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Initial Coefficient Bounds And Fekete-Szegö Problem Of Pseudo-Bazilevič Functions Involving Quasi-Subordination

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Abstract

Using quasi-subordination, we have defined a λ-pseudo Bazilevič functions of order $\gamma + i\delta$. Initial Taylor-Maclaurin coefficient bounds and the Fekete-Szegö inequality have been obtained for the newly defined Bazilevič functions of order $\gamma + i\delta$. Special cases of our main results are presented in the form of corollaries.

Keywords: Analytic functions, starlike and convex functions, Bazilevič function, quasi-subordination, coefficient inequalities, Fekete-Szegö problem.

AMS subject classifications: 30C45, 30C80.

1. INTRODUCTION

Robertson [1] introduced quasi-subordination unifying the concept of subordination and majorization. For analytic functions f and g in $\mathbb{U}$, f is quasi-subordinate to g in $\mathbb{U}$, denoted by $f \prec_q g$, if there exist a Schwarz function w and an analytic function ϕ satisfying $|\phi(z)| < 1$ and $f(z) = \phi(z)g(w(z))$ in $\mathbb{U}$. If $\phi(z) = 1$, quasi-subordination reduces to subordination. If we let $w(z) = z$, then quasi-subordination reduces to the concept of majorization.

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Let $\mathcal{H}(\mathbb{U})$ be the class of functions which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$ and let $\mathcal{A} \subset \mathcal{H}(\mathbb{U})$ be the class of functions having a Taylor series expansion of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (z \in \mathbb{U} = \{z : |z| < 1\}). \tag{1.1}$$

For functions $f \in \mathcal{A}$ given by (1.1) and $h \in \mathcal{A}$ of the form

$$h(z) = z + \sum_{k=2}^{\infty} \Phi_k z^k, \tag{1.2}$$

the Hadamard product (or convolution) is defined by

$$\mathcal{R}(z) = (f * h)(z) = z + \sum_{k=2}^{\infty} a_k \Phi_k z^k. \tag{1.3}$$

Using Löwner-Kufarev differential equation, Bazilevič [2] constructed a class $\mathcal{B}(\gamma, \delta)$ of analytic and univalent functions in the unit disc, which is defined by the integral

$$f(z) = \left\{ \int_0^z g^{\gamma}(\zeta) h(\zeta) \zeta^{i\delta - 1} d\zeta \right\}^{\frac{1}{\gamma + i\delta}},$$

where $h \in \mathcal{P}$, the class of analytic function with positive real part and $g \in \mathcal{S}^*$, the well-known class of starlike univalent function. The numbers $\gamma > 0$ and δ are real and all powers are chosen so that it remains single-valued. Sheil-Small [3, Theorem 2] established that $f \in \mathcal{B}(\gamma, \delta)$ if and only if

$$Re \left[\left(\frac{z f'(z)}{f(z)} \right) \left(\frac{f(z)}{z} \right)^{\gamma + i\delta} \right] > 0.$$

Throughout this paper, let ψ be an analytic function such that $Re\,[\psi(z)] > 0$, $(z \in \mathbb{U})$ and ψ maps the open unit disc $\mathbb{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. Also, let ψ has a series expansion of the form

$$\psi(z) = 1 + A_1 z + A_2 z^2 + A_3 z^3 + \cdots \qquad (A_1 \neq 0;\ z \in \mathbb{U}). \tag{1.4}$$

Motivated by the Janowski class and a class introduced by Noor and Malik [4], Karthikeyan et.al. [5, Definition 1.1.] defined a class $\mathcal{PS}_{\lambda}^{t}(\alpha, \theta;\ b;\ \psi;\ h;\ A, B)$ of analytic functions which satisfies the condition

$$1 + \frac{(1 + i \tan\theta)}{b} \left[\frac{z^{1-t} [\mathcal{R}'(z)]^{\lambda}}{[(1-\alpha)\mathcal{R}(z) + \alpha z]^{1-t}} - 1 \right] \prec \frac{(A+1)\psi(z) - (A-1)}{(B+1)\psi(z) - (B-1)},$$

where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $\lambda \geq 1$, $0 \leq \alpha \leq 1$, $t \geq 0$, $b \in \mathbb{C} \setminus \{0\}$.

