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New Results on the Existence of Positive Solutions of Atangana–Baleanu Type Fractional Differential Equations

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Abstract

This paper proves the existence and uniqueness of positive solution for a class of nonlinear fractional differential equations involving the Atangana–Baleanu operator. We apply a new approach based on building the upper and lower control functions of the nonlinear term wanting no monotone conditions. Then, by means of the Schauder-type fixed point technique and upper and lower solutions method, we investigate the sufficient conditions of existence and uniqueness of positive solutions for the suggested problem. Moreover, we prove the further existence results as special cases. Some examples in order to illustrate the validity of main results.

Keywords and Pharases: Fractional differential equation, Atangana–Baleanu fractional derivative, Positive solution, Upper and lower solutions method, Fixed point theorem.

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1. INTRODUCTION

Fractional calculus is mostly description of the integral and derivative operator with fractional order. The fractional derivative (FD) was first invented by Leibniz [1, 2, 3]. After Leibniz invented the FD, it was further elaborated by more authors. The most used FDs are the Riemann–Liouville and Caputo types. There are other types of FDs as well, we allude to [4, 5, 6, 7] and references therein.

The most normal property of the previously mentioned fractional operators is that the portions they typify contain singular kernels. This property causes numerous hindrances in applying these operators.

To sidestep these hindrances, Caputo and Fabrizio [8] suggested a new fractional operator so-called Caputo–Fabrizio. This operator contains a non-singular kernel yet still conserves the most substantial peculiarity of the classical fractional operators. Utilizing this operator created better outcomes compared with the FDs singular kernel.

However, a burden of this operator emerged in light of the fact that the associated integral can be written in terms of an integral of integer order. To avoid this drawback, Atangana and Baleanu [9] proposed a FD based on a generalized Mittag-leffler function so-called Atangana–Baleanu derivative (AB type-derivative). For short, this operator in the sense of Riemman-Liouville and Caputo are meant by ABR-derivative and ABC-derivative, respectively.

Many researchers contributed in growing the AB fractional calculus. We notice for instance [10, 11, 12] and a portion of the references they contain. As of late, the AB type-derivative has been attracting the interest of many authors, where many of its uses appeared in the field of epidemiological modeling and the theory of differential equations, e.g., Koca [13] analysed the rubella disease by employ the AB-derivative. Atangana and Gomez-Aguilar in [14] constructed a nature model in the frame of AB-derivative. Toufik and Atangana [15] applied a new numerical scheme to solve fractional differential equations under AB-derivative. Khan et al. [16] discussed the existence and stability of the hepatitis B epidemic model with a AB-derivative. The class of fractional differential equations (FDEs) with AB-derivative has been studied by Jarad et al. [17]. Abdo et al. [18] investigated the existence and uniqueness results of an impulsive FDE under AB-derivative.

On the other hand, there have been some recent papers managing the existence and uniqueness of positive solutions of different types of FDEs by the usage of various techniques of fixed point, see [19, 20, 21, 22, 23, 24, 25, 26, 27, 28].

Motivated by the aforementioned discussions and papers, and inspired by [9, 20], we investigate the existence and uniqueness of positive solution of the following ABC-type FDF

$$\begin{cases} {}_0^{\mathcal{ABC}}\mathcal{D}_\varkappa^\delta \varphi(\varkappa) = \mathrm{f}(\varkappa, \varphi(\varkappa)), & \varkappa \in (0,1), \\ \varphi(0) = 0, \end{cases} \tag{1.1}$$

where $0 < \delta < 1$, ${}_0^{\mathcal{ABC}}\mathcal{D}_\varkappa^\delta$ is the ABC-derivative of order δ, $\mathrm{f} : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function.

We give interesting results for ABC-type FDE. Most of our derivations are made utilizing theorems of significant importance such as Arzelá–Ascoli's theorem, Schauder's fixed point theorem, Banach's fixed point theorem and upper and lower solutions method. Moreover, we construct the upper and lower control functions of the nonlinear term without monotone conditions to obtain the desired results.

The paper is marshaled as follows: In Section 2, we give some basic results. In Section 3, we prove the existence and uniqueness of positive solutions to the problem (1.1). Two examples are given in the last section.

2. FUNDAMENTAL RESULTS

In this section, we provide some notions and basic definitions of AB fractional calculus which are needed whole this paper. Suppose $\mathrm{X} = \mathcal{C}[0,1]$ be a Banach space with the norm $\|\varphi\| = \max\{|\varphi(\varkappa)|\,;\ \varkappa \in [0,1]\}\,;\ \varphi \in \mathrm{X}$. Define the cone

$$\mathrm{K} = \{\varphi \in \mathrm{X} : \varphi(\varkappa) \geq 0, \qquad 0 \leq \varkappa \leq 1\}.$$

