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Initial Coefficient Estimates for Certain Subclasses of m-fold Symmetric bi-univalent Functions

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Abstract

This paper provides the two new subclasses of the function class $\mathcal{S}_{\Sigma_m}(\alpha, \tau, \lambda)$ and $\mathcal{S}_{\Sigma_m}(\beta, \tau, \lambda)$ of analytic and bi-univalent functions defined in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$. Besides, Find estimates on the coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in these new subclasses. Many interesting new and already existing corollaries are also presented.

Key words and phrases: m-Fold symmetry, bi-univalent functions, coefficient estimates

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1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

which are univalent in $\mathbb{U}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S}$ subclass class of function of $f \in \mathcal{A}$ consisting of the form (1.1) which are also univalent in $\mathbb{U}$.

The Koebe one-quarter theorem [8] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$. Thus every univalent function f has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z$,$(z \in \mathbb{U})$ and

$$f\left(f^{-1}(w)\right) = w, \left(|w| < r_0(f), r_0(f) \geq \frac{1}{4}\right)$$

where

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \quad (1.2)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). Lewin [12] investigated the class Σ of bi-univalent functions and showed that $|a_2| < 1.51$ for the functions belonging to Σ. Subsequently, Brannan and Clunie [5] conjectured that $|a_2| \leq \sqrt{2}$. An analytic function f is subordinate to an analytic function g,written $f(z) \prec g(z)$, provided there is a schwarz function w defined on $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$. Ma and Minda [13], unified various subclasses of starlike and convex functions for which either of the quantity $\frac{zf'(z)}{f(z)}$ or $1 + \frac{zf''(z)}{f'(z)}$ is subordinate to a more general superordinate function.

In recent years, the study of bi-univalent functions has gained momentum mainly due to the work of Srivastava et al. [15], which has apparently revived the subject. Motivated by their work [15], many researchers (see, for example, [1, 2, 5, 9, 10, 11, 12]); see also the various closely-related papers on the subject, which are cited in some of these works) have recently investigated several interesting subclasses of the bi-univalent function class Σ and found non-sharp estimates on the first two Taylor-Maclaurin coefficients of functions belonging to these subclasses.

Let $m \in \mathbb{N} = 1, 2, 3, ...$ A domain D is said to be m-fold symmetric if a relation of D about the origin through an angle $\frac{2\pi}{n}$ carries D on itself. It fowwos that, a function $f(z)$ analytic in $\mathbb{U}$ is said to be m-fold symmetric $(m \in \mathbb{N})$ if

$$f(e^{\frac{2\pi i}{m}} z) = e^{\frac{2\pi i}{m}} f(z)$$

.

In Particular, every $f(z)$ is 1-fold symmatric and odd $f(z)$ is 2-fold symmtric. We denote by $\mathcal{S}_m$ the class of m-fold symmtric univalent functions in $\mathbb{U}$ if it has the following normalized form

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}, \qquad (z \in \mathbb{U}, m \in \mathbb{N}) \quad (1.3)$$

Analogous to the concept of m-fold symmetric univalent functions, we here introduced the concept of m-fold symmetric bi-univalent functions. Each function $f \in \Sigma$ generates an m-fold symmetric bi-univalent function for each integer $m \in \mathbb{N}$. The normalized form of f is given as in (1.3) and the series expansion for f^{-1} is given as follows

$$
\begin{aligned}
g(w) = f^{-1}(w) &= w - a_{m+1}w^{m+1} + [(m+1)a_{m+1}^2 - a_{2m+1}]w^{2m+1} \\
-[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 &- (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}]w^{3m+1} + \cdots .
\end{aligned} \tag{1.4}
$$

where $f^{-1} = g$. We denote by Σ_m the class of m-fold symmetric bi-univalent functions in $\mathbb{U}$. For m=1, the formula (1.4) coincides with the formula (1.2) of the class Σ.

