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Existence of Asymptotically polynomial type solutions for some 2-dimensional Coupled Nonlinear ODEs using Banach's Theorem

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Abstract

In this paper we have considered the following coupled system of non-linear ordinary differential equations.

$$\begin{aligned} x_1^{n_1}(t) &= f_1(t, x_2(t)), \\ x_2^{n_2}(t) &= f_2(t, x_1(t)), \end{aligned} \tag{0.1}$$

where f_1, f_2 are real valued continuous functions on $[t_0, \infty) \times R^+$, $t \geq t_0 > 0$. We have given sufficient conditions on the non-linear functions f_1, f_2, such that a unique solutions pair x_1, x_2 exists, which asymptotically behaves like a pair of real polynomials.

Key Words: Non-linear Coupled Ordinary Differential Equations, Fixed-point Theorem, Asymptotically Polynomial like solutions.

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1. INTRODUCTION

Studying qualitative nature of solutions of differential equations is very useful when we expect solutions to have certain properties which have practical implications. Asymptotic representation of solutions of differential equations was extensively studied by many authors. Very recently the authors in [1] studied solutions, which are asymptotic at infinity to real polynomials of degree at most $n-1$, for the n^{th} order $(n>1)$ non-linear ordinary differential equation

$$x^{(n)} = f(t, x(t)) \qquad t \geq t_0 > 0 \tag{1.1}$$

where f is a continuous real valued function on $[t_0, \infty) \times R$. This work in [1] is essentially motivated by the recent one by Lipovan [2] concerning the special case of the second order non-linear ordinary differential equation

$$x^{''} = f(t, x(t)) \qquad t \geq t_0 > 0 \tag{1.2}$$

The application of the main results in [1] to the second order non-linear ordinary differential equation (mentioned above) leads to improvised versions of the ones given in [2]. Some closely related results for second order ordinary differential equations involving the derivative of the unknown function have been given in [9].
Systems of differential equations arise in many areas of science. Particularly systems of ODEs of second order are encountered while solving elliptic systems. Interested reader may look in to [6, 7] and references therein. In this paper we investigated the solutions of the coupled system (0.1), which behave asymptotically at ∞ like real polynomials in t. We have given sufficient conditions for the solution pair x_1, x_2 to behave like real polynomial pair of at most degree m_1, m_2 respectively, where $1 \leq m_1 \leq n_1 - 1,\ \ 1 \leq m_2 \leq n_2 - 1$. We mention here that the non-linear terms in the system are explicitly dependent on only one variable, this gives a scope for further findings where these non-linear terms could be dependent on both the variables.

2. MAIN RESULT

We investigate the solutions of (0.1) which are defined for large t i.e on the interval $[T, \infty)$, where $T \geq t_0$ may depend on the solution.
Before we prove our main result, we give some preliminaries which we use in the proof.
Let $E = B([0, \infty))$, where $B([0, \infty))$ is the Banach space of all continuous and bounded real valued functions on the interval $[0, \infty]$, endowed with the sup-norm $\|.\|$: $\|h\| = sup_{t \geq 0} |h(t)|$ for $h \in B([0, \infty))$

Banach's theorem:

Let E be a Banach Space and let S be a contraction mapping that maps E to itself. Then S has a unique fixed point in E

Theorem 2.1. *Let $K > 0$ be given and fixed. Assume that f_1 and f_2 are are functions from R^+XR^+ to R^+ and satisfy*

$$\int_{t_0}^{\infty} t^{n_1-1} f_i(t,z)dt \leq K \tag{2.1}$$

for $i = 1, 2$ and any $z \in R$. Also let,

$$\begin{aligned} \left|f_1(t,z) - f_1(t,z')\right| &\leq a(t)\left|z - z'\right|, \\ \left|f_2(t,z) - f_2(t,z')\right| &\leq b(t)\left|z - z'\right|, \end{aligned} \tag{2.2}$$

where $a(t)$, $b(t)$ be continuous functions from R^+ to R^+ such that for some fixed $T > t_0$,

$$\begin{aligned} \int_T^{\infty} t^{n_2-1} b(t) \left[\int_T^{\infty} s^{n_1-1} a(s)ds\right] dt &< 1, \\ \int_T^{\infty} t^{n_1-1} a(t) \left[\int_T^{\infty} s^{n_2-1} b(s)ds\right] dt &< 1. \end{aligned} \tag{2.3}$$