Definition 2.1. [9] Let $0 < \delta < 1$, and $\varphi \in \mathrm{H}^1(a,b)$, $a < b$. The $\mathcal{ABC}$ fractional derivative for function φ of order δ is given by

$${}_0^{\mathcal{ABC}}\mathcal{D}_\varkappa^\delta \varphi(\varkappa) = \frac{\mathbb{M}(\delta)}{1-\delta} \int_0^\varkappa \varphi'(\xi) \mathrm{E}_\delta \left(\frac{-\delta(\varkappa - \xi)^\delta}{1-\delta} \right) d\xi. \tag{2.1}$$

Further, the $\mathcal{ABR}$ fractional derivative is defined by

$${}_0^{\mathcal{ABR}}\mathcal{D}_\varkappa^\delta \varphi(\varkappa) = \frac{\mathbb{M}(\delta)}{1-\delta} \frac{d}{d\varkappa} \int_0^\varkappa \varphi(\xi) \mathrm{E}_\delta \left(\frac{-\delta(\varkappa - \xi)^\delta}{1-\delta} \right) d\xi. \tag{2.2}$$

Here, $\mathbb{M}(\delta) > 0$ is a normalization function satisfies $\mathbb{M}(0) = \mathbb{M}(1) = 1$ and E_δ represents the well known Mittag- Liffler function.

Definition 2.2. [9] Let $0 < \delta < 1$ and φ be function, then $\mathcal{AB}$ fractional integral of order δ is given by

$$\begin{aligned} {}_{0}^{\mathcal{AB}}\mathcal{I}_{\varkappa}^{\delta}\varphi(\varkappa) &= \frac{1-\delta}{\mathbb{M}(\delta)}\varphi(\varkappa) + \frac{\delta}{\mathbb{M}(\delta)}\,{}_{0}^{\mathcal{RL}}\mathcal{I}_{\varkappa}^{\delta}\,\varphi(\varkappa) \\ &= \frac{1-\delta}{\mathbb{M}(\delta)}\varphi(\varkappa) + \frac{\delta}{\mathbb{M}(\delta)\Gamma(\delta)}\int_{0}^{\varkappa}\varphi(\xi)(\varkappa-\xi)^{\delta-1}d\xi, \end{aligned} \tag{2.3}$$

where

$$ {}_{0}^{\mathcal{RL}}\mathcal{I}_{\varkappa}^{\delta}\,\varphi(\varkappa) = \frac{1}{\Gamma(\delta)}\int_{0}^{\varkappa}\varphi(\xi)(\varkappa-\xi)^{\delta-1}d\xi $$

is called the Riemann-Liouville fractional integral [1].

Definition 2.3. [12] Let $n < \delta \leq n+1,\ \ n = 0,1,...,$ and φ be a function such that $\varphi^{(n)} \in \mathrm{H}^{1}(a,b)$. Then $\mathcal{ABC}$ derivative satisfies ${}_{0}^{\mathcal{ABC}}\mathcal{D}_{\varkappa}^{\delta}\varphi(\varkappa) = {}_{0}^{\mathcal{ABC}}\mathcal{D}_{\varkappa}^{\eta}\,\varphi^{(n)}(\mathrm{t})$, where $\eta = \delta - n$.

Lemma 2.4. *[12] For $n < \delta \leq n+1,\ \ n = 0,1,...,$ the following result holds for the FDEs*

$$ {}_{0}^{\mathcal{AB}}\mathcal{I}_{\varkappa}^{\delta}\,{}_{0}^{\mathcal{ABC}}\mathcal{D}_{\varkappa}^{\delta}\,\varphi(\varkappa) = \varphi(\varkappa) + \mathrm{d}_0 + \mathrm{d}_1\varkappa + \mathrm{d}_2\varkappa^2 + \cdots + \mathrm{d}_n\varkappa^n, $$

for arbitrary constant d_i with $i = 0,1,2,....,n$.

Theorem 2.5. *[29] Let Φ be a Banach space with a contraction mapping $\mathrm{T} : \Phi \to \Phi$. Then, T has a unique fixed-point φ in Φ.*

Theorem 2.6. *[29] Let Φ be a Banach space and let S a closed, convex, bounded subset of Φ. If $\mathrm{T} : \mathrm{S} \longrightarrow \mathrm{S}$ is a continuous map such that the set $\{\mathrm{T}\varphi : \varphi \in \mathrm{S}\}$ is relatively compact in Φ. Then T has at least one fixed point.*

3. MAIN RESULTS

In this portion, we prove the existence and uniqueness of positive solutions for (1.1). Before starting, we introduce the following lemma:

