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The Standard Deviation of the Least Monopoly Energy of Graphs

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Abstract

Consider a graph $G = (V, E)$ to be simple, containing a set of vertices $V(G) = \{v_1, v_2, v_3, \dots, v_p\}$ we call a set $S \subseteq V(G)$ to be a monopoly set (MS) if for each vertex in $V - S$ has atleast $\frac{d(p)}{2}$ neighbours in S. In between all (MS) sets of the graph G, the set which is containing smallest number of elements is named as the monopoly size of G and it is identified by mo(G). In this paper, we introduce a new concept called the standard deviation of the least monopoly energy $E^{\sigma}{}_{LM}(G)$ pertaining to graph G. The standard deviation of the least monopoly energies of a few various kinds of graphs is obtained. We also obtain the boundary values for $E^{\sigma}{}_{LM}(G)$.

Keywords: Monopoly Set, Monopoly Size, Matrix of the Minimum Monopoly, Minimum Monopoly Eigen values, Minimum Monopoly Energy, Minimum Mean Monopoly Energy, Standard deviation of the least monopoly energy of a graph.

1. INTRODUCTION

Consider a graph $G = (V, E)$ to be simple with non-empty vertex set. The cardinality of non-empty vertex and edge set of G be denoted by p and q respectively. Also, we represent d(p) to be vertex degree for any vertex p of G; it is the total number of

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vertices adjacent to vertex p. Open neighbourhood $N(p)$, for any vertex p is formalized as $N(p) = \{q \in V: < q, p > \in E(G)\}$. The vertex degree of any vertex p belonging to vertex set of G, regarding to a subset K of vertex set of G is $d_k(V) = |N(p) \cap K|$. We refer Harary for other standard graph terminologies [2].

A set $S \subseteq V(G)$ in a graph G is said to be a monopoly set if for each vertex in $V - S$ possesses at least $\frac{d(p)}{2}$ neighbours in S. Among all monopoly sets in a graph G, the set which is containing smallest number of elements is named as the monopoly size of G and it is identified by mo(G).

In [1], Peleg introduced the concept of Dynamos. Monopolies are usually referred as dynamos meaning dynamic monopoly because at any fixed time steps, if we color the graph to black it will convert the entire graph to black under an irreversible majority conversion process in the next time step.

In [3], I. Gutman introduced the notion of energy of a graph in 1978. Consider a graph having the vertex set V of cardinality **p** and edge set E of cardinality **q.** The adjacency matrix of the graph be $K = (k_{i,j})$. The characteristic roots $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_p$ of K, are presumed to be in decreasing order and they are also the characteristic roots of the graph. Let $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_k$ for $k \leq p$ be the definite characteristic roots of S having a set $m_1, m_2, m_3, \dots, m_k$ times of repeated characteristic roots for each k. The eigen values of K along with its multiplicity is known as the spectrum of any given graph G0.

The term $E(G)$ is regarded as the energy for any graph G, and is defined as modulus of all eigenvalues of G that is, $(G) = \sum_{i=1}^{p} |\lambda_i|$. In [4], Laura Buggy et al. , put forward the concept of minimum $\bar{\lambda}$ energy of G, is defined as $E^M(G) = \sum_{i=1}^{p} |\lambda_i - \bar{\lambda}|$,where $\bar{\lambda}$ is the mean of the eigenvalues. Taking the inspiration from this article, we introduce a new concept called the standard deviation of the least monopoly energy for any graph G which is represented by $E^{\sigma}{}_{LM}(G)$. We obtain the standard deviation of the least monopoly energy of some standard graphs. In addition to this, bounds for $E^{\sigma}{}_{LM}(G)$ are also established. The standard deviation of the least monopoly energy that we are evaluating in this paper is feasible for many application purposes in chemistry as well as in other fields.

2. THE STANDARD DEVIATION OF THE LEAST MONOPOLY ENERGY OF GRAPHS

Definition 2.1: Consider G along with set of vertices V(G) = $\{V_1, V_2, \dots, V_p\}$ having cardinality p and set of edges E. Let S be a minimum monopoly set of G. The

Matrix of the minimum monopoly (MM) of a graph G is the $p \times p$ matrix, denoted by $K_{MM}(G) = (K_{ij})$ where

$$K_{ij} = \begin{cases} 1, if\ V_iV_j \epsilon E(G) \\ 1,\ if\ i = j, V_i \epsilon S \\ 0,\ otherwise \end{cases}$$

Since, K_{MM}(G) is symmetric all its eigen values $\lambda_1, \lambda_2, \dots, \lambda_p$ are real and non-negative. The minimum monopoly energy E_{MM}(G), for a graph G is determined by E_{MM}(G) $= \sum_{i=1}^{p} |\lambda_i|$, where $\lambda_1, \lambda_2, \dots, \lambda_p$ are eigen values of minimum monopoly matrix K_{MM}(G).

Definition 2.2: The standard deviation of the least monopoly energy(E_{LM}^{σ}), is formulated as E_{LM}^{σ}(H) $= \sum_{i=1}^{p} |\lambda_i$-$\sigma|$, where σ is the standard deviation of the eigen values of the graph H. Here, $\bar{\lambda}$= mean

$$\sigma = \sqrt{\frac{(\lambda_1 - \bar{\lambda})^2 + (\lambda_2 - \bar{\lambda})^2 + \cdots (\lambda_p - \bar{\lambda})^2}{p}}$$

For other basic concepts refer [5], [4].