Then the system (0.1) *has a solution pair $\{x_1, x_2\}$ on the interval (T, ∞) such that for $1 \leq m_1 \leq n_1 - 1,\ 1 \leq m_2 \leq n_2 - 1$,*

$$\begin{aligned} x_1(t) &= c_{10} + c_{11}t + ... + c_{1m_1}t^{m_1} + o(1), \\ x_2(t) &= c_{20} + c_{21}t + ... + c_{2m_2}t^{m_2} + o(1) \end{aligned} \tag{2.4}$$

for $t \to \infty$.

Proof. By substituting

$$\begin{aligned} y_1(t) &= x_1(t) - (c_{10} + c_{11}t + ... + c_{1m_1}t^{m_1}), \\ y_2(t) &= x_2(t) - (c_{20} + c_{21}t + ... + c_{2m_2}t^{m_2}), \end{aligned}$$

the system (0.1) gets transformed in to

$$\begin{aligned} y_1^{(n_1)}(t) &= f_1\left(t, y_2(t) + \sum_{i=0}^{m_2} c_{2i}t^i\right), \\ y_2^{(n_2)}(t) &= f_2\left(t, y_1(t) + \sum_{i=0}^{m_1} c_{1i}t^i\right). \end{aligned} \tag{2.5}$$

Therefore we see that it is sufficient to prove that the system (2.5) has a solution pair y_1, y_2 on the interval $[T, \infty)$ with

$$\begin{aligned} \lim_{t\to\infty} y_1^{(\rho_1)}(t) &= 0, \\ \lim_{t\to\infty} y_2^{(\rho_2)}(t) &= 0 \end{aligned} \tag{2.6}$$

where $\rho_1 = 0, 1, ..., n_1 - 1$ and $\rho_2 = 0, 1, ..., n_2 - 1$.
Now consider the Banach Space $E = B([T, \infty))$ with the sup-norm $\|.\|$, and define

$$Y = \{y \in E : \|y\| \leq K\}.$$

Clearly Y is a non-empty closed convex subset of E. Let y_1 and y_2 be arbitrary functions in Y.
We now define mapping S on Y as

$$(Sy_1)(t) = (-1)^{n_1} \int_t^\infty \frac{(s-t)^{n_1-1}}{(n_1-1)!} f_1\left(s, y_2(s) + \sum_{i=0}^{m_2} c_{2i} s^i\right) ds \tag{2.7}$$

with

$$y_2(s) = \int_s^\infty \frac{(r-t)^{n_2-1}}{(n_2-1)!} f_2\left(r, y_1(r) + \sum_{i=0}^{m_1} c_{1i} r^i\right) dr \tag{2.8}$$

for every $t \geq T$.
Clearly we see that S maps Y into itself and is valid on account of (2.1). Now we show that this mapping has a fixed point using the Banach's Fixed Point Theorem. To this end we now need to show that S_1 is a contraction.
Consider

$$\left\|Sy_1 - Sy_1^{'}\right\| = \|(-1)^{n_1} \int_t^\infty \frac{(s-t)^{n_1-1}}{(n_1-1)!} \times$$

$$f_1\left(s, \int_s^\infty \frac{(r-t)^{n_2-1}}{(n_2-1)!} f_2\left(r, y_1(r) + \sum_{i=0}^{m_1} c_{1i} r^i\right) dr + \sum_{i=0}^{m_2} c_{2i} s^i\right) ds$$

$$-(-1)^{n_1} \int_t^\infty \frac{(s-t)^{n_1-1}}{(n_1-1)!} f_1\left(s, \int_s^\infty \frac{(r-T)^{n_2-1}}{(n_2-1)!} f_2\left(r, y_1^{'}(r) + \sum_{i=0}^{m_1} c_{1i} r^i\right) dr + \sum_{i=0}^{m_2} c_{2i} s^i\right) ds\|$$