Some examples of m-fold symmetric bi-univalent functions are given as follows

$$
\left(\frac{z^m}{1-z^m}\right)^{\frac{1}{m}}, \quad \left[\frac{1}{2}log\left(\frac{1+z^m}{1-zm}\right)\right]^{\frac{1}{m}} \quad and \quad \left[-log\left(1-z^m\right)\right]^{\frac{1}{m}}
$$

with the corresponding inverse functions

$$
\left(\frac{w^m}{1+w^m}\right)^{\frac{1}{m}}, \quad \left[\frac{e^{2w^m}-1}{e^{2w^m}+1}\right]^{\frac{1}{m}} \quad and \quad \left(\frac{e^{w^m}-1}{e^{w^m}}\right)^{\frac{1}{m}}
$$

respectively. Recently, many authors investigated bounds for various subclasses of m-fold bi-univalent functions (see [3, 15, 16, 17, 18, 19, 20]).

The aim of the present paper is to introdues the certain subclasses $\mathcal{S}_{\Sigma_m}(\alpha, \tau, \lambda)$ and $\mathcal{S}_{\Sigma_m}(\beta, \tau, \lambda)$. Derive the estimates on initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in these subclasses.

1.1. The class $\mathcal{S}_{\Sigma_m}(\alpha, \tau, \lambda)$

Definition 1.1. For $\tau \in \mathbb{C}\backslash\{0\}, 0 \leq \lambda \leq 1, 0 < \alpha \leq 1, m \in \mathbb{N}$, a function $f \in \Sigma_m$ is said to be in class $\mathcal{S}_{\Sigma_m}(\alpha, \tau, \lambda)$ if the following conditions are satisfied

$$
\left|arg\left[1 + \frac{1}{\tau}\left(\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} - 1\right)\right]\right| < \frac{\alpha\pi}{2} \tag{1.5}
$$

and

$$
\left|arg\left[1 + \frac{1}{\tau}\left(\frac{zg'(z) + \lambda z^2 g''(z)}{(1-\lambda)g(z) + \lambda z g'(z)} - 1\right)\right]\right| < \frac{\alpha\pi}{2} \tag{1.6}
$$

where function $g = f^{-1}$.

Remark 1.2. On specializing the parameter τ, λ, m one can state the various new as well as known subclasses of analytic bi-univalent functions studied earlier in the literature.

(i) For $m = 1$, we obtain new class of bi-univalent function.

$$
\mathcal{S}_{\Sigma_m}(\alpha, \tau, \lambda) = \mathcal{S}_{\Sigma}(\alpha, \tau, \lambda).
$$

(ii) For $\lambda = 0$, we obtain new class which consists m-fold symmetric bi starlike function.

$$\mathcal{S}_{\Sigma_m}(\alpha, \tau, \lambda) = \mathcal{S}^*_{\Sigma_m}(\alpha, \tau).$$

(iii) For $\lambda = 1$, we obtain new class which consists m-fold symmetric convex bi univalent function.

$$\mathcal{S}_{\Sigma_m}(\alpha, \tau, \lambda) = \mathcal{C}_{\Sigma_m}(\alpha, \tau).$$

(iv) For $\lambda = 0, \tau = 1$, we obtain class which consists m-fold symmetric bi-univalent function by S. Altinkaya, S. Yalcin [3].

$$\mathcal{S}_{\Sigma_m}(\alpha, \tau, \lambda) = \delta^{\alpha}_{\Sigma,m}$$

(v) For $\lambda = 0, m = 1, \tau = 1$, we obtain class of bi-univalent function introduced by Brannan and Taha [7].

$$\mathcal{S}_{\Sigma_m}(\alpha, \tau, \lambda) = \delta^*_{\Sigma}(\alpha).$$

(vi) For $\lambda = 1, \tau = 1$, we obtain class which consists m-fold symmetric convex bi univalent function by A. K. Wanas and A. H. Majeed [20].

$$\mathcal{S}_{\Sigma_m}(\alpha, \tau, \lambda) = E_{\Sigma_m}(0, 1, 1, \alpha).$$

(vii) For $\lambda = 1, m = 1, \tau = 1$, we obtain class which consists convex bi univalent function introduced by Brannan and Taha [7].