Remark 1: The $E_{MM}(G)$ of a graph G depends on the set S which we choose. Therefore, $E_{MM}(G)$ is not a graph invariant.

3. THE STANDARD DEVIATION OF THE LEAST MONOPOLY ENERGY OF SOME STANDARD GRAPHS

Theorem 3.1: For p≥2, the standard deviation of the least monopoly energy of complete graph K_p is

$$E^{\sigma}{}_{LM}(K_p) = \begin{cases} \frac{(p-2)}{2}\left[\sqrt{(4p-3)} + 1\right] + \sqrt{p^2+1} & ; p = even \\ (p-2)\sqrt{\frac{(8p^3+2p^2+2p)(p-1)}{8p^3}} + \frac{(p-1)}{2} + \sqrt{p^2-1} & ; p = odd \end{cases}$$

Proof. Consider K_p graph possessing set of vertices as V(G)=$\{V_1, V_2, \dots\dots, V_p\}$. Then the size of (MM) is

$$\text{mmo}(K_p) = \left\lfloor \frac{p}{2} \right\rfloor = \begin{cases} \frac{p}{2} & , p = even \\ \frac{(p-1)}{2}, & p = odd \end{cases}$$

The minimum monopoly set S=$\left\{v_1, v_2, \dots \dots, v_{\frac{p}{2}}\right\}$, when p=even (or) S=$\left\{v_1, v_2, \dots, v_{\frac{(p-1)}{2}}\right\}$, when p=odd.

Case 1: When p=even

The minimum monopoly matrix

$$K_{MM}(K_p) = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 11 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \vdots & 1 & 11 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \vdots & 1 & 11 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\vdots & \vdots & \cdots & \vdots \\ 1 & 1 & 1 & \vdots & 1 & 11 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 & 01 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \vdots & 1 & 10 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \vdots & 1 & 11 & 0 & \vdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 11 & 1 & \cdots & 0 \end{bmatrix}$$

Then the characteristic equation

$|K_{MM}\ (K_p) - \lambda I| = 0$ is

$$\left(\lambda^{\frac{p-2}{2}}(\lambda+1)^{\frac{p-2}{2}}\left(\lambda^2-(p-1)\lambda-\frac{p}{2}\right)\right) = 0$$

Then, $\lambda = 0\left[\frac{p-2}{2}\ times\right]$, $\lambda = -1\left[\frac{p-2}{2}\ times\right]$ and $\lambda = \frac{(p-1)\pm\sqrt{p^2+1}}{2}$

Mean $\bar{\lambda} = \frac{1}{2}$

The minimum monopoly energy $E_{MM}(K_p) = \frac{(p-2)}{2} + \sqrt{p^2+1}$

The standard deviation $\sigma = \sqrt{\frac{(\lambda_1-\bar{\lambda})^2+(\lambda_2-\bar{\lambda})^2+\cdots+(\lambda_p-\bar{\lambda})^2}{p}}$

$$= \sqrt{\frac{(0-\bar{\lambda})^2+\cdots+(0-\bar{\lambda})^2+(-1-\bar{\lambda})^2+\cdots(-1-\bar{\lambda})^2+\left(\frac{(p-1)+\sqrt{p^2+1}}{2}-\bar{\lambda}\right)^2+\left(\frac{(p-1)-\sqrt{p^2+1}}{2}-\bar{\lambda}\right)^2}{p}}$$

$$= \sqrt{\frac{p\left(p-\frac{3}{4}\right)}{p}} = \sqrt{\left(p-\frac{3}{4}\right)}$$

The standard deviation of the least monopoly energy

$$E_{LM}^{\sigma}(K_p) = \sum_{i=1}^{p} |\lambda_i - \sigma|$$

$$= \sum_{i=1}^{\frac{p-2}{2}} |0-\sigma| + \sum_{i=1}^{\frac{p-2}{2}} |-1-\sigma| + \left|\frac{(p-1)\pm\sqrt{p^2+1}}{2} - \sigma\right|$$

$$= \frac{(p-2)}{2} \times \sqrt{p-\frac{3}{4}} + \frac{(p-2)}{2} \times \left|-1-\sqrt{p-\frac{3}{4}}\right| + \left|\frac{(p-1)\pm\sqrt{p^2+1}-2\sqrt{p-\frac{3}{4}}}{2}\right|$$

$$= \frac{\frac{(2p-4)}{2}\sqrt{4p-3}}{4} + \frac{\frac{(2p-4)}{2}[\sqrt{(4p-3)}+2]}{4} + \left|\frac{\frac{(2p-2)}{2}\pm\sqrt{p^2+1}-\sqrt{4p-6+3}}{2}\right|$$

$$= \frac{(p-2)\sqrt{4p-3}}{4} + \frac{(p-2)[\sqrt{4p-3}+2]}{4} + \frac{(p-1)+\sqrt{p^2+1}-\sqrt{4p-3}}{2} + \frac{\sqrt{p^2+1}}{2} + \frac{\sqrt{4p-3}}{2} - \frac{(-p-1)}{2}$$

$$= \frac{(p-2)\sqrt{4p-3}}{4} + \frac{(p-2)\sqrt{4p-3}}{4} + \frac{(p-2)}{2} + \frac{(p-1)}{2} + \frac{\sqrt{p^2+1}}{2} - \frac{\sqrt{4p-3}}{2} + \frac{\sqrt{p^2+1}}{2} + \frac{\sqrt{4p-3}}{2} - \frac{(p-1)}{2}$$