Lemma 3.1. *[9] Let $0 < \delta < 1$ and $\mathrm{h} : [0,1] \to \mathbb{R}^{+}$ is a continuous function with $\mathrm{h}(0) = 0$. Then the linear $\mathcal{ABC}$-type FDF*

$$\begin{aligned} {}_{0}^{\mathcal{ABC}}\mathcal{D}_{\varkappa}^{\delta}\varphi(\varkappa) &= \mathrm{h}(\varkappa), \ \ \varkappa \in (0,1), \\ \varphi(0) &= 0, \end{aligned} \tag{3.1}$$

is equivalent to

$$ \varphi(\varkappa) = \frac{1-\delta}{\mathbb{M}(\delta)}\mathrm{h}(\varkappa) + \frac{\delta}{\mathbb{M}(\delta)\Gamma(\delta)}\int_{0}^{\varkappa}\mathrm{h}(\xi)(\varkappa-\xi)^{\delta-1}d\xi. \tag{3.2}$$

As result of Lemma 3.1, we get the following Lemma:

Lemma 3.2. *Assume that* $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ *is a continuous with* $\mathrm{f}(0, \varphi(0)) = 0$, *and* φ *be a function. Then the nonlinear* $\mathcal{ABC}$*-type FDF (1.1) is equivalent to*

$$\varphi(\varkappa) = \frac{1-\delta}{\mathbb{M}(\delta)}\mathrm{f}(\varkappa, \varphi(\varkappa)) + \frac{\delta}{\mathbb{M}(\delta)\Gamma(\delta)} \int_0^{\varkappa} \mathrm{f}(\xi, \varphi(\xi))(\varkappa - \xi)^{\delta-1} d\xi. \tag{3.3}$$

Note that, The equation (3.3) is also equivalent to

$$(\mathrm{T}\varphi)(\varkappa) = \frac{1-\delta}{\mathbb{M}(\delta)}\mathrm{f}(\varkappa, \varphi(\varkappa)) + \frac{\delta}{\mathbb{M}(\delta)\Gamma(\delta)} \int_0^{\varkappa} \mathrm{f}(\xi, \varphi(\xi))(\varkappa - \xi)^{\delta-1} d\xi, \tag{3.4}$$

where $\mathrm{T} : \mathrm{K} \to \mathrm{K}$ is an operator such that $(\mathrm{T}\varphi)(\varkappa) = \varphi(\varkappa)$, $\varphi(\varkappa) \in \mathrm{X}$.

Definition 3.3. [20] The upper and lower control functions are defined by

$$\overline{\Delta}(\varkappa, \varphi) = \sup_{a \le \eta \le \varphi} \mathrm{f}(\varkappa, \eta) \text{ and } \underline{\Delta}(\varkappa, \varphi) = \inf_{\varphi \le \eta \le b} \mathrm{f}(\varkappa, \eta),$$

respectively, where $a, b \in \mathbb{R}^+ (b > a)$ and $\varphi \in [a, b]$.

It is clear that $\overline{\Delta}(\varkappa, \varphi)$ and $\underline{\Delta}(\varkappa, \varphi)$ are non-decreasing on φ and

$$\underline{\Delta}(\varkappa, \varphi) \le \mathrm{f}(\varkappa, \varphi) \le \overline{\Delta}(\varkappa, \varphi).$$

Definition 3.4. [20] Let $\overline{\varphi}(\varkappa)$,$\underline{\varphi}(\varkappa) \in \mathrm{K}$ and $a \le \underline{\varphi}(\varkappa) \le \overline{\varphi}(\varkappa) \le b$ satisfy

$$\overline{\varphi}(\varkappa) \ge \frac{1-\delta}{\mathbb{M}(\delta)}\overline{\Delta}(\varkappa, \overline{\varphi}(\varkappa)) + \frac{\delta}{\mathbb{M}(\delta)\Gamma(\delta)} \int_0^{\varkappa} \overline{\Delta}(\xi, \overline{\varphi}(\xi))(\varkappa - \xi)^{\delta-1} d\xi$$

and

$$\underline{\varphi}(\varkappa) \le \frac{1-\delta}{\mathbb{M}(\delta)}\underline{\Delta}(\varkappa, \underline{\varphi}(\varkappa)) + \frac{\delta}{\mathbb{M}(\delta)\Gamma(\delta)} \int_0^{\varkappa} \underline{\Delta}(\xi, \underline{\varphi}(\xi))(\varkappa - \xi)^{\delta-1} d\xi$$

Then, $\overline{\varphi}(\varkappa)$ and $\underline{\varphi}(\varkappa)$ are called upper and lower solutions for (1.1).

Now, we give the following hypothese:

(H_1) $\mathrm{f} : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous.