$$\mathcal{S}_{\Sigma_m}(\alpha, \tau, \lambda) = \delta_{\Sigma_1}(\alpha).$$

1.2. The class $\mathcal{S}_{\Sigma_m}(\beta, \tau, \lambda)$

Definition 1.3. For $\tau \in \mathbb{C}\backslash\{0\}, 0 \leq \lambda \leq 1, 0 < \beta \leq 1, m \in \mathbb{N}$, a function $f \in \Sigma_m$ is said to be in class $\mathcal{S}_{\Sigma_m}(\beta, \tau, \lambda)$ if the following conditions are satisfied

$$\mathcal{R}\left[1 + \frac{1}{\tau}\left(\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda) f(z) + \lambda z f'(z)} - 1\right)\right] > \beta \tag{1.7}$$

and

$$\mathcal{R}\left[1 + \frac{1}{\tau}\left(\frac{zg'(w) + \lambda z^2 g''(w)}{(1-\lambda) g(w) + \lambda z g'(w)} - 1\right)\right] > \beta \tag{1.8}$$

where function $g = f^{-1}$.

Remark 1.4. On specializing the parameter τ, λ, m one can state the various new as well as known subclasses of analytic bi-univalent functions studied earlier in the literature.

(i) For $m = 1$, we obtain new class of bi-univalent function.

$$\mathcal{S}_{\Sigma_m}(\beta, \tau, \lambda) = \mathcal{S}_{\Sigma}(\beta, \tau, \lambda).$$

(ii) For $\lambda = 0$, we obtain new class which consists m-fold symmetric bi starlike function.

$$\mathcal{S}_{\Sigma_m}(\beta, \tau, \lambda) = \mathcal{S}^*_{\Sigma_m}(\beta, \tau).$$

(iii) For $\lambda = 1$, we obtain new class which consists m-fold symmetric convex bi univalent function.

$$\mathcal{S}_{\Sigma_m}(\beta, \tau, \lambda) = \mathcal{C}_{\Sigma_m}(\beta, \tau).$$

(iv) For $\lambda = 0, \tau = 1$, we obtain class which consists m-fold symmetric bi-univalent function by S. Altinkaya, S. Yalcin [3].

$$\mathcal{S}_{\Sigma_m}(\beta, \tau, \lambda) = \mathbb{N}^0_{\Sigma,m}(\beta, 1).$$

(v) For $\lambda = 0, m = 1, \tau = 1$, we obtain class of bi-univalent function introduced by Brannan and Taha [7].

$$\mathcal{S}_{\Sigma_m}(\beta, \tau, \lambda) = \delta^*_{\Sigma}(\beta).$$

(vi) For $\lambda = 1, \tau = 1$, we obtain class which consists m-fold symmetric convex bi univalent function by A. K. Wanas and A. H. Majeed [20].

$$\mathcal{S}_{\Sigma_m}(\beta, \tau, \lambda) = E^*_{\Sigma_m}(0, 1, 1, \beta).$$

(vii) For $\lambda = 1, m = 1, \tau = 1$, we obtain class which consists convex bi univalent function introduced by Brannan and Taha [7].

$$\mathcal{S}_{\Sigma_m}(\beta, \tau, \lambda) = \delta_{\Sigma_1}(\beta).$$

In order to prove our main results, we required the following lemma.

Lemma 1.5. *(see [8]) If $\mathcal{P}(z) = 1 + p_1 z + p_2 z^2 + p_2 z^2 + \cdots$ is an analytic function in $\mathbb{U}$ with positive real part, then*

$$|p_n| \leq 2 \quad (n \in \mathbb{N} = 1, 2, 3, \cdots)$$

2. COEFFICIENT ESTIMATES

Theorem 2.1. *If $f \in \mathcal{S}_{\Sigma_m}(\alpha, \tau, \lambda) \quad (\tau \in \mathbb{C}\backslash\{0\}, 0 \leq \lambda \leq 1, 0 < \alpha \leq 1, m \in \mathbb{N})$, then*

$$|a_{m+1}| \leq \frac{2\alpha|\tau|}{\sqrt{2m\alpha\tau\left[(m+1)(1+2\lambda m) - (1+\lambda m)^2\right] + m^2(1-\alpha)(1+\lambda m)^2}} \tag{2.9}$$

and

$$|a_{2m+1}| \leq \frac{\alpha\tau}{m(1+2\lambda m)} + \frac{2\alpha^2\tau^2(m+1)}{m^2(1+\lambda m)^2}. \tag{2.10}$$

