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Numerical Solutions of Nonlinear Wave-Like Equations Using Laplace-Adomian Decomposition and SBA Methods

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Abstract

In this paper, we use the Laplace-Adomian and SBA methods (combination of Adomian method and Picard successive approximations) has been applied for solving the nonlinear Wave-Like equation

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1. INTRODUCTION

Although the problem of resistance to moving of a ship's hull has always been of interest to sailors, it is still difficult to solve in a general context. Indeed, it is currently impossible to calculate this resistance in the case of any hull advancing in a formed sea [1].

However, considering that the ship is moving in a water at rest, we can simplify the problem by adopting the Froude hypothesis, which consists in saying that the resistance to the progress is the sum of three particular components :

- wave resistance due to the energy required to maintain the wave train accompanying the ship;

- viscous resistance due to friction;

- aerodynamic resistance which, practically, can be neglected (2 to 3% of the total resistance). However, it is very important to predict the amplitudes of waves. Nowadays, it is known that for the two-dimensional problem, the wave resistance is described by the formula :

$$R = \frac{1}{4}\rho g A^2[1 - \frac{2kh}{\sinh(2kh)}]$$

where is ρ = water density ,g = gravitational acceleration, h= depth, k= wave number ($\frac{2\pi}{\lambda}$), A = amplitude of waves [2].

In order to predict this wave resistance, we have empirical laws deduced from the total resistance obtained in model-scale test basins or from numerical models based on linearized or non-linearized theories. In the numerical context, some authors [3], [4],[5] have developed analytical methods of computations to solve Wave-Like equations, notably the decompositional methods of Adomian (ADM) and Sumudu. In order to verify the effectiveness of the analytical methods in comparison with those used by the previous authors, the present work uses the Laplace-Adomian method and the SBA method to solve analytically some Wave-Like equations of type:

$$\begin{cases} \frac{\partial^2 u(X,t)}{\partial t^2} = \sum_{i,j=1}^{n} F_{1ij}(X,t,u)\frac{\partial^{k+m}F_{2ij}(u_{x_i},u_{x_j})}{\partial x_i^k \partial x_i^m} + \sum_{i=1}^{n} G_{1i}(X,t,u)\frac{\partial^p G_{2i}(u_{x_i},u_{x_j})}{\partial x^p} + \\ \qquad H(X,t,u) + S(X,t) \\ u(X,0) = a_1(x) \\ \frac{\partial u(X,0)}{\partial t} = a_2(x) \end{cases} \tag{1}$$

where

$$\begin{cases} X = (x_1, x_2, \cdots, x_n) \\ F_{1ijX}, F_{2ij} \text{ non linear functions as } X, t, u \\ G_{1i}, G_{2i} \text{ non linear functions as } x_i, x_j \\ H, S \text{ non linear functions} \\ k, m, p \ \in \mathbb{N} \end{cases} \tag{2}$$

2. DESCRIBING OF BOTH METHOD

2.1. The Laplace transform [6]

Let's note the Laplace transform by

$$\mathcal{L}(u(x,t)) = \int_0^\infty u(x,t)e^{-st}dt \tag{3}$$

From (3), we get:

$$\begin{cases} \mathcal{L}(\frac{\partial u(x,t)}{\partial t}) = s\mathcal{L}\left(u(x,t)\right) - u(x,0) \\ \mathcal{L}(\frac{\partial^2 u(x,t)}{\partial t^2}) = s^2\mathcal{L}\left(u(x,t)\right) - su(x,0) - \frac{\partial u(x,0)}{\partial t} \end{cases} \tag{4}$$

2.2. Laplace-Adomian Decomposition method (LADM) [7] , [8]
Suppose that we need to solve the following equation:

$$u_{tt}(x,t) = \alpha u(x,t) + \beta N(u(x,t)) \tag{5}$$

The initial conditions are :

$$u(x,0) = f(x); \quad u_t(x,0) = g(x) \tag{6}$$

in a Banach space E, where $F : E \longrightarrow E$ is a linear or a nonlinear operator, $h \in E$ and u is the unknown function.

Let's suppose that operator F can be decomposed under the following form:

$$F = L - R - N \tag{7}$$

where $L - R$ is linear, N nonlinear. Let's suppose that L is invertible to the sense of Adomian with L^{-1} as inverse.

We get:

$$\mathcal{L}\left[u_{tt}(x,t)\right] = \alpha\mathcal{L}\left[u(x,t)\right] + \beta\mathcal{L}\left[N(u(x,t))\right] \tag{8}$$

Equation (8) is given by:

$$s^2\mathcal{L}\left(u(x,t)\right) - su(x,0) - u_t(x,0) = \alpha\mathcal{L}\left[u(x,t)\right] + \beta\mathcal{L}\left[N(u(x,t))\right] \tag{9}$$

$\Longleftrightarrow$

$$\left(s^2 - \alpha\right)\mathcal{L}\left(u(x,t)\right) = su(x,0) + u_t(x,0) + \beta\mathcal{L}\left[N(u(x,t))\right] \tag{10}$$

From (10), we get :

$$\mathcal{L}\left(u(x,t)\right) = \frac{s}{s^2-\alpha}u(x,0) + \frac{1}{s^2-\alpha}u_t(x,0) + \frac{\beta}{s^2-\alpha}\mathcal{L}\left[N(u(x,t))\right] \tag{11}$$

$\Longleftrightarrow$

$$u(x,t) = \mathcal{L}^{-1}\left(\frac{s}{s^2-\alpha}u(x,0)\right) + \mathcal{L}^{-1}\left(\frac{1}{s^2-\alpha}u_t(x,0)\right) + \mathcal{L}^{-1}\left(\frac{\beta}{s^2-\alpha}\mathcal{L}\left[N(u(x,t))\right]\right) \tag{12}$$

We look for the solution of $(5\ )$ in the following series expansion form

$$u(x,t)=\sum_{n=0}^{+\infty}u_n(x,t) \tag{13}$$

and we consider

$$Nu(x,t)=\sum_{n=0}^{+\infty}A_n \tag{14}$$

where A_n are the Adomian polynomials of $u_0, u_1, \cdots, u_n$ and it can be calculated by formula given below :

$$A_n=\frac{1}{n!}\left[\frac{d^n}{dp^n}N(\sum_{i=0}^{\infty}\lambda^i u_i\right]_{\lambda=0}, n=0,1,2,3,\cdots \tag{15}$$