(H_2) There exists a constant $L_f > 0$ such that

$$|\mathrm{f}(\varkappa, \varphi_1) - \mathrm{f}(\varkappa, \varphi_2)| \le \mathrm{L_f} |\varphi_1 - \varphi_2|, \ \forall \varkappa \in [0,1], \ \varphi_1, \varphi_2 \in \mathbb{R}^+$$

(H_3) $\mathrm{f}(\varkappa,\varphi)$ is a completely continuous such that for $\varkappa \in [0,1]$ and $\varphi \in \mathrm{X}$, there exists a constant $\rho > 0$ satisfying

$$|\mathrm{f}(\varkappa,\varphi)| \leq \rho, \;\; (\varkappa,\varphi) \in [0,1] \times \mathbb{R}^{+}.$$

Theorem 3.5. *Suppose that (H_1) and (H_3) hold. Let $\overline{\varphi}(\varkappa)$, $\underline{\varphi}(\varkappa)$ are upper, lower solutions of problem (1.1). Then the nonlinear $\mathcal{ABC}$-type FDF (1.1) has at least a solution $\varphi(\varkappa)$. Moreover,*

$$\underline{\varphi}(\varkappa) \leq \varphi(\varkappa) \leq \overline{\varphi}(\varkappa), \;\; \varkappa \in [0,1].$$

Proof. Convert the problem (1.1) into a fixed point problem. Consider the operator $\mathrm{T}:\mathrm{K}\to\mathrm{K}$ defined by (3.4). Then we shall make use of Theorem 2.6 to verify that T has a fixed point. The proof will be presented in some steps.

Step1: The operator $\mathrm{T}:\mathrm{K}\to\mathrm{K}$ is compact.

From the continuity and nonnegativity of $\mathrm{f}(\varkappa,\varphi)$, the operator T is continuous. Define a ball

$$\mathrm{B_r} = \{\varphi \in \mathrm{K} : \|\varphi\| \leq \mathrm{r}, \;\; \varkappa \in [0,1]\},$$

Let $\varphi \in \mathrm{B_r}$ and $\varkappa \in [0,1]$. Then we get

$$\begin{aligned} |(\mathrm{T}\varphi)(\varkappa)| &\leq \frac{1-\delta}{\mathbb{M}(\delta)}|\mathrm{f}(\varkappa,\varphi(\varkappa))| + \frac{\delta}{\mathbb{M}(\delta)\Gamma(\delta)}\int_0^{\varkappa}|\mathrm{f}(\xi,\varphi(\xi))|(\varkappa-\xi)^{\delta-1}d\xi, \\ &\leq \frac{1-\delta}{\mathbb{M}(\delta)}\rho + \frac{\delta\rho}{\mathbb{M}(\delta)\Gamma(\delta)}\int_0^{\varkappa}(\varkappa-\xi)^{\delta-1}d\xi \\ &\leq \frac{\rho}{\mathbb{M}(\delta)}\left(1-\delta+\frac{1}{\Gamma(\delta)}\right). \end{aligned}$$

Hence

$$\|\mathrm{T}\varphi\| \leq \frac{\rho}{\mathbb{M}(\delta)}\left(1-\delta+\frac{1}{\Gamma(\delta)}\right).$$

This show that $\mathrm{T}:\mathrm{B_r}\to\mathrm{B_r}$ is uniformly bounded.

Now, we prove that T is equicontinuous. For each $\varphi \in \mathrm{B_r}$. Then for $\varkappa_1, \varkappa_2 \in [0,1]$

with $\varkappa_1 < \varkappa_2$, we have

$$\begin{aligned}
|(\mathrm{T}\varphi)(\varkappa_2) - (\mathrm{T}\varphi)(\varkappa_1)| \leq\ & \Bigg|\frac{1-\delta}{\mathbb{M}(\delta)}\mathrm{f}(\varkappa_2, \varphi(\varkappa_2)) + \frac{\delta}{\mathbb{M}(\delta)\Gamma(\delta)}\int_0^{\varkappa_2} \mathrm{f}(\xi, \varphi(\xi))(\varkappa_2 - \xi)^{\delta-1} d\xi \\
& - \frac{1-\delta}{\mathbb{M}(\delta)}\mathrm{f}(\varkappa_1, \varphi(\varkappa_1)) + \frac{\delta}{\mathbb{M}(\delta)\Gamma(\delta)}\int_0^{\varkappa_1} \mathrm{f}(\xi, \varphi(\xi))(\varkappa_1 - \xi)^{\delta-1} d\xi\Bigg| \\
\leq\ & \frac{1-\delta}{\mathbb{M}(\delta)}\left|\mathrm{f}(\varkappa_2, \varphi(\varkappa_2)) - \mathrm{f}(\varkappa_1, \varphi(\varkappa_1))\right| \\
& + \frac{\delta}{\mathbb{M}(\delta)\Gamma(\delta)}\int_0^{\varkappa_1}\left|(\varkappa_2 - \xi)^{\delta-1} - (\varkappa_1 - \xi)^{\delta-1}\right| |\mathrm{f}(\xi, \varphi(\xi))|\, d\xi \\
& + \frac{\delta}{\mathbb{M}(\delta)\Gamma(\delta)}\int_{\varkappa_1}^{\varkappa_2} (\varkappa_2 - \xi)^{\delta-1} |\mathrm{f}(\xi, \varphi(\xi))|\, d\xi \qquad (3.5)
\end{aligned}$$