$$\leq \int_t^\infty \frac{(s-t)^{n_1-1}}{(n_1-1)!} \times$$
$$\|f_1\left(s, \int_s^\infty \frac{(r-t)^{n_2-1}}{(n_2-1)!} f_2\left(r, y_1(r) + \sum_{i=0}^{m_1} c_{1i} r^i\right) dr + \sum_{i=0}^{m_2} c_{2i} s^i\right) ds$$
$$-f_1\left(s, \int_s^\infty \frac{(r-t)^{n_2-1}}{(n_2-1)!} f_2\left(r, y_1^{'}(r) + \sum_{i=0}^{m_1} c_{1i} r^i\right) dr + \sum_{i=0}^{m_2} c_{2i} s^i\right) \|ds$$

$$\leq \int_t^\infty a(s)\frac{(s-t)^{n_1-1}}{(n_1-1)!}\times$$
$$\int_s^\infty \frac{(r-t)^{n_2-1}}{(n_2-1)!}\|f_2\left(r, y_1(r)+\sum_{i=0}^{m_1} c_{1i}r^i\right)ds$$
$$-f_2\left(r, y_1^{'}(r)+\sum_{i=0}^{m_1} c_{1i}r^i\right)\|ds$$

$$\leq \int_t^\infty a(s)\frac{(s-t)^{n_1-1}}{(n_1-1)!}\times$$
$$\left(\int_s^\infty b(s)\frac{(r-t)^{n_2-1}}{(n_2-1)!}\left|y_1(r)-y_1^{'}(r)\right|dr\right)ds.$$

Taking (2.3) in to consideration, we conclude that S is a contraction, hence by Contraction Principle S has a unique fixed point $y_1 \in Y$ such that $Sy_1 = y_1$. That implies

$$y_1(t) = (-1)^{n_1}\int_t^\infty \frac{(s-t)^{n_1-1}}{(n_1-1)!}f_1\left(s, y_2(s)+\sum_{i=0}^{m_1} c_{2i}s^i\right)ds,$$

where

$$y_2(s) = \int_s^\infty \frac{(r-t)^{n_2-1}}{(n_2-1)!}f_2\left(r, y_1(r)+\sum_{i=0}^{m_1} c_{1i}r^i\right)dr.$$

By applying the Leibnitz's differentiation of integral rule and differentiating the above equations we get back the two equations of the transformed system (2.5)

$$\begin{aligned} y_1^{(n_1)}(t) &= f_1\left(t, y_2(t)+\sum_{i=0}^{m_2} c_{2i}t^i\right), \\ y_2^{(n_2)}(t) &= f_2\left(t, y_1(t)+\sum_{i=0}^{m_1} c_{1i}t^i\right) \end{aligned}$$

for all $t \geq T$.

Therefore we conclude that the y_1 is a solution of the first equation of the system (2.5) iff its a fixed point of the operator S. Now, we can find the other function y_2 by substituting the fixed point y_1 in (2.8). Consequently y_1, y_2 satisfy the transformed system (2.5). Since,

$$y_i^{(\rho_i)}(t) = (-1)^{n_i-\rho_i}\int_t^\infty \frac{(s-t)^{n_i-1-\rho_i}}{(n_i-1-\rho_i)!}f_i\left(s, y_{3-i}(s)+\sum_{k=0}^{m_i} c_{(3-i)k}s^k\right)ds$$

for $i = 1, 2.$ and $\rho_i = 0, 1, .., n_i - 1$. It is easy to verify that y_1, y_2 satisfy the condition (2.6). This completes the proof of the theorem. □

Theorem 2.2. *Let $K > 0$ be given and fixed. Assume that f_1 and f_2 are two functions from $R^+ X R^+$ to R^+ and satisfy*

$$f_i(t,z) \leq h_i(t) g_i(z) \tag{2.9}$$

for $i = 1,2$ and, where $h_i : R^+ \to R^+$ are continuous functions such that

$$\int_{t_0}^{\infty} s^{n_i - 1} h_i(s) ds < 1 \tag{2.10}$$

for $i = 1,2$ and assume that $g_i s$ are non-negative continuous real valued functions which are not identically zero on R^+ such that for any real constants $c_{i0}, c_{i1}, ..., c_{im_i}$, $i = 1,2$ and for some fixed $T \geq t_0$ we have

$$\sup \left\{ g_i(z) : 0 < z < K + \sum_{j=0}^{m_i} \left| c_{ij} T^j \right| \right\} < K \tag{2.11}$$