Using eq (13) and eq (14) in eq (12), gives :

$$\left\{\begin{array}{c} \sum_{n=0}^{+\infty}u_n(x,t)=\mathcal{L}^{-1}\left(\frac{s}{s^2-\alpha}u(x,0)\right)+\mathcal{L}^{-1}\left(\frac{1}{s^2-\alpha}u_t(x,0)\right)+ \\ \sum_{n=0}^{+\infty}\mathcal{L}^{-1}\left(\frac{\beta}{s^2-\alpha}\mathcal{L}\left(A_n(x,t)\right)\right) \end{array}\right. \tag{16}$$

From (16), we have the following Adomian algorithm:

$$\left\{\begin{array}{l} u_0(x,t)=\mathcal{L}^{-1}\left(\frac{s}{s^2-\alpha}u(x,0)\right)+\mathcal{L}^{-1}\left(\frac{1}{s^2-\alpha}u_t(x,0)\right)=K(x,t) \\ u_{n+1}(x,t)=\mathcal{L}^{-1}\left(\frac{\beta}{s^2-\alpha}\mathcal{L}\left(\sum_{n=0}^{+\infty}A_n(x,t)\right)\right), n\geq 0 \end{array}\right. \tag{17}$$

$\Leftrightarrow$

$$\left\{\begin{array}{l} u_0(x,t)=K(x,t) \\ u_{n+1}(x,t)=\mathcal{L}^{-1}\left(\frac{\beta}{s^2-\alpha}\mathcal{L}\left(\sum_{n=0}^{+\infty}A_n(x,t)\right)\right), n\geq 0 \end{array}\right. \tag{18}$$

2.3. **SBA method** $[9]$, $[10]$.

Let's consider the following functional equation

$$Au=f \tag{19}$$

Where $A: H \to H$ is an operator not necessarily linear and H is a Hilbert space adequately chosen given the operator A.

Let:

$$A = L - R - N \tag{20}$$

Where L is an invertible operator in the Adomian sense, R the linear remainder and N a nonlinear operator. Equation (19) therefore becomes:

$$Lu - Ru - Nu = f \Leftrightarrow u = \theta + L^{-1}(f) + L^{-1}(Ru) + L^{-1}(Nu) \tag{21}$$

Where θ is such that $L\theta = 0$. Equation (21) is the Adomian canonical form [1]. Using the successive approximations [2], we get:

$$u^k = \theta + L^{-1}(f) + L^{-1}(R(u^k)) + L^{-1}(N(u^{k-1}));\ k \geq 1 \tag{22}$$

This yields the following Adomian algorithm :

$$\begin{cases} u_0^k = \theta + L^{-1}(f) + L^{-1}(N(u^{k-1})) \\ \\ u_n^k = L^{-1}(R(u_{n-1}^k)), \forall n \geq 1 \end{cases} \tag{23}$$

The Picard principle is then applied to equation (23) let u^0 be such that $N(u^0) = 0$, for $k = 1$, we get:

$$\begin{cases} u_0^1 = \theta + L^{-1}(f\ ) + L^{-1}(N(u^0)) \\ \\ u_n^1 = L^{-1}(R(u_{n-1}^1)), \forall n \geq 1 \end{cases} \tag{24}$$

If the series $\left(\sum\limits_{n=0}^{\infty} u_n^1\right)$ converges, then $u^1 = \left(\sum\limits_{n\geq 1}^{\infty} u_n^1\right)$

For $k = 2$,we get:

$$\begin{cases} u_0^2 = \theta + L^{-1}(f\ ) + L^{-1}(N(u^1)) \\ \\ u_n^2 = L^{-1}(R(u_{n-1}^2)), n\forall \geq 1 \end{cases} \tag{25}$$

If the series $\left(\sum\limits_{n=0}^{\infty} u_n^2\right)$ converges, then $u^2 = \left(\sum\limits_{n\geq 1}^{\infty} u_n^2\right)$.

This process is repeated to k.

If the series $\left(\sum\limits_{n=0}^{\infty} u_n^k\right)$ converges, then $u^k = \left(\sum\limits_{n\geq 1}^{\infty} u_n^k\right)$.

Therefore, $u = \lim\limits_{k\to+\infty} u^k$ is a solution of the problem.

3. TEST EXAMPLES

In this section, we present some examples with analytical solution to show the efficiency of method described in previons section for solving equation (1).

3.1. Example 1

Solve the following probem :

$$\begin{cases} \dfrac{\partial^2 u(x,t)}{\partial t^2} = (u(x,t))^2 \dfrac{\partial^2}{\partial x^2}\left(\dfrac{\partial u(x,t)}{\partial x}\dfrac{\partial^2 u(x,t)}{\partial x^2}\dfrac{\partial^3 u(x,t)}{\partial x^3}\right) + \\ \qquad \left(\dfrac{\partial u(x,t)}{\partial x}\right)^2 \frac{\partial^2}{\partial x^2}\left(\dfrac{\partial^2 u(x,t)}{\partial x^2}\right)^3 - 18u^5(x,t) + u(x,t) \\ u(x,0) = e^x \\ \dfrac{\partial u(x,0)}{\partial t} = e^x \end{cases} \tag{26}$$

Laplace-Adomian decomposition method

Applying Laplace-Adomian to (26) gives :

$$\begin{aligned} \mathcal{L}\left(u(x,t)\right) = \tfrac{s}{s^2-1}u(x,0) + \tfrac{1}{s^2-1}\frac{\partial u(x,0)}{\partial t} + \tfrac{1}{s^2-1}\mathcal{L}\left(N\left(u(x,t)\right)\right) + \\ \tfrac{1}{s^2-1}\mathcal{L}\left(M(u(x,t))\right) - 18\tfrac{1}{s^2-1}\mathcal{L}\ \left(K(u(x,t))\right) \end{aligned} \tag{27}$$

So that

$$\begin{aligned} \mathcal{L}\left(u(x,t)\right) = \tfrac{s}{s^2-1}e^x + \tfrac{1}{s^2-1}e^x + \tfrac{1}{s^2-1}\mathcal{L}\left(N\left(u(x,t)\right)\right) + \\ \tfrac{1}{s^2-1}\mathcal{L}\left(M(u(x,t))\right) - 18\tfrac{1}{s^2-1}\mathcal{L}\ \left(K(u(x,t))\right) \end{aligned} \tag{28}$$

Invertible transform gives canonic form :

$$\begin{aligned} u(x,t) = \mathcal{L}^{-1}\left(\tfrac{s+1}{s^2-1}\right)e^x + \mathcal{L}^{-1}\left(\tfrac{1}{s^2-1}\mathcal{L}\left(N\left(u(x,t)\right)\right)\right) + \\ \mathcal{L}^{-1}\left(\tfrac{1}{s^2-1}\mathcal{L}\left(M(u(x,t))\right)\right) - 18\mathcal{L}^{-1}\left(\tfrac{1}{s^2-1}\mathcal{L}\ \left(K(u(x,t))\right)\right) \end{aligned} \tag{29}$$