Motivated by the $\mathcal{PS}_{\lambda}^{t}(\alpha, \theta;\, b;\, \psi;\, h;\, A, B)$ and a recent study by Mundalia and Sivaprasad Kumar in [6](also see [7]), we now introduce the following the class of functions.

Definition 1.1. *For $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $\lambda \geq 1$, $\gamma \geq 0$, $\delta \in \mathbb{R}$, $b \in \mathbb{C} \setminus \{0\}$ and $\mathcal{R} = f * h$ defined as in (1.3), let $\mathcal{B}_{\theta}^{\lambda}(b;\, \alpha, \beta, \gamma, \delta;\, \psi)$ be the class of functions defined by*

$$\frac{(1 + i\tan\theta)}{b}\left[\mathcal{F}^{\lambda}(b;\, \alpha, \beta, \gamma, \delta;\, \psi) - 1\right] \prec_q \psi(z) - 1, \tag{1.5}$$

where

$$\mathcal{F}^{\lambda}(b;\, \alpha, \beta, \gamma, \delta;\, \psi) = \left(\frac{z[\alpha z\mathcal{R}''(z) + \mathcal{R}'(z)]^{\lambda}}{(1-\alpha)\mathcal{R}(z) + \alpha z\mathcal{R}'(z)}\right)\left(\beta\,\mathcal{R}'(z) + (1-\beta)\frac{\mathcal{R}(z)}{z}\right)^{\gamma + i\delta}.$$

Remark 1.1. *The left hand side of* (1.5) *was motivated by the so-called λ-pseudo starlike functions introduced and studied by Babalola in [8]. Recently, the so-called λ-pseudo-starlike functions of complex order was extensively studied by Karthikeyan et. al. in [5].*

Remark 1.2. *The class $\mathcal{B}_{\theta}^{\lambda}(b;\, \alpha, \beta, \gamma, \delta;\, \psi)$ is very closely related to well-known class studied by various authors. For example, if we let $\theta = 0$, $\lambda = \alpha = 1$, $\phi(z) = 1$, $\psi(z) = z(1-z)^{-1}$ and $h(z) = z + \sum_{n=2}^{\infty} z^n$, we get the well-known convex function of complex order introduced and studied by Wiatrowski in [9]. Similarly, if we let $\alpha = \theta = 0$, $\lambda = 1$, $\phi(z) = 1$, $\psi(z) = z(1-z)^{-1}$ and $h(z) = z + \sum_{n=2}^{\infty} z^n$, we get the class of starlike functions of complex order b introduced and studied by Nasr and Aouf in [10].*

2. PRELIMINARIES

In this section, we state the results that would be used to establish our main results which can be found in the standard text on univalent function theory.

Lemma 2.1. *[11, p. 56] If the function $f(z) \in \mathcal{A}$ given by (1.1) and $g(w)$ given by*

$$g(w) = w + \sum_{k=2}^{\infty} b_k w^k \tag{2.1}$$

are inverse functions, then for $k \geq 2$

$$b_k = \frac{(-1)^{k+1}}{k!}\begin{vmatrix} ka_2 & 1 & 0 & \cdots & 0 \\ 2ka_3 & (k+1)a_2 & 2 & \cdots & 0 \\ 3ka_4 & (2k+1)a_3 & (k+2)a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & (k-2) \\ (k-1)ka_k & [k(k-2)+1]\,a_{k-1} & [k(k-3)+2]\,a_{k-2} & \cdots & (2k-2)a_2 \end{vmatrix}. \tag{2.2}$$

Remark 2.1. *The elements in the determinant $|\Gamma_{ij}|$ in (2.2) are given by*

$$\Gamma_{ij} = \begin{cases} [(i-j+1)n+j-1]\, a_{i-j+2}, & \text{if } i+1 \geq j \\ 0, & \text{if } i+1 < j. \end{cases}$$