Proof. Let $f \in \mathcal{S}_{\Sigma_m}(\tau, \lambda, \alpha)$. Then

$$1 + \frac{1}{\tau}\left(\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda) f(z) + \lambda z f'(z)} - 1\right) = [p(z)]^\alpha \tag{2.11}$$

and

$$1 + \frac{1}{\tau}\left(\frac{zg'(z) + \lambda z^2 g''(z)}{(1-\lambda) g(z) + \lambda z g'(z)} - 1\right) = [q(w)]^\alpha \tag{2.12}$$

where $p(z)$ and $q(z)$ are in familiar Caratheodory class $\mathcal{P}$ and following series expansions:

$$p(z) = 1 + p_m z^m + p_{2m} z^{2m} + p_{3m} z^{3m} + \cdots \tag{2.13}$$

and

$$q(w) = 1 + q_m w^m + q_{2m} w^{2m} + q_{3m} w^{3m} + \cdots \tag{2.14}$$

Now, equating the coefficients of (2.11) and (2.12), we get

$$\frac{m}{\tau}(1 + m\lambda) a_{m+1} = \alpha p_m \tag{2.15}$$

$$\frac{m}{\tau}\left[2(1 + 2m\lambda) a_{2m+1} - (1 + m\lambda)^2 a_{m+1}^2\right] = \alpha p_{2m} + \frac{\alpha(\alpha-1)}{2} p_m^2 \tag{2.16}$$

and

$$-\frac{m}{\tau}(1 + m\lambda) a_{m+1} = \alpha q_m \tag{2.17}$$

$$\frac{m}{\tau}\left[\left\{2(m+1)(1 + 2m\lambda) - (1 + m\lambda)^2\right\} a_{m+1}^2 - 2(1 + 2m\lambda) a_{2m+1}\right] = \alpha q_{2m} + \frac{\alpha(\alpha-1)}{2} q_m^2 \tag{2.18}$$

Now considering (2.15) and (2.17), we get

$$p_m = -q_m \tag{2.19}$$

and

$$\frac{2m^2}{\tau^2}(1 + m\lambda)^2 a_{m+1}^2 = \alpha^2\left(p_m^2 + q_m^2\right) \tag{2.20}$$

Now from (2.16), (2.18) and (2.20) we get

$$a_{m+1}^2 = \frac{\alpha^2 \tau^2 (p_{2m} + q_{2m})}{\left[2m\tau\alpha\left\{(m+1)(1 + 2m\lambda) - (1 + m\lambda)^2\right\} + m^2(1-\alpha)(1 + m\lambda)^2\right]} \tag{2.21}$$

Now, taking absolute value of (2.21) and applying lemma 1.1 for the coefficients p_{2m} and q_{2m}, we obtain

$$|a_{m+1}| \leq \frac{2\alpha|\tau|}{\sqrt{\left[2m\tau\alpha\left\{(m+1)(1 + 2m\lambda) - (1 + m\lambda)^2\right\} + m^2(1-\alpha)(1 + m\lambda)^2\right]}} \tag{2.22}$$

This gives the desired estimate for $|a_{m+1}|$ as asserted in (2.9). In order to find the bound on $|a_{2m+1}|$, by subtracting (2.18) from (2.16), we get

$$\frac{m}{\tau}\left[4\left(1+2m\lambda\right)a_{2m+1}-2\left(m+1\right)\left(1+2m\lambda\right)a_{m+1}^2\right]=\alpha\left(p_{2m}-q_{2m}\right)+\frac{\alpha\left(\alpha-1\right)}{2}\left(p_m^2-q_m^2\right) \tag{2.23}$$

