis is an important branch of Mathematics and it has many applications in Mathematics and Sciences. Metric space is the beginning of functional analysis and it has several applications in many branch of functional analysis. In this paper an extension of the concept of metric is made by using multi set and multi number instead of crisp set and crisp real number. There is an ample scope for further research on multi metric space. Research on Multi norm and multi inner product can be of special interest.

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