Lemma 2.2. *[12] If $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k \in \mathcal{P}$, then $|p_k| \leq 2$ for all $k \geq 1$, and the inequality is sharp.*

Lemma 2.3. *[13] Let $p(z) \in \mathcal{P}$ and also let μ be a complex number, then*

$$|p_2 - \mu p_1^2| \leq 2 \ \max\{1, |2\mu - 1|\}, \tag{2.3}$$

the result is sharp for functions given by

$$p(z) = p_2(z) = \frac{1+z^2}{1-z^2}, \qquad p(z) = p_1(z) = \frac{1+z}{1-z}.$$

3. COEFFICIENTS ESTIMATES FOR FUNCTIONS IN $\mathcal{B}_\theta^\lambda(b;\, \alpha,\, \beta,\, \gamma,\, \delta;\, \psi)$

Let $g = f^{-1}$ defined by $f^{-1}(f(z)) = z = f(f^{-1}(z))$ be inverse of f and

$$g(w) = f^{-1}(w) = w + \sum_{k=2}^{\infty} b_k w^k \quad (|w| < r_0;\ r_0 > \frac{1}{4}). \tag{3.1}$$

The class of all functions in $\mathcal{B}_\theta^\lambda(b;\, \alpha,\, \beta,\, \gamma,\, \delta;\, \psi)$ is not univalent, so the inverse is not guaranteed. However, there exist an inverse function in some small disk with center at $w = 0$ depending on the parameters involved. Let $\phi(z) = d_0 + d_1 z + d_2 z^2 + \cdots$ $(d_0 \neq 0)$ and $|d_0| \leq 1$.

3.1. Estimates Of The Inverse Coefficients

Theorem 3.1. *If the function $f(z)$ given by (1.1) and $g(w)$ given by (2.1) are inverse functions and if $f \in \mathcal{B}_\theta^\lambda(b;\, \alpha,\, \beta,\, \gamma,\, \delta;\, \psi)$ with $\psi(z) = 1 + A_1 z + A_2 z^2 + A_3 z^3 + \cdots$, $(A_1 \neq 0;\ z \in \mathbb{U})$, then the estimates of the inverse coefficients of f are*

$$|b_2| \leq \frac{|b||A_1|}{\sec\theta\, |(2\lambda - 1)(1+\alpha) + (\gamma + i\delta)(1+\beta)|\, |\phi_2|} \tag{3.2}$$

and

$$|b_3| \leq \frac{|A_1||b|}{\sec\theta\, |(\gamma + i\delta)(1+2\beta) + (3\lambda+1)(1+2\alpha)|\, |\Phi_3|} \left[\left|\frac{d_1}{d_0}\right| + \max\{1;\ |2\mu - 1|\}\right]. \tag{3.3}$$

with

$$\mu = \frac{1}{2}\left(1 - \frac{A_2}{A_1} + \frac{2bA_1 d_0\, [(\gamma+i\delta)(1+2\beta) + (3\lambda+1)(1+2\alpha)]\, \Phi_3}{\Phi_2^2 (1 + i\tan\theta)\, [(2\lambda-1)(1+\alpha) + (\gamma+i\delta)(1+\beta)]^2} + A_1 \frac{bd_0(\gamma+i\delta)(1+\beta)\, [(\gamma + i\delta - 1)(1+\beta) + 2(2\lambda-1)(1+\alpha)] + 4\lambda(\lambda-2)(1+\alpha)}{2(1+i\tan\theta)\, [(2\lambda-1)(1+\alpha) + (\gamma+i\delta)(1+\beta)]^2}\right). \tag{3.4}$$

Proof. Let $f \in \mathcal{B}_\theta^\lambda(b;\, \alpha,\, \beta,\, \gamma,\, \delta;\, \psi)$. Then by the definition of quasi-subordination, there is a function $w(z)$ such that

$$\frac{1+i\tan\theta}{b}\left[\left(\frac{z[\alpha z R''(z)+R'(z)]^\lambda}{(1-\alpha)R(z)+\alpha z R'(z)}\right)\left(\beta R'(z)+(1-\beta)\frac{R(z)}{z}\right)^{\gamma+i\delta}-1\right] = \phi(z)\left[\psi(w(z))-1\right].$$