Since $\mathrm{f}(\varkappa, \varphi)$ is completely continuous due to (H_3),

$$|\mathrm{f}(\varkappa_2, \varphi(\varkappa_2)) - \mathrm{f}(\varkappa_1, \varphi(\varkappa_1))| \to 0, \text{ as } \varkappa_1 \to \varkappa_2.$$

Consequently,

$$\begin{aligned}
|(\mathrm{T}\varphi)(\varkappa_2) - (\mathrm{T}\varphi)(\varkappa_1)| \leq\ & \frac{\delta\rho}{\mathbb{M}(\delta)\Gamma(\delta)}\int_0^{\varkappa_1} (\varkappa_1 - \xi)^{\delta-1} - (\varkappa_2 - \xi)^{\delta-1} d\xi \\
& + \frac{\delta\rho}{\mathbb{M}(\delta)\Gamma(\delta)}\int_{\varkappa_1}^{\varkappa_2} (\varkappa_2 - \xi)^{\delta-1} d\xi \\
\leq\ & \frac{\rho}{\mathbb{M}(\delta)\Gamma(\delta)}\left[(\varkappa_2 - \varkappa_1)^{\delta} + \varkappa_1^{\delta} - \varkappa_2^{\delta}\right] \\
& + \frac{\rho}{\mathbb{M}(\delta)\Gamma(\delta)}(\varkappa_2 - \varkappa_1)^{\delta} \\
\leq\ & \frac{2\rho}{\mathbb{M}(\delta)\Gamma(\delta)}(\varkappa_2 - \varkappa_1)^{\delta},
\end{aligned}$$

which implies

$$|(\mathrm{T}\varphi)(\varkappa_2) - (\mathrm{T}\varphi)(\varkappa_1)| \to 0, \text{ as } \varkappa_1 \to \varkappa_2.$$

Thus, $(\mathrm{TB_r})$ is equicontinuous. Hence, $\mathrm{T} : \mathrm{B_r} \to \mathrm{B_r}$ is completely continuous due to Arzela–Ascoli theorem.

Step2: To apply Theorem 2.6, we need to prove $\mathrm{T} : \mathcal{N} \to \mathcal{N}$, where

$$\mathcal{N} = \{\varphi(\varkappa) : \varphi(\varkappa) \in \mathrm{K},\ \underline{\varphi}(\varkappa) \leq \varphi(\varkappa) \leq \overline{\varphi}(\varkappa),\ \varkappa \in [0, 1]\}. \qquad (3.6)$$

It is clear that $\mathcal{N}$ is a closed, convex, and bounded subset of $C([0, 1], \mathbb{R}^+)$. For any $\varphi(\varkappa) \in \mathcal{N}$, we get $\underline{\varphi}(\varkappa) \leq \varphi(\varkappa) \leq \overline{\varphi}(\varkappa)$, it follows from Definitions 3.3 and 3.4 that

$$\begin{aligned}
(\mathrm{T}\varphi)(\varkappa) =\ & \frac{1-\delta}{\mathbb{M}(\delta)}\mathrm{f}(\varkappa, \varphi(\varkappa)) + \frac{\delta}{\mathbb{M}(\delta)\Gamma(\delta)}\int_0^{\varkappa} \mathrm{f}(\xi, \varphi(\xi))(\varkappa - \xi)^{\delta-1} d\xi \\
\leq\ & \frac{1-\delta}{\mathbb{M}(\delta)}\overline{\Delta}(\varkappa, \overline{\varphi}(\varkappa)) + \frac{\delta}{\mathbb{M}(\delta)\Gamma(\delta)}\int_0^{\varkappa} \overline{\Delta}(\xi, \overline{\varphi}(\xi))(\varkappa - \xi)^{\delta-1} d\xi \\
\leq\ & \overline{\varphi}(\varkappa),
\end{aligned}$$

and

$$\begin{aligned}(\mathrm{T}\varphi)(\varkappa) &= \frac{1-\delta}{\mathbb{M}(\delta)}\mathrm{f}(\varkappa,\varphi(\varkappa)) + \frac{\delta}{\mathbb{M}(\delta)\Gamma(\delta)}\int_0^{\varkappa} \mathrm{f}(\xi,\varphi(\xi))(\varkappa-\xi)^{\delta-1}d\xi \\ &\geq \frac{1-\delta}{\mathbb{M}(\delta)}\underline{\Delta}(\varkappa,\underline{\varphi}(\varkappa)) + \frac{\delta}{\mathbb{M}(\delta)\Gamma(\delta)}\int_0^{\varkappa} \underline{\Delta}(\xi,\underline{\varphi}(\xi))(\varkappa-\xi)^{\delta-1}d\xi \\ &\geq \underline{\varphi}(\varkappa).\end{aligned}$$