$$= \frac{(p-2)\sqrt{(4p-3)}}{2} + \sqrt{p^2+1} + \frac{(p-2)}{2}$$

$$= \frac{(p-2)}{2}\left[\sqrt{(4p-3)}+1\right] + \sqrt{p^2+1}$$

Case 2: When p=odd

The minimum monopoly of matrix

$$K_{MM}(K_p) = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 11 & 1 & \dots & 1 \\ 1 & 1 & 1 & \vdots & 1 & 11 & 1 & \dots & 1 \\ 1 & 1 & 1 & \cdots & 1 & 11 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\vdots & \vdots & \dots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 11 & 1 & \dots & 1 \\ 1 & 1 & 1 & \cdots & 1 & 01 & 1 & \dots & 1 \\ 1 & 1 & 1 & \vdots & 1 & 10 & 1 & \dots & 1 \\ 1 & 1 & 1 & \vdots & 1 & 11 & 0 & \vdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots\vdots & \vdots & \cdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 11 & 1 & \cdots & 0 \end{bmatrix}$$

The characteristic equation $|K_{MM}(K_p) - \lambda I| = 0$ is

$$\left(\lambda^{\frac{p-3}{2}}(\lambda+1)^{\frac{p-1}{2}}\left(\lambda^2-(p-1)\lambda-\frac{(p-1)}{2}\right)\right) = 0$$

Then, $\lambda = 0\left[\frac{(p-3)}{2}\,times\right]$; $\lambda = -1\left[\frac{(p-1)}{2}\,times\right]$; and

$\lambda = \frac{(p-1)\pm\sqrt{p^2+1}}{2}$.

Mean $\overline{(\lambda)} = \frac{(p-1)}{2p}$.

The minimum monopoly energy $E_{MM}(K_p)$

$$= \frac{(p-1)}{2} + \sqrt{p^2-1}$$

The standard deviation $\sigma(G) = \sqrt{\frac{(\lambda_1-\bar{\lambda})^2+(\lambda_2-\bar{\lambda})^2+\cdots+(\lambda_p-\bar{\lambda})^2}{p}}$

$$= \sqrt{\frac{(\bar{\lambda})^2+\cdots+(\bar{\lambda})^2+(-\bar{\lambda}-1)^2+\cdots+(-1-\bar{\lambda})^2+\left(\frac{(p-1)+\sqrt{p^2-1}}{2}-\bar{\lambda}\right)^2+\left(\frac{(p-1)-\sqrt{p^2-1}}{2}-\bar{\lambda}\right)^2}{p}}$$

$$= \sqrt{\frac{\frac{(p-3)}{2}\frac{(p-1)^2}{4p^2}+\frac{(p-1)}{2}\frac{(1-3p)^2}{4p^2}+\frac{(p-1)^2(p^2+1)+p^2(p^2-1)-2p(p-1)^2}{2p^2}}{p}}$$

$$= \sqrt{(8p^3+2p^2+2p)\frac{(p-1)}{8p^2}\times\frac{1}{p}}$$

$$\sigma = \sqrt{\frac{(8p^3+2p^2+2p)(p-1)}{8p^3}}$$

The standard deviation of the least monopoly energy

$$E_{LM}^{\sigma}(K_p) = \sum_{i=1}^{p}|\lambda_i-\sigma|$$

$$= \sum_{i=1}^{\frac{p-3}{2}}|0-\sigma| + \sum_{i=1}^{\frac{p-1}{2}}|-1-\sigma| + \left|\frac{(p-1)\pm\sqrt{p^2-1}}{2}-\sigma\right|$$

$$= \frac{(p-3)}{2}\sqrt{\frac{(8p^3+2p^2+2p)(p-1)}{8p^3}} + \frac{(p-1)}{2}\left|-1-\sqrt{\frac{(8p^3+2p^2+2p)(p-1)}{8p^3}}\right| + \left|\frac{(p-1)\pm\sqrt{p^2-1}}{2} - \sqrt{\frac{(8p^3+2p^2+2p)(p-1)}{8p^3}}\right|$$

$$= \frac{(p-3)}{2}\sqrt{\frac{(8p^3+2p^2+2p)(p-1)}{8p^3}} + \frac{(p-1)}{2} + \frac{(p-1)}{2}\sqrt{\frac{(8p^3+2p^2+2p)(p-1)}{8p^3}} + \frac{(p-1)}{2} + \frac{\sqrt{p^2-1}}{2} - \sqrt{\frac{(8p^3+2p^2+2p)(p-1)}{8p^3}} + \frac{\sqrt{p^2-1}}{2} + \sqrt{\frac{(8p^3+2p^2+2p)(p-1)}{8p^3}} - \frac{(p-1)}{2}$$

$$= \frac{(p-3)}{2}\sqrt{\frac{(8p^3+2p^2+2p)(p-1)}{8p^3}} + \frac{(p-1)}{2} + \frac{(p-1)}{2}\sqrt{\frac{(8p^3+2p^2+2p)(p-1)}{8p^3}} + \sqrt{p^2-1}$$

$$= \sqrt{\frac{(8p^3+2p^2+2p)(p-1)}{8p^3}} + \frac{[p-3+p-1]}{2} + \frac{(p-1)}{2} + \sqrt{p^2-1}$$

$$= (p-2)\sqrt{\frac{(8p^3+2p^2+2p)(p-1)}{8p^3}} + \frac{(p-1)}{2} + \sqrt{p^2-1}$$

Theorem 3.2: For p≥2, the standard deviation of the least monopoly energy of star graph $K_{1,p-1}$ is

$$E_{LM}^{\sigma}(K_{1,p-1}) = (p-2)\sqrt{\frac{(2p^3-p^2-p)}{p^3}} + \sqrt{4p-3}$$

Proof. Consider $K_{1,p-1}$ graph holding the vertex set V(G)= $\{v_1, v_2, \ldots, v_p\}$. The minimum monopoly set S= $\{v_1\}$. [We assume v_1 to be the centre vertex].