$\Leftrightarrow$

$$\begin{aligned} u(x,t) = e^{t+x} + \mathcal{L}^{-1}\left(\tfrac{1}{s^2-1}\mathcal{L}\left(N\left(u(x,t)\right)\right)\right) + \mathcal{L}^{-1}\left(\tfrac{1}{s^2-1}\mathcal{L}\left(M(u(x,t))\right)\right) - \\ 18\mathcal{L}^{-1}\left(\tfrac{1}{s^2-1}\mathcal{L}\ \left(K(u(x,t))\right)\right) \end{aligned} \tag{30}$$

and Laplace-Adomian algorithm

$$\begin{cases} u_0(x,t) = e^{t+x} \\ u_{n+1}(x,t) = \mathcal{L}^{-1}\left(\frac{1}{s^2-1}\mathcal{L}\left(A_n(x,t)\right)\right) + \mathcal{L}^{-1}\left(\frac{1}{s^2-1}\mathcal{L}\left(B_n(x,t)\right)\right) - \\ \qquad 18\mathcal{L}^{-1}\left(\frac{1}{s^2-1}\mathcal{L}\ \left(C_n(x,t)\right)\right), \forall n \geq 0 \end{cases} \tag{31}$$

Then

$$\begin{cases} \begin{cases} A_0 = (u_0(x,t))^2 \frac{\partial^2}{\partial x^2}(\frac{\partial u_0(x,t)}{\partial x}\frac{\partial^2 u_0(x,t)}{\partial x^2}\frac{\partial^3 u_0(x,t)}{\partial x^3}) \\ = (e^{t+x})^2 \frac{\partial^2}{\partial x^2}(\frac{\partial}{\partial x}(e^{t+x})\frac{\partial^2}{\partial x^2}(e^{t+x})\frac{\partial^3}{\partial x^3}(e^{t+x})) \\ = 9e^{5t+5x} \end{cases} \\ \begin{cases} B_0 = \left(\frac{\partial u_0(x,t)}{\partial x}\right)^2 \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 u_0(x,t)}{\partial x^2}\right)^3 \\ = \left(\frac{\partial}{\partial x}(e^{t+x})\right)^2 \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2}{\partial x^2}(e^{t+x})\right)^3 \\ = 9e^{5t+5x} \end{cases} \\ \begin{cases} C_0 = u_0^5(x,t) \\ \quad = e^{5t+5x} \end{cases} \end{cases} \tag{32}$$

Thus, we see that

$$\begin{cases} u_0(x,t) = e^{t+x} \\ u_1(x,t) = 18e^{5x}\left[\mathcal{L}^{-1}\left(\frac{1}{s^2-1}\mathcal{L}\left(e^{5t}\right)\right)\right] - 18e^{5x}\left[\mathcal{L}^{-1}\left(\frac{1}{s^2-1}\mathcal{L}\ \left(e^{5t}\right)\right)\right] = 0 \\ u_n(x,t) = 0, \forall n \geq 2 \end{cases} \tag{33}$$

so

$$u(x,t) = u_0(x,t) = e^{t+x} \tag{34}$$

b) **Solving by SBA method**

Consider state equation :

$$\begin{aligned} \frac{\partial^2 u(x,t)}{\partial t^2} = (u(x,t))^2 \frac{\partial^2}{\partial x^2}(\frac{\partial u(x,t)}{\partial x}\frac{\partial^2 u(x,t)}{\partial x^2}\frac{\partial^3 u(x,t)}{\partial x^3}) + \\ \left(\frac{\partial u(x,t)}{\partial x}\right)^2 \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)^3 - 18u^5(x,t) + u(x,t) \end{aligned} \tag{35}$$

Let's make

$$L(.) = \frac{\partial^2(.)}{\partial t^2} \Leftrightarrow L^{-1} = \int_0^t \left(\int_0^s (.)dz\right) ds \tag{36}$$

and

$$N(u(x,t)) = (u(x,t))^2 \frac{\partial^2}{\partial x^2}\left(\frac{\partial u(x,t)}{\partial x}\frac{\partial^2 u(x,t)}{\partial x^2}\frac{\partial^3 u(x,t)}{\partial x^3}\right)+ \left(\frac{\partial u(x,t)}{\partial x}\right)^2 \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)^3 - 18u^5(x,t) \tag{37}$$

Equation (35) gives :

$$\frac{\partial^2 u(x,t)}{\partial t^2} = u(x,t) + N(u(x,t)) \tag{38}$$

Applying L^{-1} to (38) gives canonic form following :

$$u(x,t) = u(x,0) + \frac{\partial u(x,0)}{\partial t} t + \int_0^t \left(\int_0^s (u(x,z))dz\right) ds + \int_0^t \left(\int_0^s (N(u(x,z)))dz\right) ds \tag{39}$$

So

$$u(x,t) = (1+t)\, e^x + \int_0^t \left(\int_0^s (u(x,z))dz\right) ds + \int_0^t \left(\int_0^s (N(u(x,z)))dz\right) ds \tag{40}$$

Successive approximation theorem gives :

$$u^k(x,t) = (1+t)\, e^x + \int_0^t \left(\int_0^s (u^k(x,z))dz\right) ds + \int_0^t \left(\int_0^s (N(u^{k-1}(x,z)))dz\right) ds \tag{41}$$

By replacing $u^k(x,t)$ by $\sum_{n=0}^{+\infty} u_n^k(x,t)$ and we let $\widetilde{N}(u^{k-1}(x,t)) = \int_0^t \left(\int_0^s N(u(x,z))dz\right) ds$, we have :

$$\sum_{n=0}^{+\infty} u_n^k(x,t) = (1+t)\, e^x + \sum_{n=0}^{+\infty} \int_0^t \left(\int_0^s (u_n^k(x,z))dz\right) ds + \widetilde{N}(u^{k-1}(x,t)) \tag{42}$$

Then algorithm form is :

$$\begin{cases} u_0^k(x,t) = (1+t)\, e^x + \widetilde{N}(u^{k-1}(x,t)) \\ u_{n+1}^k(x,t) = \int_0^t \left(\int_0^s (u_n^k(x,z))dz\right) ds \end{cases} \tag{43}$$