The left hand side of the above expression is given by

$$\begin{aligned}
&\frac{1+i\tan\theta}{b}\left[\left(\frac{z[\alpha z R''(z)+R'(z)]^\lambda}{(1-\alpha)R(z)+\alpha z R'(z)}\right)\left(\beta R'(z)+(1-\beta)\frac{R(z)}{z}\right)^{\gamma+i\delta}-1\right]\\
&= \frac{1+i\tan\theta}{b}\Bigg(\left[(2\lambda-1)(1+\alpha)+(\gamma+i\delta)(1+\beta)\right]\Phi_2 a_2 z\\
&+\Bigg\{\left[(\gamma+i\delta)(1+2\beta)+(3\lambda+1)(1+2\alpha)\right]\Phi_3 a_3+\Bigg[\frac{(\gamma+i\delta)(1+\beta)}{2}\\
&[(\gamma+i\delta-1)(1+\beta)+2(2\lambda-1)(1+\alpha)]+2\lambda(\lambda-2)(1+\alpha)\Bigg]\Phi_2^2 a_2^2\Bigg\}z^2+\cdots.\Bigg)
\end{aligned} \tag{3.5}$$

where Φ_k's are the corresponding coefficients from the power series expansion of h, which may be real or complex. Define the function p by

$$p(z) = 1+p_1z+p_2z^2+\cdots = \frac{1+w(z)}{1-w(z)} \prec \frac{1+z}{1-z} \quad (z\in\mathbb{U}). \tag{3.6}$$

We can note that $p(0) = 1$ and $p \in \mathcal{P}$ (see Lemma 2.2). Using (3.6), it is easy to see that

$$w(z) = \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left[p_1z+\left(p_2-\frac{p_1^2}{2}\right)z^2+\left(p_3-p_1p_2+\frac{p_1^3}{4}\right)z^3+\cdots\right].$$

So we have

$$\begin{aligned}
\phi(z)\left[\psi(w(z))-1\right] &= 1+\frac{1}{2}A_1d_0p_1z\\
&+\left[d_0\left(\frac{1}{2}A_1\left(p_2-\frac{p_1^2}{2}\right)+\frac{1}{4}A_2p_1^2\right)+\frac{d_1A_1p_1}{2}\right]z^2+\cdots.
\end{aligned} \tag{3.7}$$

By using (3.1) and (3.7), we have

$$a_2 = \frac{bA_1d_0p_1}{2\Phi_2(1+i\tan\theta)\left[(2\lambda-1)(1+\alpha)+(\gamma+i\delta)(1+\beta)\right]}, \tag{3.8}$$

$$a_3 = \frac{A_1 d_0 b}{2(1+i\tan\theta)\left[(\gamma+i\delta)(1+2\beta)+(3\lambda+1)(1+2\alpha)\right]\Phi_3}\left[p_2 - \frac{1}{2}\left(1-\frac{A_2}{A_1}\right.\right.$$
$$\left.+A_1\frac{(\gamma+i\delta)(1+\beta)\left[(\gamma+i\delta-1)(1+\beta)+2(2\lambda-1)(1+\alpha)\right]+4\lambda(\lambda-2)(1+\alpha)}{2(1+i\tan\theta)\left[(2\lambda-1)(1+\alpha)+(\gamma+i\delta)(1+\beta)\right]^2}\right)p_1^2 \tag{3.9}$$
$$\left.+\frac{d_1 p_1}{d_0}\right].$$