for $i = 1,2$ and also assume like in the previous theorem

$$\begin{aligned} \left| f_1(t,z) - f_1(t,z^{'}) \right| &\leq a(t) \left| z - z^{'} \right|, \\ \left| f_2(t,z) - f_2(t,z^{'}) \right| &\leq b(t) \left| z - z^{'} \right|, \end{aligned} \tag{2.12}$$

where $a(t)$, $b(t)$ be continuous functions from R^+ to R^+ such that

$$\begin{aligned} \int_{t_0}^{\infty} t^{n_2 - 1} b(t) \left[\int_{t_0}^{\infty} s^{n_1 - 1} a(s) ds \right] dt &< 1, \\ \int_{t_0}^{\infty} t^{n_1 - 1} a(t) \left[\int_{t_0}^{\infty} s^{n_2 - 1} b(s) ds \right] dt &< 1. \end{aligned} \tag{2.13}$$

Then the system (0.1) *has a solution pair $\{x_1, x_2\}$ on the interval (T, ∞) such that for $1 \leq m_1 \leq n_1 - 1,\ \ 1 \leq m_2 \leq n_2 - 1$,*

$$\begin{aligned} x_1(t) &= c_{10} + c_{11} t + ... + c_{1m_1} t^{m_1} + o(1), \\ x_2(t) &= c_{20} + c_{21} t + ... + c_{2m_2} t^{m_2} + o(1) \end{aligned} \tag{2.14}$$

for $t \to \infty$.

Proof. We proceed as in the previous proof, except that for proving that the operator S is a self map, we use (2.9), (2.10) and (2.11) instead of (2.1) □

Example 2.3. Consider the following system of equations

$$\begin{aligned} x_1^{(n)}(t) &= a(t) \left| x_2(t) \right|^{2p-1} sgn\ x_2(t), \\ x_2^{(n)}(t) &= b(t) \left| x_1(t) \right|^{2p-1} sgn\ x_1(t) \end{aligned} \tag{2.15}$$

for $p > 1$ is an integer, where a and b are non-negative continuous real valued functions on $[0, \infty)$. and and let $c_0, c_1, c_2, ..., c_m$ and $d_0, d_1, d_2, ..., d_m$ be real numbers and T be a point with $T > 0$ and suppose that there exists a positive constant K such that

$$\sup\left\{ z^{2p-1} : 0 \le z \le \big(K + \sum_{i=0}^{m} \left|c_i T^i\right|\big) \right\} \le K,$$
$$\sup\left\{ z^{2p-1} : 0 \le z \le \big(K + \sum_{i=0}^{m} \left|d_i T^i\right|\big) \right\} \le K. \qquad (2.16)$$

We see that, for $p = 2, T = 1$ and $K = \frac{1}{2}$, above inequalities are definitely satisfied for $c_0 = d_0 = 0$, $c_1 = d_1 = 0$. Note that z^{2p-1} is Lipschitz with Lipschitz constant $2p - 1$, so consider $a(t), b(t)$ such that

$$\int_0^{\infty} t^{n-1} |a(t)| \left[\int_0^{\infty} s^{n-1} |b(s)|\, ds\right] dt < \frac{1}{(2p-1)^2},$$
$$\int_0^{\infty} t^{n-1} |b(t)| \left[\int_0^{\infty} s^{n-1} |a(s)|\, ds\right] dt < \frac{1}{(2p-1)^2}.$$

Then by invoking Theorem 2.2, (2.15) has a pair of solutions x_1 and x_2 which asymptotically behave like m^{th} degree polynomials with coefficients c_i, d_i respectively where $1 \le m \le n - 1$.
As a special case when we consider $n = 2$, $m = 1$, $p = 2$, $T = 1$, $a(t) = 2e^{-t-2}$, $b(t) = e^{-t}$, $c_0 = d_0 = 0$, $c_1 = d_1 = 0$. We see that

$$\int_1^{\infty} t\, |a(t)| \left[\int_1^{\infty} s\, |b(s)|\, ds\right] dt < \frac{1}{9},$$
$$\int_1^{\infty} t\, |b(t)| \left[\int_1^{\infty} s\, |a(s)|\, ds\right] dt < \frac{1}{9}.$$

Therefore Theorem 2.2 guarantees a solution such that $x_i(t) = 0(1), i = 1, 2$ as $t \to \infty$.

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