The minimum monopoly matrix is

$$K_{MM}(K_{1,p-1}) = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 11 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 & 00 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 00 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\vdots & \vdots & \cdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 00 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 00 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \vdots & 0 & 00 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \vdots & 0 & 00 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 00 & 0 & \cdots & 0 \end{bmatrix}$$

Then the characteristic equation

$|K_{MM}(K_{1,p-1}) - \lambda I| = 0$ is

$\lambda^{(p-2)}(\lambda^2 - \lambda - (p-1)) = 0$

Then, $\lambda = 0\ [(p-2) times]$; $\lambda = \frac{1}{2} \pm \frac{\sqrt{4p-3}}{2}$

Mean $\overline{\lambda} = \frac{1}{p}$.

The minimum monopoly energy of $K_{1,p-1}$ is

$$E_{MM}(K_{1,p-1}) = \sum_{i=1}^{(p-2)} 0(p-2) + \left|\frac{1}{2} \pm \frac{\sqrt{4p-3}}{2}\right|$$

$$=\sqrt{4p-3}$$

The standard deviation $\sigma(G) = \sqrt{\frac{(\lambda_1-\bar{\lambda})^2+(\lambda_2-\bar{\lambda})^2+\cdots+(\lambda_p-\bar{\lambda})^2}{p}}$

$$= \sqrt{\frac{(0-\bar{\lambda})^2+\cdots+(0-\bar{\lambda})^2+\left(\frac{1+\sqrt{4p-3}}{2}-\bar{\lambda}\right)^2\left(\frac{1-\sqrt{4p-3}}{2}-\bar{\lambda}\right)^2}{p}}$$

$$\sigma(G) = \sqrt{\frac{(4p^3-2p^2-2p)}{2p^2}\times\frac{1}{p}} = \sqrt{\frac{(2p^3-p^2-p)}{p^3}}$$

The standard deviation of the least monopoly energy

$$E_{LM}^{\sigma}(K_{1,p-1}) = \sum_{i=1}^{p} |\lambda_i - \sigma|$$

$$=\sum_{i=1}^{(p-2)}|0-\sigma| + \left|\frac{1\pm\sqrt{4p-3}}{2} - \sigma\right|$$

$$= (p-2)\sqrt{\frac{(2p^3-p^2-p)}{p^3}} + \left|\frac{1+\sqrt{4p-3}}{2} - \sqrt{\frac{(2p^3-p^2-p)}{p^3}}\right| + \left|\frac{1-\sqrt{4p-3}}{2} - \sqrt{\frac{(2p^3-p^2-p)}{p^3}}\right|$$

$$= (p-2)\sqrt{\frac{(2p^3+p^2-p)}{p^3}} + \frac{1}{2} + \frac{\sqrt{4p-3}}{2} - \sqrt{\frac{(2p^3-p^2-p)}{p^3}} + \frac{\sqrt{4p-3}}{2} + \sqrt{\frac{(2p^3-p^2-p)}{p^3}} - \frac{1}{2}$$

$$= (p-2)\sqrt{\frac{(2p^3+p^2-p)}{p^3}} + \sqrt{4p-3}$$

Theorem 3.3: For p ≤ q, the standard deviation of the least monopoly energy of the complete bipartite graph$\left(K_{p,q}\right)$ is

$$E_{LM}^{\sigma}\left(K_{p,q}\right) = (p+q-2)\sqrt{\frac{pq^2(1+2q)+p^2q(1+2p)+4p^2q^2}{(p+q)^3}} + \sqrt{(4pq+1)} - (q-1)\ .$$

Proof: For the complete bipartite graph$\left(K_{p,q}\right)$, p ≤ q with vertex set

V= $\{v_1, v_2, \ldots v_p, u_1, u_{2,} \ldots, u_q\}$. The minimum monopoly set $S = \{v_1, v_2, \ldots., v_p\}$.

The minimum monopoly set S= $\{v_1, v_2, \ldots v_p\}$.

Then, the minimum monopoly matrix is

$$K_{MM}\left(K_{p,q}\right) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 11 & 1 & \ldots & 1 \\ 0 & 1 & 0 & \cdots & 0 & 11 & 1 & \ldots & 1 \\ 0 & 0 & 1 & \cdots & 0 & 11 & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots\vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 11 & 1 & \ldots & 1 \\ 1 & 1 & 1 & \cdots & 1 & 00 & 0 & \ldots & 0 \\ 1 & 1 & 1 & \vdots & 1 & 00 & 0 & \ldots & 0 \\ 1 & 1 & 1 & \vdots & 1 & 00 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots\vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 00 & 0 & \cdots & 0 \end{bmatrix}$$