Détermine $u^k(x,t)$ for $k \geq 0$

For $k = 1$, we have the following SBA algorithm. If we choice u^0 so that

$$\begin{cases} u_0^1(x,t) = (1+t)\,e^x \ + \ \widetilde{N}(u^0(x,t)) \\ u_{n+1}^1(x,t) = \displaystyle\int_0^t \left(\int_0^s (u_n^1(x,z))dz \right) ds \ , \forall n \geq 0 \end{cases} \tag{44}$$

Let's suppose that one can find $u^0(x,t)$ as $\widetilde{N}(u^0(x,t)) = 0$, we obtain the following SBA algorithm:

$$\begin{cases} u_0^1(x,t) = (1+t)\,e^x \\ u_{n+1}^1(x,t) = \displaystyle\int_0^t \left(\int_0^s (u_n^1(x,z))dz \right) ds, \ \forall n \geq 0 \end{cases} \tag{45}$$

From (45), we get

$$\begin{cases} u_0^1(x,t) = e^x + te^x \\ u_1^1(x,t) = \frac{1}{2}t^2e^x + \frac{1}{3!}t^3e^x \\ u_2^1(x,t) = \frac{1}{4!}t^4e^x + \frac{1}{5!}t^5e^x \\ u_3^1(x,t) = \frac{1}{6!}t^6e^x + \frac{1}{7!}t^7e^x \\ \vdots \end{cases} \tag{46}$$

Thus

$$\begin{cases} u^1(x,t) = u_0^1(x,t) + u_1^1(x,t) + u_2^1(x,t) + u_3^1(x,t) + \cdots \\ \qquad = e^x + te^x + \frac{1}{2}t^2e^x + \frac{1}{3!}t^3e^x + \frac{1}{4!}t^4e^x + \frac{1}{5!}t^5e^x + \cdots \\ \qquad = \left(1 + t + \frac{1}{2}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \frac{1}{5!}t^5 + \cdots\right) e^x \\ \qquad = e^te^x \end{cases} \tag{47}$$

(47)$\Leftrightarrow$

$$u^1(x,t) = e^{t+x} \tag{48}$$

For $k = 2$, we get following SBA algorithm:

$$\begin{cases} u_0^2(x,t) = (1+t)\,e^x \ + \ \widetilde{N}(u^1(x,t)) \\ u_{n+1}^2(x,t) = \displaystyle\int_0^t \left(\int_0^s (u_n^2(x,z))dz \right) ds, \forall n \geq 0 \end{cases} \tag{49}$$

From (49), therefore we get :

$$\begin{aligned}
N(u^1(x,t)) &= \left(u^1(x,t)\right)^2 \frac{\partial^2}{\partial x^2}\left(\frac{\partial u^1(x,t)}{\partial x}\frac{\partial^2 u^1(x,t)}{\partial x^2}\frac{\partial^3 u^1(x,t)}{\partial x^3}\right)+ \\
&\quad \left(\frac{\partial u^1(x,t)}{\partial x}\right)^2 \tfrac{\partial^2}{\partial x^2}\left(\frac{\partial^2 u^1(x,t)}{\partial x^2}\right)^3 - 18\left(u^1(x,t)\right)^5 \\
&= \left(e^{t+x}\right)^2 \frac{\partial^2}{\partial x^2}\left(\frac{\partial\left(e^{t+x}\right)}{\partial x}\frac{\partial^2\left(e^{t+x}\right)}{\partial x^2}\frac{\partial^3\left(e^{t+x}\right)}{\partial x^3}\right) \\
&\quad + \left(\frac{\partial\left(e^{t+x}\right)}{\partial x}\right)^2 \tfrac{\partial^2}{\partial x^2}\left(\frac{\partial^2\left(e^{t+x}\right)}{\partial x^2}\right)^3 - 18\left(e^{t+x}\right)^5 \\
&= 9e^{3t+3x}e^{2(t+x)} + 9e^{3t+3x}e^{2(t+x)} - 18e^{5(t+x)} \\
&= 18e^{5t+5x} - 18e^{5t+5x} \\
&= 0
\end{aligned} \tag{50}$$

$\Longrightarrow$

$$\widetilde{N}(u^1(x,t)) = \int_0^t \left(\int_0^s N(u^1(x,z))dz\right) ds = 0 \tag{51}$$

From (49) and (50), we obtain

$$\begin{cases}
u_0^2(x,t) = e^x + te^x \\
u_1^2(x,t) = \frac{1}{2}t^2e^x + \frac{1}{3!}t^3e^x \\
u_2^2(x,t) = \frac{1}{4!}t^4e^x + \frac{1}{5!}t^5e^x \\
u_3^2(x,t) = \frac{1}{6!}t^6e^x + \frac{1}{7!}t^7e^x \\
\vdots
\end{cases} \tag{52}$$

and

$$\begin{aligned}
u^2(x,t) &= u_0^2(x,t) + u_1^2(x,t) + u_2^2(x,t) + u_3^2(x,t) + \cdots \\
&= e^x + te^x + \tfrac{1}{2}t^2e^x + \tfrac{1}{3!}t^3e^x + \tfrac{1}{4!}t^4e^x + \tfrac{1}{5!}t^5e^x + \cdots \\
&= \left(1 + t + \tfrac{1}{2}t^2 + \tfrac{1}{3!}t^3 + \tfrac{1}{4!}t^4 + \tfrac{1}{5!}t^5 + \cdots\right)e^x \\
&= e^te^x \\
&= e^{t+x}
\end{aligned} \tag{53}$$

Thus,

$$u^2(x,t) = e^{t+x} \tag{54}$$

Using the same procedure for $k \geq 3$, we have

$$u^3(x,t) = \cdots = u^k(x,t) = e^{t+x} \tag{55}$$

From which, we obtain

$$u(x,t) = \lim_{k\to+\infty} u^k(x,t) = e^{t+x} \tag{56}$$

This table give comparative solutions for example 1 :

Laplace-Adomian Method	$u(x,t) = e^{t+x}$
SBA Method	$u(x,t) = e^{t+x}$

(57)

3.1.1 Example 2

Lets us consider the following problem :

$$\begin{cases} \dfrac{\partial^2 u(x,t)}{\partial t^2} = x^2 \dfrac{\partial}{\partial x}\left(\dfrac{\partial u(x,t)}{\partial x}\dfrac{\partial^2 u(x,t)}{\partial x^2}\right) - \ x^2\left(\dfrac{\partial^2 u(x,t)}{\partial x^2}\right)^2 - u(x,t) \\ u(x,0) = 0 \\ \dfrac{\partial u(x,0)}{\partial t} = x^2 \end{cases} \tag{58}$$

where $u = u(x,t)$ and $0 \leq x \leq 1$, $t \geq 0$

Détermine solution of problem (58) by LADM and SBA method.