From (2.2), we see that $b_2 = -a_2$. Hence, applying Lemma 2.3 in (3.8), we have (3.2). Also from (2.2), we have

$$b_3 = \frac{(-1)^4}{3!}\begin{vmatrix} 3a_2 & 1 \\ 6a_3 & 4a_2 \end{vmatrix} = 2a_2^2 - a_3 = \frac{b^2 A_1^2 d_0^2 p_1^2}{2\Phi_2^2(1+i\tan\theta)^2\left[(2\lambda-1)(1+\alpha)+(\gamma+i\delta)(1+\beta)\right]^2}$$
$$-\frac{A_1 d_0 b}{2(1+i\tan\theta)\left[(\gamma+i\delta)(1+2\beta)+(3\lambda+1)(1+2\alpha)\right]\Phi_3}\left[p_2 - \frac{1}{2}\left(1-\frac{A_2}{A_1}\right.\right.$$
$$\left.\left.+A_1\frac{bd_0(\gamma+i\delta)(1+\beta)\left[(\gamma+i\delta-1)(1+\beta)+2(2\lambda-1)(1+\alpha)\right]+4\lambda(\lambda-2)(1+\alpha)}{2(1+i\tan\theta)\left[(2\lambda-1)(1+\alpha)+(\gamma+i\delta)(1+\beta)\right]^2}\right)p_1^2 + \frac{d_1p_1}{d_0}\right]$$

$$= -\frac{A_1 d_0 b}{2(1+i\tan\theta)\left[(\gamma+i\delta)(1+2\beta)+(3\lambda+1)(1+2\alpha)\right]\Phi_3}\left[p_2 - \frac{1}{2}\left(1-\frac{A_2}{A_1}\right.\right.$$
$$+A_1\frac{bd_0(\gamma+i\delta)(1+\beta)\left[(\gamma+i\delta-1)(1+\beta)+2(2\lambda-1)(1+\alpha)\right]+4\lambda(\lambda-2)(1+\alpha)}{2(1+i\tan\theta)\left[(2\lambda-1)(1+\alpha)+(\gamma+i\delta)(1+\beta)\right]^2}$$
$$\left.\left.+\frac{2bA_1d_0\left[(\gamma+i\delta)(1+2\beta)+(3\lambda+1)(1+2\alpha)\right]\Phi_3}{\Phi_2^2(1+i\tan\theta)\left[(2\lambda-1)(1+\alpha)+(\gamma+i\delta)(1+\beta)\right]^2}\right)p_1^2 + \frac{d_1p_1}{d_0}\right].$$

Applying Lemma 2.3 to the above expression, we can establish the assertion of the Theorem 3.1.

Corollary 3.2. *If the function $f(z)$ given by (1.1) and $g(w)$ given by (2.1) are inverse functions and if $f \in \mathcal{B}(\gamma, \delta)$, then the estimates of the inverse coefficients of f are*

$$|b_2| \leq \frac{2}{\sqrt{(1+\gamma)^2+\delta^2}}$$

and

$$|b_3| \leq \frac{2}{\sqrt{(4+\gamma)^2+\delta^2}}\max\left\{1; \left|\frac{(\gamma+i\delta)(\gamma+i\delta-1)-2}{(2+\gamma+i\delta)^2} - \frac{2(\gamma+i\delta+4)}{(1+\gamma+i\delta)^2} - 1\right|\right\}.$$

Remark 3.1. *The impact of the well-known Janowski function on*

$$\kappa(z) = 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, \qquad (z \in \mathbb{U}) \tag{3.10}$$

was recently studied by Malik et. al. [14]

Theorem 3.3. *Suppose that $f \in \mathcal{B}_\theta^\lambda(b;\, \alpha,\, \beta,\, \gamma,\, \delta;\, \psi)$ with $\psi(z)$ of the form*

$$\psi(z) = \frac{(A+1)\kappa(z) + (A-1)}{(B+1)\kappa(z) + (B-1)},$$

where $-1 \leq B < A \leq 1$ and $\kappa(z)$ is defined as in(3.10), then

$$|b_2| \leq \frac{4|b|(A-B)}{\pi^2 \sec\theta \,|(2\lambda-1)(1+\alpha) + (\gamma+i\delta)(1+\beta)|\,|\phi_2|}$$

and

$$|b_3| \leq \frac{4(A-B)|b|}{\pi^2 \sec\theta \,|(\gamma+i\delta)(1+2\beta) + (3\lambda+1)(1+2\alpha)|\,|\Phi_3|} \left[\left|\frac{d_1}{d_0}\right| + \max\{1;\, |\kappa|\}\right]$$