Then the characteristic equation

$\left|K_{MM}\left(K_{p,q}\right) - \lambda I\right| = 0$ is

$\lambda^{p-1}(\lambda-1)^{q-1}(\lambda^2-\lambda-pq) = 0$

Then, $\lambda = 0\ [(p-1) times]$; $\lambda = 1[(q-1) times]$;

and $\lambda = \frac{1 \pm \sqrt{4pq+1}}{2}$

Mean $\bar{\lambda} = \frac{q}{p+q}$

The minimum monopoly energy of complete bipartite graph

$E_{MM}\left(K_{p,q}\right) = (p-1) + \sqrt{4pq+1}$

The standard deviation $\sigma(G) = \sqrt{\frac{(\lambda_1-\bar{\lambda})^2+(\lambda_2-\bar{\lambda})^2+\cdots+(\lambda_{p+q}-\bar{\lambda})^2}{(p+q)}}$

$$= \sqrt{\frac{(0-\bar{\lambda})^2+\cdots+(0-\bar{\lambda})^2+(1-\bar{\lambda})^2+\cdots+(1-\bar{\lambda})^2+\left(\frac{1+\sqrt{4pq+1}}{2}-\bar{\lambda}\right)^2+\left(\frac{1-\sqrt{4pq+1}}{2}-\bar{\lambda}\right)^2}{(p+q)}}$$

$$= \sqrt{\frac{(p-1)\frac{q^2}{(p+q)^2}+(q-1)\frac{p^2}{(p+q)^2}+\left[2\left(\frac{1}{4}+\frac{(4pq+1)}{4}+\frac{q^2}{(p+q)^2}\right)-\frac{2q}{p+q}\right]}{(p+q)}}$$

$$= \sqrt{\frac{2p^3q+2pq^3+4p^2q^2+pq^2+p^2q}{(p+q)^3}}$$

$$\sigma(G) = \sqrt{\frac{pq^2(1+2q)+p^2q(1+2q)+4p^2q^2}{(p+q)^3}}$$

The standard deviation of the least monopoly energy

$$E_{LM}^{\sigma}(K_{p,q}) = \sum_{i=1}^{(p-1)} |0-\sigma| + \sum_{i=1}^{(q-1)} |1-\sigma| + \left|\frac{1 \pm \sqrt{4pq+1}}{2} - \sigma\right|$$

$$=(p-1)\sqrt{\frac{pq^2(1+2q+p^2q(1+2p)+4p^2q^2}{(p+q)^3}} - (q-1) + \sqrt{4pq+1}$$

$$+(q-1)\sqrt{\frac{pq^2(1+2q)+p^2q(1+2p)+4p^2q^2}{(p+q)^3}}$$

$$=\sqrt{\frac{pq^2(1+2q)+p^2q(1+2p)+4p^2q^2}{(p+q)^3}}(p+q-2) + \sqrt{4pq+1} -$$

$(q-1)$

Theorem 3. 4: For a double star graph $S_{p,p}$, the standard deviation of the least monopoly energy of the graph is

$$E_{LM}^{\sigma}(S_{p,p})=(2p-2)\sqrt{\frac{p^4+p^3-p}{p^3}} + 2\sqrt{p}$$

Proof: Let $S_{p,p}$ be a double star having the vertex set V= $\{v_{0,}v_1, v_2, \ldots v_{p-1,}u_0, u_1, u_{2,\ldots.}u_{p-1}\}$.Then the minimum monopoly set S=$\{u_{0,}v_0\}$.

The minimum monopoly matrix is

$$K_{MM}(S_{p,p}) = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 10 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \vdots & 0 & 00 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \vdots & 0 & 00 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots\vdots & \vdots & \cdots & \vdots \\ 1 & 0 & 0 & \vdots & 0 & 00 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 11 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \vdots & 0 & 10 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \vdots & 0 & 10 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 10 & 0 & \cdots & 0 \end{bmatrix}$$

The characteristic equation

$|K_{MM}(S_{p,p}) - \lambda I| = 0$ is

$\lambda^{2p-4}(\lambda^2 - (p-1))(\lambda^2 - 2\lambda - (p-1)) = 0.$

Then, $\lambda = 0[(2p-4)\text{times}]; \lambda = \pm\sqrt{\frac{(2p-2)}{2}}; \lambda = 1 \pm \sqrt{p}$.

Mean $\bar{\lambda} = \frac{2}{2p}$.

The minimum monopoly energy

$E_{MM}(S_{p,p})= 2\sqrt{\frac{(2p-2)}{2}} + 2\sqrt{\frac{2p}{2}}$

The standard deviation σ(G)= $\sqrt{\frac{(\lambda_1-\bar{\lambda})^2+(\lambda_1-\bar{\lambda})^2+\cdots+(\lambda_{2p}-\bar{\lambda})^2}{2p}}$

$= \sqrt{\frac{(0-\bar{\lambda})^2+\cdots+(0-\bar{\lambda})^2+(\sqrt{p-1}-\bar{\lambda})^2+(-\bar{\lambda}-\sqrt{p-1})^2+(-\bar{\lambda}+1+\sqrt{p})^2+(-\bar{\lambda}+1-\sqrt{p})^2}{2p}}$