a) <u>Solving by Laplace-Adomian Decomposition method</u>

Applying Laplace transform to $(58\ )$ gives :

$$\mathcal{L}(u(x,t)) = \tfrac{s}{s^2+1}u(x,0) + \tfrac{1}{s^2+1}\frac{\partial u(x,0)}{\partial t} + \tfrac{1}{s^2+1}\mathcal{L}\left(x^2\frac{\partial}{\partial x}(N(u(x,t)))\right) - \tfrac{1}{s^2+1}\mathcal{L}\left(\ x^2 M(u(x,t))\right) \tag{59}$$

so

$$\begin{cases} N(u(x,t)) = \dfrac{\partial u(x,t)}{\partial x}\dfrac{\partial^2 u(x,t)}{\partial x^2} \\ M(u(x,t) = \left(\dfrac{\partial^2 u(x,t)}{\partial x^2}\right)^2 \end{cases}$$

We can rearrange the equation (59) as either

$$\mathcal{L}(u(x,t)) = \tfrac{1}{s^2+1}x^2 + \tfrac{1}{s^2+1}\mathcal{L}\left(x^2\frac{\partial}{\partial x}(N(u(x,t)))\right) - \tfrac{1}{s^2+1}\mathcal{L}\left(\ x^2 M(u(x,t))\right) \tag{60}$$

Applying invertible Laplace transform to $(60\ )$, we obtain canonic form following:

$$u(x,t) = \mathcal{L}^{-1}\left(\tfrac{1}{s^2+1}\right)x^2 + \mathcal{L}^{-1}\left(\tfrac{1}{s^2+1}\mathcal{L}\left(x^2\frac{\partial}{\partial x}(N(u(x,t)))\right)\right) - \mathcal{L}^{-1}\left(\tfrac{1}{s^2+1}\mathcal{L}\left(\ x^2 M(u(x,t))\right)\right) \tag{61}$$

$\Leftrightarrow$

$$u(x,t) = x^2\sin t + \mathcal{L}^{-1}\left(\tfrac{1}{s^2+1}\mathcal{L}\left(x^2\frac{\partial}{\partial x}(N(u(x,y,t)))\right)\right) - \mathcal{L}^{-1}\left(\tfrac{1}{s^2+1}\mathcal{L}\left(\ x^2 M(u(x,y,t))\right)\right) \tag{62}$$

Where

$$\begin{cases} N(u(x,t)) = \sum\limits_{n=0}^{+\infty} A_n(x,t) \\ M(u(x,t) = \sum\limits_{n=0}^{+\infty} B_n(x,t) \end{cases} \tag{63}$$

Using eq (15), we get Adomian's polynomials as follows

$$\begin{cases} \begin{cases} A_0(x,t) = \dfrac{\partial u_0(x,t)}{\partial x}\dfrac{\partial^2 u_0(x,t)}{\partial x^2} \\ A_1(x,t) = \dfrac{\partial u_0(x,t)}{\partial x}\dfrac{\partial^2 u_1(x,t)}{\partial x^2} + \dfrac{\partial u_1(x,t)}{\partial x}\dfrac{\partial^2 u_0(x,t)}{\partial x^2} \\ A_2(x,t) = \dfrac{\partial u_0(x,t)}{\partial x}\dfrac{\partial^2 u_2(x,t)}{\partial x^2} + \dfrac{\partial u_1(x,t)}{\partial x}\dfrac{\partial^2 u_1(x,t)}{\partial x^2} + \dfrac{\partial u_2(x,t)}{\partial x}\dfrac{\partial^2 u_0(x,t)}{\partial x^2} \\ \vdots \end{cases} \\ \text{and} \\ \begin{cases} B_0(x,t) = \left(\dfrac{\partial^2 u_0(x,t)}{\partial x^2}\right)^2 \\ B_1(x,t) = 2\left(\dfrac{\partial^2 u_0(x,t)}{\partial x^2}\right)\left(\dfrac{\partial^2 u_1(x,t)}{\partial x^2}\right) \\ B_2(x,t) = \left(\dfrac{\partial^2 u_1(x,t)}{\partial x^2}\right)^2 + 2\left(\dfrac{\partial^2 u_0(x,t)}{\partial x^2}\right)\left(\dfrac{\partial^2 u_2(x,t)}{\partial x^2}\right) \end{cases} \end{cases} \tag{64}$$

That is $(62\ \ )$, we get Laplace-Adomian algorithm following :

$$\begin{cases} u_0(x,t) = x^2\sin t \\ u_{n+1}(x,t) = \mathcal{L}^{-1}\left(\tfrac{1}{s^2+1}\mathcal{L}\left(x^2\frac{\partial}{\partial x}(A_n(x,t))\right)\right) - \mathcal{L}^{-1}\left(\tfrac{1}{s^2+1}\mathcal{L}\left(\ x^2 B_n(x,t)\right)\right) \end{cases} \tag{65}$$

So

$$\begin{cases} A_0(x,t) = \dfrac{\partial\left(x^2\sin t\right)}{\partial x}\dfrac{\partial^2\left(x^2\sin t\right)}{\partial x^2} = 4x\sin^2 t \\ B_0(x,t) = \left(\dfrac{\partial^2\left(x^2\sin t\right)}{\partial x^2}\right)^2 = 4\sin^2 t \\ u_0(x,t) = x^2\sin t \\ u_1(x,t) = 4x^2\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\mathcal{L}\left(\sin^2 t\right)\right) - \ 4x^2\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\mathcal{L}\left(\ \sin^2 t\right)\right) = 0 \\ A_n(x,t) = 0, \forall n \geq 1 \\ B_n(x,t) = 0, \forall n \geq 1 \\ u_n(x,t) = 0, \forall n \geq 1 \end{cases} \tag{66}$$

Therefore, solution of problem is :

$$u(x,t) = u_0(x,t) = x^2\sin t \tag{67}$$

b) **Solving by SBA method**

Let us consider equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = -u(x,t) + C(u(x,t)) \tag{68}$$

where

$$C(u(x,t)) = x^2\frac{\partial}{\partial x}\left(\frac{\partial u(x,t)}{\partial x}\frac{\partial^2 u(x,t)}{\partial x^2}\right) - \ x^2\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)^2 \tag{69}$$