with

$$\kappa = \left(\frac{4(B+1)}{\pi^2} - \frac{2}{3}\right) + \left(\frac{4(A-B)}{\pi^2}\right)\left(\frac{2bd_0\,[(\gamma+i\delta)(1+2\beta) + (3\lambda+1)(1+2\alpha)]\,\Phi_3}{\Phi_2^2(1+i\tan\theta)\,[(2\lambda-1)(1+\alpha) + (\gamma+i\delta)(1+\beta)]^2}\right.$$
$$\left. + \frac{bd_0(\gamma+i\delta)(1+\beta)\,[(\gamma+i\delta-1)(1+\beta) + 2(2\lambda-1)(1+\alpha)] + 4\lambda(\lambda-2)(1+\alpha)}{2(1+i\tan\theta)\,[(2\lambda-1)(1+\alpha) + (\gamma+i\delta)(1+\beta)]^2}\right).$$

Proof. Following the steps as in Theorem 1 of [15], we get

$$\psi(z) = 1 + \frac{4(A-B)}{\pi^2}z + \frac{8(A-B)}{3\pi^2}\left[1 - \frac{6(B+1)}{\pi^2}\right]z^2 + \cdots\,. \tag{3.11}$$

Now replacing $A_1,\, A_2$ and A_3 in Theorem 3.1 with the corresponding coefficients of the series given in (3.11), we have the assertion of the Theorem.

If we let $h(z) = z + \sum_{k=2}^{\infty} z^k$, $\phi(z) = 1$, $\lambda = \alpha = 1$, $\gamma + i\delta = 0$, $b = 1 + 0i$ and $\theta = 0$ in Theorem 3.1, we have

Corollary 3.4. *[14, Theorem 4] Suppose that $f \in UP[A,\, B]$, $-1 \leq B < A \leq 1$, then*

$$|b_2| \leq \frac{2(A-B)}{\pi^2},$$

and

$$|b_3| \leq \frac{4(A-B)}{6\pi^2}.$$

3.2. Fekete-Szegö Problem

The Fekete-Szegö problem which is related to the Bieberbach conjecture represents various geometric quantities.

Theorem 3.5. *Suppose* $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{B}_\theta^\lambda(b;\, \alpha,\, \beta,\, \gamma,\, \delta;\, \psi)\ (z \in \mathbb{U})$. *Then, for any* $\rho \in \mathbb{C}$

$$\left|a_3 - \rho a_2^2\right| \leq \frac{|A_1||b|}{\sec\theta\, |(\gamma + i\delta)(1+2\beta) + (3\lambda+1)(1+2\alpha)|\, |\Phi_3|} \left[\left|\frac{d_1}{d_0}\right| + \max\{1;\, |2\nu - 1|\}\right], \tag{3.12}$$

where ν *is given by*

$$\nu = \frac{1}{2}\left[1 - \frac{A_2}{A_1} + A_1\left(\mathcal{M}_1 + \mathcal{M}_2\right)\right]. \tag{3.13}$$

with

$$\mathcal{M}_1 = \frac{(\gamma + i\delta)(1+\beta)\left[(\gamma + i\delta - 1)(1+\beta) + 2(2\lambda - 1)(1+\alpha)\right] + 4\lambda(\lambda - 2)(1+\alpha)}{2(1 + i\tan\theta)\left[(2\lambda - 1)(1+\alpha) + (\gamma + i\delta)(1+\beta)\right]^2}$$

and

$$\mathcal{M}_2 = \frac{\rho b A_1 d_0 \left[(\gamma + i\delta)(1+2\beta) + (3\lambda+1)(1+2\alpha)\right]\Phi_3}{\Phi_2^2(1 + i\tan\theta)\left[(2\lambda - 1)(1+\alpha) + (\gamma + i\delta)(1+\beta)\right]^2}.$$

The inequalities are sharp for each ρ.