$= \sqrt{\frac{2p^4+2p^3-2p}{2p^3}} = \sqrt{\frac{p^4+p^3-p}{p^3}}$

The standard deviation of the least monopoly energy

$$E_{LM}^{\sigma}(S_{p,p})=\sum_{i=1}^{(2p-4)}|0-\sigma|+\left|\pm\sqrt{p-1}-\sigma\right|+\left|1\pm\sqrt{p}-\sigma\right|$$

$$=(2p-4)\sqrt{\frac{p^4+p^3-p}{p^3}}+\left|\sqrt{p-1}-\sqrt{\frac{p^4+p^3-p}{p^3}}\right|+\left|-\sqrt{p-1}-\sqrt{\frac{p^4+p^3-p}{p^3}}\right|+$$

$$\left|1+\sqrt{p}-\sqrt{\frac{p^4+p^3-p}{p^3}}\right|+\left|1-\sqrt{p}-\sqrt{\frac{p^4+p^3-p}{p^3}}\right|$$

$$=(2p-4)\sqrt{\frac{p^4+p^3-p}{p^3}}+\sqrt{\frac{p^4+p^3-p}{p^3}}-\sqrt{p-1}+\sqrt{p-1}+\sqrt{\frac{p^4+p^3-p}{p^3}}+1+$$

$$\sqrt{p}-\sqrt{\frac{p^4+p^3-p}{p^3}}+\sqrt{p}+\sqrt{\frac{p^4+p^3-p}{p^3}}-1$$

$$=(2p-4)\sqrt{\frac{p^4+p^3-p}{p^3}}+2\sqrt{\frac{p^4+p^3-p}{p^3}}+2\sqrt{p}$$

$$=(2p-2)\sqrt{\frac{p^4+p^3-p}{p^3}}+2\sqrt{p}$$

Theorem 3. 5: For any integer p≥3, the standard deviation of the least monopoly energy of the crown graph S_p^0 is

$$E_{LM}^{\sigma}(S_p^0)=\sqrt{5}(p-1)+\sqrt{4p^2-8p+5}$$

Proof: Consider S_p^0 graph having V(G)=$\{u_1,u_{2,\dots,}u_p,v_1,v_2,\dots v_p\}$.

Then, the set S=$\{u_1,u_{2,\dots,}u_p\}$.

Then, the matrix $K_{MM}(S_p{}^0)$ is given by

$$K_{MM}(S^0{}_p)=\begin{bmatrix}1&0&0&\cdots&0&01&1&\cdots&1\\0&1&0&\vdots&0&10&1&\cdots&1\\0&0&1&&0&11&0&\cdots&1\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\vdots&\vdots&\ddots&\vdots\\\vdots&\vdots&\vdots&&\vdots&\vdots\vdots&\vdots&\cdots&\vdots\\0&0&0&\vdots&1&11&1&\cdots&1\\0&1&1&\cdots&1&00&0&\cdots&0\\1&0&1&\vdots&1&00&0&\cdots&0\\1&1&0&\vdots&1&00&0&\cdots&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\vdots&\vdots&\vdots&\vdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\vdots&\vdots&\vdots&\vdots\\1&1&1&\cdots&0&00&0&\cdots&0\end{bmatrix}$$

Then the characteristic equation

$\left|K_{MM}\left(S_p^0\right)-\lambda \mathrm{I}\right|=0$ is

$(\lambda^2-\lambda-1)^{(p-1)}(\lambda^2-\lambda-(\mathrm{p}-1)^2)=0$

Then, $\lambda=\frac{1\pm\sqrt{5}}{2}[(\mathrm{p}-1)\text{times}];\lambda=\frac{1\pm\sqrt{4p^2-8p+5}}{2}$

Mean $\bar{\lambda}=\frac{1}{\mathrm{p}}$

Then the minimum monopoly energy

$E_{MM}\left(S_p^0\right)=\sqrt{5}(\mathrm{p}-1)+\sqrt{4\mathrm{p}^2-8\mathrm{p}+5}$

The standard deviation σ(G)= $\sqrt{\frac{(\lambda_1-\bar{\lambda})^2+\cdots+(\lambda_{2p}-\bar{\lambda})^2}{2p}}$

$$=\sqrt{\frac{\left[\frac{1+\sqrt{5}}{2}-\frac{1}{\mathrm{p}}\right](\mathrm{p}-1)+\left[\frac{1-\sqrt{5}}{2}-\frac{1}{\mathrm{p}}\right]^2(\mathrm{p}-1)+\left[\frac{1+\sqrt{4\mathrm{p}^2-8\mathrm{p}+5}}{2}-\frac{1}{\mathrm{p}}\right]^2+\left[\frac{1-\sqrt{4\mathrm{p}^2-8\mathrm{p}+5}}{2}-\frac{1}{\mathrm{p}}\right]^2}{2\mathrm{p}}}$$

$$\sigma(\mathrm{G})=\sqrt{\frac{4\mathrm{p}^4-2\mathrm{p}^3-4\mathrm{p}^2+4\mathrm{p}}{4\mathrm{p}^3}}$$

The standard deviation of the least monopoly energy

$E_{LM}^{\sigma}(G)=\sum_{i=1}^{(p-1)}\left|\frac{1\pm\sqrt{5}}{2}-\sigma\right|+\left|\frac{1\pm\sqrt{4\mathrm{p}^2-8\mathrm{p}+5}}{2}-\sigma\right|$

$=(\mathrm{p}-1)\left|\frac{1+\sqrt{5}}{2}-\sigma\right|+(\mathrm{p}-1)\left|\frac{1-\sqrt{5}}{2}-\sigma\right|+\left|\frac{1+\sqrt{4\mathrm{p}^2-8\mathrm{p}+5}}{2}-\sigma\right|+\left|\frac{1-\sqrt{4\mathrm{p}^2-8\mathrm{p}+5}}{2}-\sigma\right|$