It follows from (2.19), (2.20) and (2.23)

$$a_{2m+1}=\frac{\alpha\tau\left(p_{2m}-q_{2m}\right)}{4m\left(1+2m\lambda\right)}+\frac{\alpha^2\tau^2\left(m+1\right)\left(p_m^2+q_m^2\right)}{4m^2\left(1+m\lambda\right)^2} \tag{2.24}$$

Taking the absolute value of (2.24) and applying Lemma 1.1 once again for the coefficients p_{2m} and q_{2m}, we obtain

$$|a_{2m+1}|\leq\frac{\alpha\left|\tau\right|}{m\left(1+2m\lambda\right)}+\frac{2\alpha^2\tau^2\left(m+1\right)}{m^2\left(1+m\lambda\right)^2} \tag{2.25}$$

Which completes the proof of Theorem 2.1. □

For $m=1$, in Theorem 2.1, we have the following Corollary.

Corollary 2.2. *Let f given by 1.3 is in the class $\mathcal{S}_{\Sigma}\left(\alpha,\tau,\lambda\right)$, then*

$$|a_2|\leq\frac{2\alpha\left|\tau\right|}{\sqrt{2\alpha\tau\left[2\left(1+2\lambda\right)-\left(1+\lambda\right)^2\right]+\left(1-\alpha\right)\left(1+\lambda\right)^2}}$$

and

$$|a_3|\leq\frac{\alpha\tau}{\left(1+2\lambda\right)}+\frac{4\alpha^2\tau^2}{\left(1+\lambda\right)^2}.$$

For $\lambda=0$, in Theorem 2.1, we have the following Corollary.

Corollary 2.3. *Let f given by 1.3 is in the class $\mathcal{S}^{*}_{\Sigma_m}\left(\alpha,\tau\right)$, then*

$$|a_{m+1}|\leq\frac{2\alpha\left|\tau\right|}{m\sqrt{1+\alpha\left(2\tau-1\right)}}$$

and

$$|a_{2m+1}|\leq\frac{\alpha\tau}{m}+\frac{2\alpha^2\tau^2\left(m+1\right)}{m^2}.$$

For $\lambda=1$, in Theorem 2.1, we have the following Corollary.

Corollary 2.4. *Let f given by 1.3 is in the class $\mathcal{C}_{\Sigma_m}\left(\alpha,\tau\right)$, then*

$$|a_{m+1}|\leq\frac{2\alpha\left|\tau\right|}{m\sqrt{2\alpha\tau\left(m+1\right)+\left(1-\alpha\right)\left(1+m\right)^2}}$$

and

$$|a_{2m+1}|\leq\frac{\alpha\tau}{m\left(1+2m\right)}+\frac{2\alpha^2\tau^2}{m^2\left(1+m\right)}.$$

For $\lambda = 0, \tau = 1$, in Theorem 2.1, we have the following Corollary.

Corollary 2.5. *Let f given by 1.3 is in the class $\delta^{\alpha}_{\Sigma,m}$, then*

$$|a_{m+1}| \leq \frac{2\alpha}{m\sqrt{1+\alpha}}$$

and

$$|a_{2m+1}| \leq \frac{\alpha}{m} + \frac{2\alpha^2 (m+1)}{m^2}.$$

For $\lambda = 0, m = 1, \tau = 1$, in Theorem 2.1, we have the following Corollary.

Corollary 2.6. *Let f given by 1.3 is in the class $\delta^{*}_{\Sigma}(\alpha)$, then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{1+\alpha}}$$

and

$$|a_3| \leq \alpha + 4\alpha^2 = \alpha(1+4\alpha).$$

For $\lambda = 1, \tau = 1$, in Theorem 2.1, we have the following Corollary.

Corollary 2.7. *Let f given by 1.3 is in the class $E_{\Sigma_m}(0,1,1,\alpha)$, then*

$$|a_{m+1}| \leq \frac{2\alpha}{m\sqrt{2\alpha(m+1) + (1-\alpha)(1+m)^2}}$$

and

$$|a_{2m+1}| \leq \frac{\alpha}{m(1+2m)} + \frac{2\alpha^2}{m^2(1+m)}.$$

For $\lambda = 1, m = 1, \tau = 1$, in Theorem 2.1, we have the following Corollary.