Proof. Let $f \in \mathcal{B}_\theta^\lambda(b;\, \alpha, \beta, \gamma, \delta;\, \psi)$, then in view of the equations (3.8) and (3.9), for $\mu \in \mathbb{C}$ we have
$|a_3 - \rho a_2^2| =$

$$\left|\frac{A_1 d_0 b}{2(1 + i\tan\theta)\left[(\gamma + i\delta)(1+2\beta) + (3\lambda+1)(1+2\alpha)\right]\Phi_3}\left[p_2 - \frac{p_1^2}{2} + \frac{1}{2}p_1^2\left(\frac{A_2}{A_1} - A_1\frac{(\gamma + i\delta)(1+\beta)\left[(\gamma + i\delta - 1)(1+\beta) + 2(2\lambda - 1)(1+\alpha)\right] + 4\lambda(\lambda - 2)(1+\alpha)}{2(1 + i\tan\theta)\left[(2\lambda - 1)(1+\alpha) + (\gamma + i\delta)(1+\beta)\right]^2} - \frac{\rho b A_1 d_0\left[(\gamma + i\delta)(1+2\beta) + (3\lambda+1)(1+2\alpha)\right]\Phi_3}{\Phi_2^2(1 + i\tan\theta)\left[(2\lambda - 1)(1+\alpha) + (\gamma + i\delta)(1+\beta)\right]^2}\right) + \frac{d_1 p_1}{d_0}\right]\right|. \tag{3.14}$$

Using Lemma 2.2 in (3.14), we have
$|a_3 - \rho a_2^2| \leq$

$$\frac{|A_1||b|}{2\sec\theta\, |(\gamma + i\delta)(1+2\beta) + (3\lambda+1)(1+2\alpha)|\, |\Phi_3|}\left[2 + 2\left|\frac{d_1}{d_0}\right| + \frac{1}{2}|p_1|^2\left(\left|\frac{A_2}{A_1} - A_1\frac{(\gamma + i\delta)(1+\beta)\left[(\gamma + i\delta - 1)(1+\beta) + 2(2\lambda - 1)(1+\alpha)\right] + 4\lambda(\lambda - 2)(1+\alpha)}{2(1 + i\tan\theta)\left[(2\lambda - 1)(1+\alpha) + (\gamma + i\delta)(1+\beta)\right]^2} - \frac{\rho b A_1 d_0\left[(\gamma + i\delta)(1+2\beta) + (3\lambda+1)(1+2\alpha)\right]\Phi_3}{\Phi_2^2(1 + i\tan\theta)\left[(2\lambda - 1)(1+\alpha) + (\gamma + i\delta)(1+\beta)\right]^2}\right| - 1\right)\right] \tag{3.15}$$

Now if $\left|\frac{A_2}{A_1} - A_1\mathcal{M}_1 - A_1\mathcal{M}_2\right| \leq 1$ in (3.14), then

$$\left|a_3 - \rho a_2^2\right| \leq \frac{|A_1||b|}{\sec\theta\,|(\gamma+i\delta)(1+2\beta)+(3\lambda+1)(1+2\alpha)|\,|\Phi_3|}\left[1+\left|\frac{d_1}{d_0}\right|\right]. \quad (3.16)$$

Further, if $\left|\frac{A_2}{A_1} - A_1\mathcal{M}_1 - A_1\mathcal{M}_2\right| \geq 1$ in (3.14), then

$$\left|a_3 - \rho a_2^2\right| \leq \frac{|A_1||b|}{\sec\theta\,|(\gamma+i\delta)(1+2\beta)+(3\lambda+1)(1+2\alpha)|\,|\Phi_3|} \left(\left|\frac{d_1}{d_0}\right| + \left|\frac{A_2}{A_1} - A_1\mathcal{M}_1 - A_1\mathcal{M}_2\right|\right). \quad (3.17)$$