So, $\underline{\varphi}(\varkappa) \leq \mathrm{T}\varphi(\varkappa) \leq \overline{\varphi}(\varkappa), 0 \leq \varkappa \leq 1$ which implies $\mathrm{T}\varphi(\varkappa) \in \mathcal{N}, \forall\, \varkappa \in [0,1]$. This proves that $\mathrm{T} : \mathcal{N} \to \mathcal{N}$. As an results of Theorem 2.6, T has at least one fixed point $\varphi(\varkappa) \in \mathcal{N}$, $0 \leq \varkappa \leq 1$. Therefore, $\varphi(\varkappa) \in \mathrm{X}$ is a solution of the problem (1.1), and $\underline{\varphi}(\varkappa) \leq \varphi(\varkappa) \leq \overline{\varphi}(\varkappa), 0 \leq \varkappa \leq 1$. □

Corollary 3.6. *Assume that* $\mathrm{f} : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous, and there exist* $\mathcal{L}_2 \geq \mathcal{L}_1 > 0$ *such that*

$$\mathcal{L}_1 \leq \mathrm{f}(\varkappa,\sigma) \leq \mathcal{L}_2, \;\; (\varkappa,\sigma) \in [0,1] \times \mathbb{R}^+. \tag{3.7}$$

Then there exists at least a solution $\varphi(\varkappa)$ *of the* $\mathcal{ABC}$*-type FDF (1.1). Moreover,*

$$\frac{\mathcal{L}_1}{\mathbb{M}(\delta)}\left(1-\delta+\frac{1}{\Gamma(\delta)}\varkappa^{\delta}\right) \leq \varphi(\varkappa) \leq \frac{\mathcal{L}_2}{\mathbb{M}(\delta)}\left(1-\delta+\frac{1}{\Gamma(\delta)}\varkappa^{\delta}\right), \;\; \textit{for } \varkappa \in [0,1], \tag{3.8}$$

Proof. In view of Definition 3.3 and assumption (3.7), we have

$$\mathcal{L}_1 \leq \underline{\Delta}(\varkappa,\sigma) \leq \overline{\Delta}(\varkappa,\sigma) \leq \mathcal{L}_2, \;\; (\varkappa,\sigma) \in [0,1] \times [a,b]. \tag{3.9}$$

Consider the $\mathcal{ABC}$-type FDF

$$\begin{aligned} {}^{\mathcal{ABC}}_{0}\mathcal{D}^{\delta}_{\varkappa}\overline{\varphi}(\varkappa) &= \mathcal{L}_2, \;\; 0 \leq \varkappa \leq 1, \\ \overline{\varphi}(0) &= 0, \end{aligned} \tag{3.10}$$

by virtue of Lemma 3.1, the $\mathcal{ABC}$-type FDF (3.10) has a positive solution

$$\begin{aligned}\overline{\varphi}(\varkappa) &= \left({}^{\mathcal{AB}}_{0}\mathcal{I}^{\delta}_{\varkappa}\,\mathcal{L}_2\right)(\varkappa) \\ &= \frac{1-\delta}{\mathbb{M}(\delta)}\mathcal{L}_2 + \frac{\delta\mathcal{L}_2}{\mathbb{M}(\delta)\Gamma(\delta)}\int_0^{\varkappa}(\varkappa-\xi)^{\delta-1}d\xi \\ &= \frac{\mathcal{L}_2}{\mathbb{M}(\delta)}\left(1-\delta+\frac{1}{\Gamma(\delta)}\varkappa^{\delta}\right)\end{aligned}$$

By (3.9), we get

$$\overline{\varphi}(\varkappa) = \left({}^{\mathcal{AB}}_{0}\mathcal{I}^{\delta}_{\varkappa}\,\mathcal{L}_2\right)(\varkappa) \geq {}^{\mathcal{AB}}_{0}\mathcal{I}^{\delta}_{\varkappa}\,\overline{\Delta}(\varkappa,\overline{\varphi}(\varkappa)).$$

Therefore, $\overline{\varphi}(\varkappa)$ is the upper solution of the $\mathcal{ABC}$-type FDF (1.1).

Similarly, the $\mathcal{ABC}$-type FDF

$$\begin{aligned} {}_{0}^{\mathcal{ABC}}\mathcal{D}_{\varkappa}^{\delta}\underline{\varphi}(\varkappa) &= \mathcal{L}_1, \quad 0 \le \varkappa \le 1, \\ \underline{\varphi}(0) &= 0, \end{aligned} \tag{3.11}$$

has a positive solution

$$\begin{aligned} \underline{\varphi}(\varkappa) &= \left({}_{0}^{\mathcal{AB}}\mathcal{I}_{\varkappa}^{\delta}\,\mathcal{L}_1\right)(\varkappa) \\ &= \frac{1-\delta}{\mathbb{M}(\delta)}\mathcal{L}_1 + \frac{\delta\mathcal{L}_1}{\mathbb{M}(\delta)\Gamma(\delta)}\int_0^{\varkappa}(\varkappa-\xi)^{\delta-1}d\xi \\ &= \frac{\mathcal{L}_1}{\mathbb{M}(\delta)}\left(1-\delta+\frac{1}{\Gamma(\delta)}\varkappa^{\delta}\right) \end{aligned}$$

By (3.9), we conclude that

$$\underline{\varphi}(\varkappa) = \left({}_{0}^{\mathcal{AB}}\mathcal{I}_{\varkappa}^{\delta}\,\mathcal{L}_1\right)(\varkappa) \le {}_{0}^{\mathcal{AB}}\mathcal{I}_{\varkappa}^{\delta}\,\underline{\Delta}(\varkappa,\underline{\varphi}(\varkappa)).$$

Thus, $\underline{\varphi}(\varkappa)$ is the lower solution of the $\mathcal{ABC}$-type FDF (1.1).