Corollary 2.8. *Let f given by 1.3 is in the class $\delta_{\Sigma_1}(\alpha)$, then*

$$|a_2| \leq \alpha$$

and

$$|a_3| \leq \frac{\alpha}{3} + \alpha^2.$$

3. COEFFICIENT ESTIMATES

Theorem 3.1. *If $f \in \mathcal{S}_{\Sigma_m}(\beta,\tau,\lambda)$ $(\tau \in \mathbb{C}\backslash\{0\}, 0 \leq \lambda \leq 1, 0 < \alpha \leq 1, m \in \mathbb{N})$, then*

$$|a_{m+1}| \leq \sqrt{\frac{2(1-\beta)\tau}{m\left[(m+1)(1+2m\lambda) - (1+m\lambda)^2\right]}} \tag{3.26}$$

and

$$|a_{2m+1}| \leq \frac{|\tau|(1-\beta)}{m(1+2m\lambda)} + \frac{2\tau^2(m+1)(1-\beta)^2}{m^2(1+m\lambda)^2}. \tag{3.27}$$

Proof. Let $f \in \mathcal{S}_{\Sigma_m}(\tau, \lambda, \beta)$. Then

$$1 + \frac{1}{\tau}\left(\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda) f(z) + \lambda z f'(z)} - 1\right) = \beta + (1-\beta) p(z) \tag{3.28}$$

and

$$1 + \frac{1}{\tau}\left(\frac{zg'(z) + \lambda z^2 g''(z)}{(1-\lambda) g(z) + \lambda z g'(z)} - 1\right) = \beta + (1-\beta) q(w) \tag{3.29}$$

where $p(z)$ and $q(z)$ have the forms (2.13) and (2.14) respectively. Equating the coefficients of (3.28) and (3.29), we get

$$\frac{m}{\tau}(1 + m\lambda) a_{m+1} = (1-\beta) p_m \tag{3.30}$$

$$\frac{m}{\tau}\left[2(1+2m\lambda) a_{2m+1} - (1+m\lambda)^2 a_{m+1}^2\right] = (1-\beta) p_{2m} \tag{3.31}$$

and

$$-\frac{m}{\tau}(1 + m\lambda) a_{m+1} = (1-\beta) q_m \tag{3.32}$$

$$\frac{m}{\tau}\left[\left\{2(m+1)(1+2m\lambda) - (1+m\lambda)^2\right\} a_{m+1}^2 - 2(1+2m\lambda) a_{2m+1}\right] = (1-\beta) q_{2m} \tag{3.33}$$

Now considering (3.30) and (3.32), we get

$$p_m = -q_m \tag{3.34}$$

and

$$\frac{2m^2}{\tau^2}(1+m\lambda)^2 a_{m+1}^2 = (1-\beta)^2 \left(p_m^2 + q_m^2\right) \tag{3.35}$$

Now from (3.31) and (3.33) we get

$$a_{m+1}^2 = \frac{(1-\beta)\tau(p_{2m} + q_{2m})}{2m\left[(m+1)(1+2m\lambda) - (1+m\lambda)^2\right]} \tag{3.36}$$

Now, taking absolute value of (3.36) and applying lemma 1.1 for the coefficients p_{2m} and q_{2m}, we obtain

$$|a_{m+1}| \leq \sqrt{\frac{2(1-\beta)\tau}{m\left[(m+1)(1+2m\lambda) - (1+m\lambda)^2\right]}} \tag{3.37}$$

This gives the desired estimate for $|a_{m+1}|$ as asserted in (3.26). In order to find the bound on $|a_{2m+1}|$, by subtracting (3.33) from (3.31), we get

$$\frac{m}{\tau}\left[4(1+2m\lambda) a_{2m+1} - 2(m+1)(1+2m\lambda) a_{m+1}^2\right] = (1-\beta)(p_{2m} - q_{2m}) \tag{3.38}$$