Equation (68) gives canonic form following :

$$u(x,t) = u(x,0) + \frac{\partial u(x,0)}{\partial t}t - \int_0^t\left(\int_0^s u(x,z)dz\right)ds + \int_0^t\left(\int_0^s C(u(x,z))dz\right)ds \tag{70}$$

so

$$u(x,t) = x^2 t - \int_0^t\left(\int_0^s u(x,z)dz\right)ds + \int_0^t\left(\int_0^s C(u(x,z))dz\right)ds \tag{71}$$

Successive approximation method leads to for $k \geq 1$

$$u^k(x,t) = x^2 t - \int_0^t \left(\int_0^s u^k(x,z)dz \right) ds + \widetilde{C}(u^{k-1}(x,t)) \tag{72}$$

where

$$\widetilde{C}(u^{k-1}(x,t)) = \int_0^t \left(\int_0^s C(u(x,z))dz \right) ds \tag{73}$$

SBA associate algorithm is given by :

$$\begin{cases} u_0^k(x,t) = x^2 t + \widetilde{C}(u^{k-1}(x,t)) \\ \\ u_{n+1}^k(x,t) = -\int_0^t \left(\int_0^s u_n^k(x,z)dz \right) ds, \forall n \geq 0 \end{cases} \tag{74}$$

At stage $k = 1$,we get u^0 so that $\widetilde{C}(u^0(x,t)) = 0$

$$\begin{cases} u_0^1(x,t) = x^2 t + \widetilde{C}(u^0(x,t)) \\ \\ u_{n+1}^1(x,t) = -\int_0^t \left(\int_0^s u_n^1(x,z)dz \right) ds, \forall n \geq 0 \end{cases} \tag{75}$$

Equation (75) would be :

$$\begin{cases} u_0^1(x,t) = x^2 t \\ \\ u_1^1(x,t) = -\frac{1}{6} t^3 x^2 \\ \vdots \\ u_n^1(x,t) = (-1)^n \frac{t^{2n+1}}{(2n+1)!} x^2, \forall n \geq 0 \end{cases} \tag{76}$$

$$u^1(x,t) = \left(\sum_{n=0}^{+\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!} \right) x^2 = x^2 \sin t \tag{77}$$

At stage $k = 2$, we get

$$\begin{cases} u_0^2(x,t) = x^2 t + \widetilde{C}(u^1(x,t)) \\ \\ u_{n+1}^2(x,t) = -\int_0^t \left(\int_0^s u_n^2(x,z)dz \right) ds, \forall n \geq 0 \end{cases} \tag{78}$$

Thus,

$$\begin{cases} C(u^1(x,t)) = x^2\frac{\partial}{\partial x}(\frac{\partial u^1(x,t)}{\partial x}\frac{\partial^2 u^1(x,t)}{\partial x^2}) - \ x^2\left(\frac{\partial^2 u^1(x,t)}{\partial x^2}\right)^2 \\ \qquad = x^2\frac{\partial}{\partial x}(\frac{\partial\left(x^2\sin t\right)}{\partial x}\frac{\partial^2\left(x^2\sin t\right)}{\partial x^2}) - \ x^2\left(\frac{\partial^2\left(x^2\sin t\right)}{\partial x^2}\right)^2 \\ \qquad = -2x^2\left(\cos 2t - 1\right) - \ 4x^2\sin^2 t \\ \qquad = 4x^2\sin^2 t - \ 4x^2\sin^2 t \\ C(u^1(x,t)) = 0 \end{cases} \tag{79}$$

$$\Rightarrow \widetilde{C}(u^1(x,t)) = \int_0^t\left(\int_0^s C(u^1(x,z))dz\right)ds = 0 \tag{80}$$

In a recursive way, one deduces

$$\begin{cases} u_0^2(x,t) = x^2 t \\ u_1^2(x,t) = -\frac{1}{6}t^3x^2 \\ \vdots \\ u_n^2(x,t) = (-1)^n\ \frac{t^{2n+1}}{(2n+1)!}x^2, \forall n \geq 0 \end{cases} \tag{81}$$

Therefore

$$u^2(x,t) = \left(\sum_{n=0}^{+\infty}(-1)^n\ \frac{t^{2n+1}}{(2n+1)\,!}\right)x^2 = x^2\sin t \tag{82}$$

In a recursive way, one deduces for $k > 2$ approximate solution :

$$u^k(x,t) = x^2\sin t \tag{83}$$

Thus, exact solution of problem (58) is

$$u(x,t) = \lim_{k\to+\infty} u^k(x,t) = x^2\sin t \tag{84}$$

This table compare obtained solutions :

Laplace-Adomian Method	$u(x,t) = x^2\sin t$
SBA Method	$u(x,t) = x^2\sin t$

(85)

Conclusion

The two solutions were similar.

3.1.2 Example 3

Let us consider mathematical model following

$$\begin{cases} \dfrac{\partial^2 u(x,y,t)}{\partial t^2} = \dfrac{\partial^2}{\partial x \partial y}\left(\dfrac{\partial^2 u(x,y,t)}{\partial x^2}\dfrac{\partial^2 u(x,y,t)}{\partial y^2}\right) - \frac{\partial^2}{\partial x \partial y}\left(xy\dfrac{\partial u(x,y,t)}{\partial x}\dfrac{\partial u(x,y,t)}{\partial y}\right) - u(x,y,t) \\ u(x,y,0) = e^{xy} \\ \dfrac{\partial u(x,y,0)}{\partial t} = e^{xy} \end{cases} \tag{86}$$

a) **Solving by Laplace-Adomian Decomposition method**

Consider equation following

$$\frac{\partial^2 u(x,y,t)}{\partial t^2} = \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial^2 u(x,y,t)}{\partial x^2}\frac{\partial^2 u(x,y,t)}{\partial y^2}\right) - \frac{\partial^2}{\partial x \partial y}\left(xy\frac{\partial u(x,y,t)}{\partial x}\frac{\partial u(x,y,t)}{\partial y}\right) - u(x,y,t) \tag{87}$$

Equation (87) leads to

$$\frac{\partial^2 u(x,y,t)}{\partial t^2} = \frac{\partial^2}{\partial x \partial y}(N(u(x,y,t))) - \frac{\partial^2}{\partial x \partial y}(M(u(x,y,t))) - u(x,y,t) \tag{88}$$

where

$$\begin{cases} N(u(x,y,t)) = \dfrac{\partial^2 u(x,y,t)}{\partial x^2}\dfrac{\partial^2 u(x,y,t)}{\partial y^2} \\ M(u(x,y,t)) = xy\dfrac{\partial u(x,y,t)}{\partial x}\dfrac{\partial u(x,y,t)}{\partial y} \end{cases} \tag{89}$$