An examination of the proof shows equality for (3.16) holds if $p_1 = 0,\, p_2 = 2$. Equivalently, we have $p(z) = p_2(z) = \frac{1+z^2}{1-z^2}$ by Lemma 2.3. Therefore, the extremal function in $\mathcal{B}_\theta^\lambda(b;\,\alpha,\,\beta,\,\gamma,\,\delta;\,\psi)$ is given by

$$\frac{1+i\tan\theta}{b}\left[\left(\frac{z[\alpha zR''(z)+R'(z)]^\lambda}{(1-\alpha)R(z)+\alpha zR'(z)}\right)\left(\beta R'(z)+(1-\beta)\frac{R(z)}{z}\right)^{\gamma+i\delta} - 1\right] = \phi(z)\left[p_2(z)-1\right].$$

Similarly, equality for (3.17) holds if $p_2 = 2$. Equivalently, we have $p(z) = p_1(z) = \frac{1+z}{1-z}$ by Lemma 2.3. Therefore, the extremal function in $\mathcal{B}_\theta^\lambda(b;\,\alpha,\,\beta,\,\gamma,\,\delta;\,\psi)\,(z \in \mathbb{U})$ is given by

$$\frac{1+i\tan\theta}{b}\left[\left(\frac{z[\alpha zR''(z)+R'(z)]^\lambda}{(1-\alpha)R(z)+\alpha zR'(z)}\right)\left(\beta R'(z)+(1-\beta)\frac{R(z)}{z}\right)^{\gamma+i\delta} - 1\right] = \phi(z)\left[p_1(z)-1\right].$$

If we $\theta = \gamma = \delta = 0$, $\Phi_k = 1$, $\alpha = 0$, $\lambda = 1$, $\phi(z) = 1$ and $b = 1$ in Theorem 3.5, we have the following result.

Corollary 3.6. *[16] Let $0 \leq \eta < 1 < \theta$ and let the function $f \in \mathcal{A}$ satisfies the condition*

$$\frac{zf'(z)}{f(z)} \prec \psi(z) = 1 + \frac{\theta-\eta}{\pi} i \log\left(\frac{1-e^{2\pi i((1-\eta)/(\theta-\eta))}\,z}{1-z}\right).$$

Then, for a complex number μ,

$$\left|a_3 - \mu a_2^2\right| \leq \frac{\theta-\eta}{\pi}\sin\left(\frac{\pi(1-\eta)}{\theta-\eta}\right) \max\left\{1;\, \left|\frac{1}{2} + (1-2\mu)\frac{\theta-\eta}{\pi}i + \left(\frac{1}{2} - (1-2\mu)\frac{\theta-\eta}{\pi}i\right)e^{2\pi i\frac{1-\eta}{\theta-\eta}}\right|\right\}.$$

If we $\theta = \gamma = \delta = 0$, $\Phi_k = 1$, $\alpha = 0$, $\lambda = 1$, $\phi(z) = 1$ and $b = 1$ in Theorem 3.5, we have

Corollary 3.7. *[17, Theorem 3.1] Suppose $f(z) \in \mathcal{A}$ satisfies the condition*

$$\frac{zf'(z)}{f(z)} \prec \psi(z),$$

where $\prec$ denotes the subordination and ψ is defined as in (1.4). *Then*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{A_1}{2} \max\left\{1; \left|A_1 + \frac{A_2}{A_1} - 2\mu A_1\right|\right\}, \quad (\mu \in \mathbb{C}).$$

The inequality is sharp for the function given by

$$f(z) = \begin{cases} z \exp \int_0^z \left[\psi(t) - 1\right] \frac{1}{t} dt, & \textit{if } \left|A_1 + \frac{A_2}{A_1} - 2\mu A_1\right| \geq 1 \\ z \exp \int_0^z \left[\psi(t^2) - 1\right] \frac{1}{t} dt, & \textit{if } \left|A_1 + \frac{A_2}{A_1} - 2\mu A_1\right| \leq 1. \end{cases}$$

3.3. Conclusion

By defining λ-pseudo Bazilevič functions of order $\gamma + i\delta$ using quasi-subordination and Hadamard product, we were able to unify and extend the various classes of analytic function. New extensions were discussed in detail. Theorem 3.1 and Theorem 3.5 have many applications, here we pointed out only few.

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