Theorem (3.5) shows that that the $\mathcal{ABC}$-type FDF (1.1). has at least one positive solution $\varphi(\varkappa) \in \mathcal{N}$, which verifies the inequality (3.8). □

Corollary 3.7. *Suppose that* $\mathrm{f} : [0,1] \times [0,+\infty) \to [a,+\infty)$ *is a continuous satisfies*

$$a \le \lim_{\varphi \to +\infty} \mathrm{f}(\varkappa,\varphi) \le +\infty, \tag{3.12}$$

for $\varkappa \in [0,1]$ *and* a *is a positive constant. Then there exists at least a solution* $\varphi(\varkappa)$ *of the* $\mathcal{ABC}$*-type FDF (1.1).*

Proof. By hypothesis (3.12), there are positive constants $\aleph_1, \aleph_2$, such that when $\varphi > \aleph_2$, we have

$$\mathrm{f}(\varkappa,\varphi) \le \aleph_1.$$

Let

$$\mathrm{f}_{\max} = \max_{0\le\varkappa\le1, 0\le\varphi\le\aleph_2} \mathrm{f}(\varkappa,\varphi),$$

Then

$$a \le \mathrm{f}(\varkappa,\varphi) \le \aleph_1 + \mathrm{f}_{\max}, \quad 0 < \varphi < +\infty.$$

In view of Corollary 3.6, the $\mathcal{ABC}$-type FDF (1.1) has at least one positive solution $\varphi(\varkappa) \in \mathrm{X}$ and satisfies

$$\frac{a}{\mathbb{M}(\delta)}\left(1-\delta+\frac{1}{\Gamma(\delta)}\varkappa^{\delta}\right) \le \varphi(\varkappa) \le \frac{\aleph_1+\mathrm{f}_{\max}}{\mathbb{M}(\delta)}\left(1-\delta+\frac{1}{\Gamma(\delta)}\varkappa^{\delta}\right), \quad \text{for } \varkappa \in [0,1].$$

□

The next result is based on Theorem 2.5.

Theorem 3.8. *Assume that* (H_1) *and* (H_2) *hold. If*

$$\left(1-\delta+\frac{1}{\Gamma(\delta)}\right)\frac{\mathrm{L_f}}{\mathbb{M}(\delta)}<1, \tag{3.13}$$

then the $\mathcal{ABC}$*-type FDF (1.1) has a unique positive solution* $\varphi(\varkappa)\in \mathrm{X}$.

Proof. Consider $\mathrm{T}:\mathrm{K}\to\mathrm{K}$ defined by (3.4). Now, we show that T is contraction map in X. Let $\varphi_1,\varphi_2\in\mathrm{X}$ and $\varkappa\in[0,1]$. Then

$$\begin{aligned}
\|\mathrm{T}\varphi_1-\mathrm{T}\varphi_2\| &= \max_{\varkappa\in[0,1]}|(\mathrm{T}\varphi_1)(\varkappa)-(\mathrm{T}\varphi_2)(\varkappa)| \\
&\le \max_{\varkappa\in[0,1]}\left\{\frac{1-\delta}{\mathbb{M}(\delta)}|\mathrm{f}(\varkappa,\varphi_1(\varkappa))-\mathrm{f}(\varkappa,\varphi_2(\varkappa))|\right. \\
&\quad \left.+\frac{\delta}{\mathbb{M}(\delta)\Gamma(\delta)}\int_0^{\varkappa}|\mathrm{f}(\xi,\varphi_1(\xi))-\mathrm{f}(\xi,\varphi_2(\xi))|(\varkappa-\xi)^{\delta-1}d\xi\right\} \\
&\le \max_{\varkappa\in[0,1]}\left\{\frac{1-\delta}{\mathbb{M}(\delta)}\mathrm{L_f}|\varphi_1(\varkappa)-\varphi_2(\varkappa)|\right. \\
&\quad \left.+\frac{\delta}{\mathbb{M}(\delta)\Gamma(\delta)}\mathrm{L_f}\int_0^{\varkappa}|\varphi_1(\xi)-\varphi_2(\xi)|(\varkappa-\xi)^{\delta-1}d\xi\right\} \\
&\le \frac{1-\delta}{\mathbb{M}(\delta)}\mathrm{L_f}\|\varphi_1-\varphi_2\|+\frac{1}{\mathbb{M}(\delta)\Gamma(\delta)}\mathrm{L_f}\|\varphi_1-\varphi_2\| \\
&= \left(1-\delta+\frac{1}{\Gamma(\delta)}\right)\frac{\mathrm{L_f}}{\mathbb{M}(\delta)}\|\varphi_1-\varphi_2\|
\end{aligned}$$

From (3.13), T is contraction map. Due to Theorem 2.5, we can conclude that T has a unique fixed point which is the unique positive solution of (1.1) on $[0,1]$. □

4. EXAMPLES

In this portion, we provide two examples to enlighten our results.