Applying Laplace transform to (88) gives :

$$\mathcal{L}\left(\frac{\partial^2 u(x,y,t)}{\partial t^2}\right) = \mathcal{L}\left(\frac{\partial^2}{\partial x \partial y}(N(u(x,t)))\right) - \mathcal{L}\left(\frac{\partial^2}{\partial x \partial y}(M(u(x,t)))\right) - \mathcal{L}(u(x,t)) \tag{90}$$

Thus

$$\mathcal{L}(u(x,y,t)) = \tfrac{s}{s^2+1}u(x,y,0) + \tfrac{1}{s^2+1}\frac{\partial u(x,y,0)}{\partial t} + \tfrac{1}{s^2+1}\mathcal{L}\left(\tfrac{\partial^2}{\partial x \partial y}(N(u(x,y,t)))\right) - \tfrac{1}{s^2+1}\mathcal{L}\left(\tfrac{\partial^2}{\partial x \partial y}(M(u(x,y,t)))\right) \tag{91}$$

$\Leftrightarrow$

$$\begin{aligned} u(x,y,t) = {} & \mathcal{L}^{-1}\left(\tfrac{s}{s^2+1}u(x,y,0)\right) + \mathcal{L}^{-1}\left(\tfrac{1}{s^2+1}\frac{\partial u(x,y,0)}{\partial t}\right) \\ & + \mathcal{L}^{-1}\left(\tfrac{1}{s^2+1}\mathcal{L}\left(\tfrac{\partial^2}{\partial x \partial y}(N(u(x,y,t)))\right)\right) - \\ & \mathcal{L}^{-1}\left(\tfrac{1}{s^2+1}\mathcal{L}\left(\tfrac{\partial^2}{\partial x \partial y}(M(u(x,y,t)))\right)\right) \end{aligned} \tag{92}$$

We obtain canonic form following :

$$u(x,y,t) = (\cos t + \sin t)\, e^{xy} + \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\mathcal{L}\left(\frac{\partial^2}{\partial x\partial y}(N\,(u(x,y,t)))\right)\right) - \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\mathcal{L}\left(\frac{\partial^2}{\partial x\partial y}(M\,(u(x,y,t)))\right)\right) \tag{93}$$

Chosing solution so that

$$u(x,y,t) = \sum_{n=0}^{\infty} u_n(x,y,t) \tag{94}$$

and non linear terms $N\,(u(x,y,t))$ and $M\,(u(x,y,t))$ so that

$$\begin{cases} N\,(u(x,y,t)) = \sum\limits_{n=0}^{\infty} A_n(x,y,t) \\ M(u(x,y,t)) = \sum\limits_{n=0}^{\infty} B_n(x,y,t) \end{cases} \tag{95}$$

By replacing (94) and (95) it (93), we see that :

$$\sum_{n=0}^{\infty} u_n(x,y,t) = (\cos t + \sin t)\, e^{xy} + \sum_{n=0}^{\infty} \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\mathcal{L}\left(\frac{\partial^2}{\partial x\partial y}(A_n(x,y,t)\right)\right) - \sum_{n=0}^{\infty} \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\mathcal{L}\left(\frac{\partial^2}{\partial x\partial y}(B_n(x,y,t)\right)\right) \tag{96}$$

Therefore (96) gives Laplace-Adomian algorithm following :

$$\begin{cases} u_0(x,y,t) = (\cos t + \sin t)\, e^{xy} \\ u_{n+1}(x,y,t) = \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\mathcal{L}\left(\frac{\partial^2}{\partial x\partial y}(A_n(x,y,t))\right)\right) - \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\mathcal{L}\left(\frac{\partial^2}{\partial x\partial y}(B_n(x,y,t))\right)\right), \forall n \geq 0 \end{cases} \tag{97}$$

Then

$$\begin{cases} u_0(x,y,t) = (\cos t + \sin t)\, e^{xy} \\ A_0(x,y,t) = x^2y^2\,(\cos t + \sin t)^2\, e^{2(xy)} \\ B_0(x,y,t) = (\cos t + \sin t)^2\, x^2y^2e^{2(xy)} \\ u_1(x,y,t) = \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\mathcal{L}\left((\cos t + \sin t)^2\, \frac{\partial^2}{\partial x\partial y}(x^2y^2e^{2(xy)})\right)\right) - \\ \qquad \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\mathcal{L}\left((\cos t + \sin t)^2\, \frac{\partial^2}{\partial x\partial y}(x^2y^2e^{2(xy)})\right)\right) \\ u_1(x,y,t) \quad = 0 \\ A_n(x,y,t) = 0, \forall n \geq 1 \\ B_n(x,y,t) = 0, \forall n \geq 1 \\ u_n(x,y,t) = 0, \forall n \geq 2 \end{cases} \tag{98}$$

The exact solution of problem is :

$$u(x,y,t) = u_0(x,y,t) = (\cos t + \sin t)\, e^{xy} \tag{99}$$

b) **Solving by SBA method**

Equation of problem (98) is :

$$\frac{\partial^2 u(x,y,t)}{\partial t^2} = -u(x,y,t) + C(u(x,y,t)) \tag{100}$$

where

$$C(u(x,y,t)) = \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial^2 u(x,y,t)}{\partial x^2}\frac{\partial^2 u(x,y;t)}{\partial y^2}\right) - \frac{\partial^2}{\partial x \partial y}\left(xy\frac{\partial u(x,y,t)}{\partial x}\frac{\partial u(x,y,t)}{\partial y}\right) \tag{101}$$

Twice integration of (100) gives Adomian canonic form following :

$$u(x,y,t) = (1+t)\, e^{xy} - \int_0^t \left(\int_0^s u(x,y,z)dz\right) ds + \int_0^t \left(\int_0^s C(u(x,y,z))dz\right) ds \tag{102}$$

Applying successive approximation method, we gives for $k \geq 1$:

$$u^k(x,y,t) = (1+t)\, e^{xy} - \int_0^t \left(\int_0^s u^k(x,y,z)dz\right) ds + \widetilde{C}(u^{k-1}(x,y,z)) \tag{103}$$

where

$$\widetilde{C}(u^{k-1}(x,y,z)) = \int_0^t \left(\int_0^s C(u^{k-1}(x,y,z))dz\right) ds \tag{104}$$

According to (103), we get Adomian algorithm :

$$\begin{cases} u_0^k(x,y,t) = (1+t)\, e^{xy} + \widetilde{C}(u^{k-1}(x,y,z)) \\ u_{n+1}^k(x,y,t) = (1+t)\, e^{xy} - \int_0^t \left(\int_0^s u_n^k(x,y,z)dz\right) ds, \forall n \geq 0 \end{cases} \tag{105}$$