It follows from (3.34), (3.35) and (3.38)

$$a_{2m+1} = \frac{(1-\beta)\tau(p_{2m} - q_{2m})}{4m(1+2m\lambda)} + \frac{(1-\beta)^2 \tau^2 (m+1)(p_m^2 + q_m^2)}{4m^2(1+m\lambda)^2} \tag{3.39}$$

Taking the absolute value of (3.39) and applying Lemma 1.1 once again for the coefficients p_{2m} and q_{2m}, we obtain

$$|a_{2m+1}| \leq \frac{(1-\beta)|\tau|}{m(1+2m\lambda)} + \frac{2\tau^2(1-\beta)^2(m+1)}{m^2(1+m\lambda)^2} \tag{3.40}$$

Which completes the proof of Theorem 3.1. □

For $m = 1$, in Theorem 3.1, we have the following Corollary.

Corollary 3.2. *Let f given by 1.3 is in the class $\mathcal{S}_{\Sigma}(\beta, \tau, \lambda)$, then*

$$|a_2| \leq \sqrt{\frac{2\tau(1-\beta)}{2(1+2\lambda)-(1+\lambda)^2}}$$

and

$$|a_3| \leq \frac{|\tau|(1-\beta)}{(1+2\lambda)} + \frac{4\tau^2(1-\beta)^2}{(1+\lambda)^2}.$$

For $\lambda = 0$, in Theorem 3.1, we have the following Corollary.

Corollary 3.3. *Let f given by 1.3 is in the class $\mathcal{S}^*_{\Sigma_m}(\beta, \tau)$, then*

$$|a_{m+1}| \leq \frac{1}{m}\sqrt{2\tau(1-\beta)}$$

and

$$|a_{2m+1}| \leq \frac{|\tau|(1-\beta)}{m} + \frac{2\tau^2(m+1)(1-\beta)^2}{m^2}.$$

For $\lambda = 1$, in Theorem 3.1, we have the following Corollary.

Corollary 3.4. *Let f given by 1.3 is in the class $\mathcal{C}_{\Sigma_m}(\beta, \tau)$, then*

$$|a_{m+1}| \leq \frac{1}{m}\sqrt{\frac{2\tau(1-\beta)}{m+1}}$$

and

$$|a_{2m+1}| \leq \frac{|\tau|(1-\beta)}{m(1+2m)} + \frac{2\tau^2(1-\beta)^2}{m^2(1+m)}.$$

For $\lambda = 0, \tau = 1$, in Theorem 3.1, we have the following Corollary.

Corollary 3.5. *Let f given by 1.3 is in the class $\mathbb{N}^0_{\Sigma,m}(\beta, 1)$, then*

$$|a_{m+1}| \leq \frac{1}{m}\sqrt{2(1-\beta)}$$

and

$$|a_{2m+1}| \leq \frac{(1-\beta)}{m} + \frac{2(m+1)(1-\beta)^2}{m^2}.$$

For $\lambda = 0, m = 1, \tau = 1$, in Theorem 3.1, we have the following Corollary.

Corollary 3.6. *Let f given by 1.3 is in the class $\delta_{\Sigma}^{*}(\beta)$, then*

$$|a_2| \leq \sqrt{2(1-\beta)}.$$

and

$$|a_3| \leq (1-\beta) + 4(1-\beta)^2.$$

For $\lambda = 1, \tau = 1$, in Theorem 3.1, we have the following Corollary.

Corollary 3.7. *Let f given by 1.3 is in the class $E_{\Sigma_m}^{*}(0,1,1,\beta)$, then*

$$|a_{m+1}| \leq \frac{1}{m}\sqrt{\frac{2(1-\beta)}{m+1}}$$

and

$$|a_{2m+1}| \leq \frac{(1-\beta)}{m(1+2m)} + \frac{2(1-\beta)^2}{m^2(1+m)}.$$

For $\lambda = 1, m = 1, \tau = 1$, in Theorem 3.1, we have the following Corollary.

Corollary 3.8. *Let f given by 1.3 is in the class $\delta_{\Sigma_1}(\beta)$, then*

$$|a_2| \leq \sqrt{1-\beta}.$$

and

$$|a_3| \leq \frac{1-\beta}{3} + (1-\beta)^2.$$

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