Example 4.1. Consider the following $\mathcal{ABC}$-type FDF

$$\begin{aligned}
{}^{\mathcal{ABC}}_{\ 0}\mathcal{D}^{\frac{1}{2}}_{\varkappa}\,\varphi(\varkappa) &= \tfrac{\varphi(\varkappa)}{10+\sin(\varphi(\varkappa))}, \qquad 0\le\varkappa\le 1, \\
\varphi(0) &= 0,
\end{aligned} \tag{4.1}$$

where $\delta=\frac{1}{2}$, and $\mathrm{f}(\varkappa,\varphi)=\frac{\varphi}{10+\sin(\varphi)}$. It is clear that $\mathrm{f}(\varkappa,\varphi)$ is a nonnegative continuous function with $\mathrm{f}(0,\varphi(0))=0$. For $\varkappa\in[0,1]$ and $\varphi,\vartheta\in[0,\infty)$, we get

$|\mathrm{f}(\varkappa,\varphi)-\mathrm{f}(\varkappa,\vartheta)|\leq\frac{1}{10}|\varphi-\vartheta|$. Here $\mathrm{L_f}=\frac{1}{10}$. Thus (H_1) and (H_2) are satisfied. Moreover, we have

$$\left(1-\delta+\frac{1}{\Gamma(\delta)}\right)\frac{\mathrm{L_f}}{\mathbb{M}(\delta)}=\frac{1+\frac{2}{\sqrt{\pi}}}{20\mathbb{M}(\frac{1}{2})}\approx 0.11<1.$$

So, all the assumptions for Theorem 3.8 hold. Hence, Theorem 3.8 guarantees that (4.1) has a unique positive solution $\varphi(\varkappa)$ on $[0,1]$.

Example 4.2. Consider the following $\mathcal{ABC}$-type FDF

$$\begin{aligned}&{}^{\mathcal{ABC}}_{0}\mathcal{D}^{\frac{1}{3}}_{\varkappa}\,\varphi(\varkappa)=\tfrac{1}{\varkappa+10}\left(\tfrac{\varkappa\varphi(\varkappa)}{1+\varphi(\varkappa)}+10\right),\quad 0\leq\varkappa\leq 1,\\&\varphi(0)=0,\end{aligned}\tag{4.2}$$

Here, $\delta=\frac{1}{3}$, and $\mathrm{f}(\varkappa,\varphi)=\frac{1}{\varkappa+10}\left(\frac{\varkappa\varphi}{1+\varphi}+10\right)$. Since f is continuous and

$$\frac{10}{11}\leq\mathrm{f}(\varkappa,\varphi)\leq 1$$

for all $\varkappa\in[0,1]$, $\varphi\in[0,+\infty)$. Thus, $\mathcal{L}_1=\frac{10}{11}$ and $\mathcal{L}_2=1$. As a result of Corollary 3.6, the $\mathcal{ABC}$-type FDF (4.2) has a positive solution which satisfies $\underline{\varphi}(\varkappa)\leq\varphi(\varkappa)\leq\overline{\varphi}(\varkappa)$ where

$$\overline{\varphi}(\varkappa)=\left(\frac{2}{3}+\frac{1}{\Gamma(\frac{1}{3})}\varkappa^{\delta}\right),$$

$$\underline{\varphi}(\varkappa)=\frac{10}{11}\left(\frac{2}{3}+\frac{1}{\Gamma(\frac{1}{3})}\varkappa^{\delta}\right),$$

are respectively the upper and lower solutions of $\mathcal{ABC}$-type FDF (4.2). Moreover, for all $(\varkappa,\varphi)\in[0,1]\times[0,+\infty)$, $|\mathrm{f}(\varkappa,\varphi)|\leq 1=\rho$. Thus, all assumptions in Theorem 3.5 and Corollary 3.6 are satisfied, hence our results can be applied to the $\mathcal{ABC}$-type FDF.

5. CONCLUSIONS

We have deliberated a class of IVPs for nonlinear FDEs involving the ABC-type derivative. With the control functions, the fixed point techniques of Banach and Schauder, and the upper and lower solutions method, we have proven the existence and uniqueness of positive solutions for the proposed problem. Moreover, two examples to justify the main results have been presented. Further results of the corresponding problems were given as special cases of (1.1). The reported results here are a new and significant contribution to the current literature on AB fractional calculus.

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