$=(\mathrm{p}-1)\frac{(1+\sqrt{5})}{2}-(\mathrm{p}-1)\sigma+(\mathrm{p}-1)\left[\frac{\sqrt{5}}{2}+\sigma-\frac{1}{2}\right]+\frac{1}{2}+\frac{\sqrt{4\mathrm{p}^2-8\mathrm{p}+5}}{2}-\sigma+\frac{\sqrt{4\mathrm{p}^2-8\mathrm{p}+5}}{2}+\sigma-\frac{1}{2}$

$=\frac{(\mathrm{p}-1)}{2}+\frac{\sqrt{5}}{2}(\mathrm{p}-1)-(\mathrm{p}-1)\sigma+(\mathrm{p}-1)\frac{\sqrt{5}}{2}+\sigma(\mathrm{p}-1)-\frac{1}{2}(\mathrm{p}-1)+\sqrt{4\mathrm{p}^2-8\mathrm{p}+5}$

$=\sqrt{5}(\mathrm{p}-1)+\sqrt{4\mathrm{p}^2-8\mathrm{p}+5}$

4. BOUNDS OF STANDARD DEVIATION OF THE LEAST MONOPOLY ENERGY

Remark 2: For a non-disconnected graph $\boldsymbol{G}$ containing $V(G) = \{v_1, v_2, v_3, v_4, \ldots\ldots, v_p\}$ and size q. Then,

$$\big(2q + mo(G)\big) \leq \; E_{MM}(G) \leq \; \sqrt{p(2q + mo(G)}.$$

Theorem 4.1: Consider $\boldsymbol{G}$ to be a non-disconnected graph possessing $V = \{v_1, v_2, v_3, v_4, \ldots\ldots, v_p\}$ where $i, j \in \{1,2, \ldots, p\}$ and size q. Then,

$$\sqrt{\big(2q + mo(G)\big) - 2|\sigma|\sqrt{p\big(2q + mo(G)\big)}} \leq \; {E^{\sigma}}_{LM}\,(G)$$

$$\leq \; p[\big(2q + mo(G)\big)^{\frac{1}{2}} + \; \sigma]$$

Proof: By Holders Inequality,

$$\sum_{i=1}^{p} |a_i b_i| \leq \; \sqrt{(\sum_{i=1}^{p} |a_i|^2\,)} \; \sqrt{(\sum_{i=1}^{p} |b_i|^2\,)}$$

Let $a_i = 1$, $b_i = |\lambda_i - \sigma|$

$$\sum_{i=1}^{p} |a_i b_i| \leq \; \sqrt{p} \; \sqrt{\sum_{i=1}^{p} |\lambda_i - \sigma|^2}$$

Using Minkowski's Inequality,

$$\sqrt{\sum_{j=1}^{p} |x_j \pm y_j|^2} \; \leq \; \sqrt{\sum_{j=1}^{p} |x_j|^2} + \sqrt{\sum_{j=1}^{p} |y_j|^2}$$

$$\leq \sqrt{p}[\; \sqrt{\sum_{j=1}^{p} |\lambda_j||\lambda_j|} \; + \; \sqrt{\sum_{j=1}^{p} |\sigma||\sigma|} \;]$$

$$\leq \; \sqrt{p} \; [\sqrt{p\big(2q + mo(G)\big)} \; + \; \sqrt{p} \; \sigma\,]$$

$$\leq \; p \; [\sqrt{\big(2q + mo(G)\big)} \; + \; \; \sigma \;]$$

Then,

$$E^{\sigma}{}_{LM}(G) \leq p\ [\ \sqrt{\big(2q + mo(G)\big)} +\ \sigma\ ]$$

Also,

$$[E^{\sigma}{}_{LM}(G)]^2 = \sum_{i=1}^{p} |\lambda_i - \sigma|^2$$

$$= \sum_{i=1}^{p} |\lambda_i - \sigma|^2$$

$$\geq \sum_{i=1}^{p} |\lambda_i|^2 - 2|\sigma| \sum_{i=1}^{p} |\lambda_i|$$

$$E^{\sigma}{}_{LM}(G) \geq \sqrt{\big(2q + mo(G)\big) - 2|\sigma|\sqrt{p(2q + mo(G))}}$$

Remark 3: Let ***G*** be a connected graph with set $V = \{v_1, v_2, v_3, v_4, \ldots\ldots, v_p\}$ and having size q. Then,

$$\sqrt{(p+1)} \leq \ E_{MM}(G) \leq p\sqrt{p}$$

Theorem 4.2: Let ***G*** be a connected graph with set $V = \{v_1, v_2, v_3, v_4, \ldots\ldots, v_p\}$ where $i, j \in \{1,2, \ldots, p\}$ and having size q. Then ,

$$\sqrt{(p+1) - 2|\sigma|p\sqrt{p}} \leq \ E^{\sigma}{}_{LM}(G) \leq p[p + \sigma]$$

Proof : By Holders Inequality,

$$\sum_{i=1}^{p} |a_i b_i| \leq \ \sqrt{(\sum_{i=1}^{p} |a_i|^2\ )}\ \sqrt{(\sum_{i=1}^{p} |b_i|^2\ )}$$

Let $a_i = 1$, $b_i = |\lambda_i - \sigma|$

$\sum_{i=1}^{p} |a_i b_i| \leq \ \sqrt{p}\ \sqrt{\sum_{i=1}^{p} |\lambda_i - \sigma|^2}$