For $k = 1$, we get the following SBA algorithm :

$$\begin{cases} u_0^1(x,y,t) = (1+t)\, e^{xy} + \widetilde{C}(u^0(x,y,z)) \\ u_{n+1}^1(x,y,t) = (1+t)\, e^{xy} - \int_0^t \left(\int_0^s u_n^1(x,y,z)dz\right) ds, \forall n \geq 0 \end{cases} \tag{106}$$

Taking $u^0(x,y,t)=0$, we get :

$$\widetilde{C}(u^0(x,y,z)) = \int_0^t \left(\int_0^s C(u^0(x,y,z))dz\right) ds$$
$$= \int_0^t \left(\int_0^s \left(\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial^2 u_0(x,y,z)}{\partial x^2}\frac{\partial^2 u_0(x,y;z)}{\partial y^2}\right) - \frac{\partial^2}{\partial x\partial y}\left(xy\frac{\partial u_0(x,y,z)}{\partial x}\frac{\partial u_0(x,y,z)}{\partial y}\right)\right) dz\right) ds = 0 \tag{107}$$

From (106), we get :

$$\begin{cases} u_0^1(x,y,t) = (1+t)\,e^{xy} \\ u_n^1(x,y,t) = \left[(-1)^n \frac{t^{2n}}{(2n)!} + (-1)^n \frac{t^{2n+1}}{(2n+1)!}\right] e^{xy}, \forall n \geq 1 \end{cases} \tag{108}$$

Let's put

$$\varphi_n^1(x,y,t) = \sum_{i=0}^{n} u_i^1(x,y,t) \tag{109}$$

We obtain

$$\varphi_n^1(x,y,t) = \left(\sum_{i=0}^{n} (-1)^i \frac{t^{2i}}{(2i)\,!} + \sum_{i=0}^{n} (-1)^i \frac{t^{2i+1}}{(2i+1)\,!}\right) e^{xy} \tag{110}$$

Thus, for $k=1$, we get:

$$u^1(x,y,t) = \lim_{n\to+\infty} \varphi_n^1(x,y,t) = (\cos t + \sin t)\, e^{xy} \tag{111}$$

Thus

$$\begin{aligned} C(u^1(x,y,t)) &= \frac{\partial^2}{\partial x\partial y}\left(\frac{\partial^2 u^1(x,y,t)}{\partial x^2}\frac{\partial^2 u^1(x,y;t)}{\partial y^2}\right) - \frac{\partial^2}{\partial x\partial y}\left(xy\frac{\partial u^1(x,y,t)}{\partial x}\frac{\partial u^1(x,y,t)}{\partial y}\right) \\ &= (\cos t + \sin t)^2 \left[\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial^2 (e^{xy})}{\partial x^2}\frac{\partial^2 ((e^{xy}))}{\partial y^2}\right) - \frac{\partial^2}{\partial x\partial y}\left(xy\frac{\partial (e^{xy})}{\partial x}\frac{\partial u(e^{xy})}{\partial y}\right)\right] \\ &= 2xye^{2xy}\left(2x^2y^2+5xy+2\right) - 2xye^{2xy}\left(2x^2y^2+5xy+2\right) \\ &= 0 \\ \Rightarrow \widetilde{C}(u^1(x,y,z)) &= 0 \end{aligned} \tag{112}$$

For $k=2$, we get the following SBA algorithm:

$$\begin{cases} u_0^2(x,y,t) = (1+t)\,e^{xy} + \widetilde{C}(u^1(x,y,z)) = (1+t)\,e^{xy} \\ u_{n+1}^2(x,y,t) = (1+t)\,e^{xy} - \int_0^t \left(\int_0^s u_n^2(x,y,z)dz\right) ds, \forall n \geq 0 \end{cases} \tag{113}$$

From (113), we have

$$\begin{cases} u_0^2(x,y,t) = (1+t)\, e^{xy} \\ u_n^2(x,y,t) = \left[(-1)^n \frac{t^{2n}}{(2n)!} + (-1)^n \frac{t^{2n+1}}{(2n+1)!}\right] e^{xy}, \forall n \geq 1 \end{cases} \tag{114}$$

Let's put

$$\varphi_n^2(x,y,t) = \sum_{i=0}^{n} u_i^2(x,y,t) \tag{115}$$

We obtain

$$\varphi_n^2(x,y,t) = \left(\sum_{i=0}^{n} (-1)^i \frac{t^{2i}}{(2i)\,!} + \sum_{i=0}^{n} (-1)^i \frac{t^{2i+1}}{(2i+1)\,!}\right) e^{xy} \tag{116}$$

Thus, for $k = 2$, we have:

$$u^2(x,y,t) = \lim_{n \to +\infty} \varphi_n^2(x,y,t) = (\cos t + \sin t)\, e^{xy} \tag{117}$$

So

$$u^2(x,y,t) = (\cos t + \sin t)\, e^{xy} \tag{118}$$

Using the same procedure, for $k > 2$, we get :

$$\varphi_n^k(x,y,t) = \left(\sum_{i=0}^{n} (-1)^i \frac{t^{2i}}{(2i)\,!} + \sum_{i=0}^{n} (-1)^i \frac{t^{2i+1}}{(2i+1)\,!}\right) e^{xy} \tag{119}$$

and

$$u^k(x,y,t) = \lim_{n \to +\infty} \varphi_n^k(x,y,t) = (\cos t + \sin t)\, e^{xy} \tag{120}$$

From which, we obtain:

$$u(x,y,t) = \lim_{k \to +\infty} u^k(x,y,t) = (\cos t + \sin t)\, e^{xy} \tag{121}$$

This table compare obtained solutions :

Laplace-Adomian method	$u(x,y,t) = (\cos t + \sin t)\, e^{xy}$
SBA method	$u(x,y,t) = (\cos t + \sin t)\, e^{xy}$

(122)

Conclusion

The two solutions were similar.

4. CONCLUSION

Solving the same examples of wave-like equations by Laplace-Adomian and SBA methods, we get the same' exact solution.

Through these examples, we showed again usefulness and advantage of the SBA method comparatively with other methods that have been used for solving same Wave -Like equations. The results of the present study obtained by the SBA method help us to solve certain Wave-Like equations. However, we intend on the one hand to compare by simulation numerical results and experimental results, and on the other hand to explore precisely the boundary conditions (boundary surface and background neighborhood).

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