Using Minkowski's Inequality,

$$\sqrt{\sum_{j=1}^{p}|x_j \pm y_j|^2} \leq \sqrt{\sum_{j=1}^{p}|x_j|^2} + \sqrt{\sum_{j=1}^{p}|y_j|^2}$$

$$\leq \sqrt{p}[\sqrt{\sum_{j=1}^{p}|\lambda_j||\lambda_j|} + \sqrt{\sum_{j=1}^{p}|\sigma||\sigma|}]$$

$$E^{\sigma}{}_{LM}(G) \leq \sqrt{p}\,[\sqrt{p^3} + \sqrt{p}\sigma]$$

Then,

$$E^{\sigma}{}_{LM}(G) \leq p[p + \sigma]$$

Also,

$$[E^{\sigma}{}_{LM}(G)]^2 = \sum_{i=1}^{p}|\lambda_i - \sigma|^2$$

$$= \sum_{i=1}^{p}|\lambda_i - \sigma|^2$$

$$\geq \sum_{i=1}^{p}|\lambda_i|^2 - 2|\sigma|\sum_{i=1}^{p}|\lambda_i|$$

$$\geq (p+1) - 2|\sigma|p\sqrt{p}$$

Then,

$$E^{\sigma}{}_{LM}(G) \geq \sqrt{(p+1) - 2|\sigma|p\sqrt{p}}$$

Remark 4: For a non-disconnected graph $\boldsymbol{G}$ containing $V = \{v_1, v_2, v_3, v_4, \ldots\ldots, v_p\}$ and size q and $D = \det(K_{MM}(G))$. Then,

$$\sqrt{2q + mo(G) + p(p-1)D^{\frac{2}{p}}} \leq E_{MM}(G) \leq \frac{(2q + mo(G))}{p} + \sqrt{(p-1)\left[(2q + mo(G)) - \left(\frac{2q+mo(G)}{p}\right)^2\right]}.$$

Theorem 4.3: Let $\boldsymbol{G}$ be a connected graph with set $V = \{v_1, v_2, v_3, v_4, \ldots.., v_p\}$ and size q and $D = \det(K_{MM}(G))$. Then,

$$\sqrt{(2q + mo(G) + p(p-1)D^{\frac{2}{p}} - 2|\sigma| \frac{(2q+mo(G))}{p} + \sqrt{(p-1)\left[(2q + mo(G)) - \left(\frac{2q+mo(G)}{p}\right)^2\right]}}$$

$$E^{\sigma}{}_{LM}(G) \le \sqrt{p}\, [\ \frac{(2q+mo(G))}{p} + \sqrt{(p-1)\left[(2q + mo(G)) - \left(\frac{2q+mo(G)}{p}\right)^2\right]} + \sqrt{p}\, \sigma\,]$$

Proof : By Holders Inequality,

$$\sum_{i=1}^{p} |a_i b_i| \le \sqrt{(\sum_{i=1}^{p} |a_i|^2\,)}\ \sqrt{(\sum_{i=1}^{p} |b_i|^2\,)}$$

Let $a_i = 1$, $b_i = |\lambda_i - \sigma|$

$\sum_{i=1}^{p} |a_i b_i| \le \sqrt{p}\ \sqrt{\sum_{i=1}^{p} |\lambda_i - \sigma|^2}$

Using Minkowski's Inequality,

$$\sqrt{\sum_{i=1}^{p} |x_i \pm y_i|^2} \le \sqrt{\sum_{i=1}^{p} |x_i|^2} + \sqrt{\sum_{i=1}^{p} |y_i|^2}$$

$$\le \sqrt{p}[\ \sqrt{\sum_{i=1}^{p} |\lambda_i||\lambda_i|} + \sqrt{\sum_{i=1}^{p} |\sigma||\sigma|}\ ]$$

$$\le \sqrt{p}[\ \sqrt{\sum_{i=1}^{p} |\lambda_i|^2} + \sqrt{\sum_{i=1}^{p} |\sigma|^2}\ ]$$

$$\le \sqrt{p}[\ \frac{(2q + mo(G))}{p} + \sqrt{(p-1)\left[(2q + mo(G)) - \left(\frac{2q + mo(G)}{p}\right)^2\right]} + \sqrt{p}\, \sigma\,]$$

Also,

$$[E^{\sigma}{}_{LM}(G)]^2 = \sum_{i=1}^{p} |\lambda_i - \sigma|^2$$

$$= \sum_{i=1}^{p} |\lambda_i - \sigma|^2$$

$$\geq \sum_{i=1}^{p} |\lambda_i|^2 - 2|\sigma| \sum_{i=1}^{p} |\lambda_i|$$

$$\geq (2q + mo(G) + p(p-1)D^{\frac{2}{p}} - 2|\sigma| \frac{(2q + mo(G))}{p} + \sqrt{(p-1)\left[(2q + mo(G)) - \left(\frac{2q + mo(G)}{p}\right)^2\right]}$$

Then,

$$E^{\sigma}{}_{LM}(G) \geq \sqrt{(2q + mo(G) + p(p-1)D^{\frac{2}{p}} - 2|\sigma| \frac{(2q+mo(G))}{p} + \sqrt{(p-1)\left[(2q + mo(G)) - \left(\frac{2q+mo(G)}{p}\right)^2\right]}}$$

Hence the proof.

5. CONCLUSION

In this paper, we have studied the standard deviation of the least monopoly energy $E^{\sigma}{}_{LM}(G)$ pertaining to graph G. The numerical value of standard deviation of the least monopoly energies of a few various kinds of graphs is obtained. We also found the boundary values for $E^{\sigma}{}_{LM}(G)$.

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