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Oscillation and Non-Oscillation of Fourth Order Neutral Distributed Delay Generalized Difference Equation

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Abstract

The oscillatory and non-oscillatory properties of fourth order neutral generalized difference equations with distributed delay are discussed in this paper. Sufficient conditions for the oscillation and asymptotic behavior of non-oscillation of all solutions of the given equation are obtained. To substantiate our findings, adequate examples are presented.

Keywords: Generalized difference operator, Oscillation, Non-oscillation, Delay, Neutral, Fourth order.

AMS Subject Classification [2000]: 39A10, 39A11.

1. INTRODUCTION

In the recent past, the study of oscillation and non oscillation of generalized difference equations gains momentum and is an active area of research. The analysis of asymptotic behaviour of non-oscillatory solutions has already been studied by many researchers. A few authors have studied the behaviour of such solutions for delay difference equations. One can refer [1]-[3],[9]-[11] for the required literature on this topic.

As in differential equation with symmetries, Different schemes can be constructed which preserves the symmetries. In Lie theory, difference equations play a significant role. In Lie theory, discretizion of the continuum equation which preserves the symmetries leads to a class of exact solutions. For an in-depth understanding of this areas one can refer [6]-[8].

Researchers mainly concentrated on second and third order equations with delay. But there is very little literature available for neutral delay difference equations of order higher than three. Very recently, delay difference equations are used to study the stability properties of electrical power systems and for this application one can refer [14]. Many such applications are avialable in Applied Sciences, Mathematical Biology, Engineering and Technology, Economics and so on.

The study of the oscillation of solutions of different types of difference equations such as fractional difference equations, dynamic equations on a time scale, and equation of fractional partial differences are being actively carried out by many scholars at present. Many researchers [5], [12]-[17] have studied the oscillation and non oscillation of the third order difference equations with delay and neutral delay difference equations. There is only little literature available on the study of the oscillation of difference equations of order more than three. On the study of fourth order fractional difference equation one can refer [18]. This motivated us to take up the study with difference equations of order four of the generalized type with distributed delay. For our study, we consider the fourth-order generalized difference equations with distributed delay of the form

$$\Delta_\ell \left(a_3(\xi)\Delta_\ell \left(a_2(\xi)\Delta_\ell \left(a_1(\xi)\Delta_\ell y_1(\xi)\right)\right)\right) + y_2(\xi) = y_3(\xi), \;\; \xi \geq \xi_0, \tag{1}$$

where

$$y_1(\xi) = \sum_{t=a}^{b} p(\xi,t)x(\xi + t\ell - \tau\ell) + x(\xi), \tag{2}$$

$$y_2(\xi) = \sum_{t=c}^{d} q(\xi,t)f(x(\xi + t\ell - \sigma\ell)), \tag{3}$$

$$y_3(\xi) = \sum_{t=g}^{h} r(\xi,t), \tag{4}$$

and a, b, c, d, g, h τ and $\sigma \in \mathbb{Z}$. Here, Δ_ℓ is the forward generalized difference operator for any real sequence $\{x(\xi)\}$ defined by $\Delta_\ell x(\xi) = x(\xi + \ell) - x(\xi)$ for $\xi \in \mathbb{N}_{\xi_0} = \{\xi_0, \xi_0 + \ell, \xi_0 + 2\ell, \cdots\}$, where $\xi_0, \ell \in \mathbb{N} = \{0, 1, 2, \cdots\}$ and satisfies the following assumptions.

(A_1) $\{a_i(\xi)\}$ are real positive sequences such that $\sum\limits_{t=\xi_0}^{\infty} \frac{1}{a_i(t)} = \infty$ or $< \infty$, $i = 1, 2, 3$.

(A_2) $\{p(\xi,t)\}$ is a real sequence for all $\xi \geq \xi_0$ with $0 \leq p(\xi) = \sum\limits_{t=a}^{b} p(\xi,t) \leq p < \infty$.

(A_3) $\{q(\xi,t)\}$ is a positive sequence of real numbers for all $\xi \geq \xi_0$ with $\lim_{\xi\to\infty} q(\xi,t) < \infty$.

(A_4) f is a continuous real-valued function such that $\dfrac{f(y)}{y} \geq M_1$ for $y \neq 0$ and M_1 is a constant.

(A_5) $\{r(\xi,t)\}$ is a positive sequence of real numbers for all $\xi \geq \xi_0$.

(A_6) $\lim_{\xi\to\infty} \phi_i(\xi) = 0$, where $\phi_i(\xi) = \sum_{n=\xi+\ell}^{\infty} \frac{\phi_{i-1}(n)}{a_i(n)}$ with $\phi_0(\xi) \equiv 1$ for $i = 1, 2, 3$.

(A_7) $m_i(\xi) = \left[\frac{\xi-\xi_i-j-\ell}{\ell}\right]$, $\bar{\xi_i} = \xi_i + j$ and $j = \xi - \xi_0 - \left[\frac{\xi-\xi_0}{\ell}\right]\ell$.

By a solution of (1), we mean a real sequence $\{x(\xi)\}$ which is defined for all $\xi \geq \xi_0$, and satisfying (1) for sufficiently large values of ξ. If a nontrivial solution $x(\xi)$ is either eventually positive or negative, it is non-oscillatory; otherwise, it is oscillatory. If all of the solutions to (1) are oscillatory or $\lim_{\xi\to\infty} x(\xi) = 0$, it is called almost oscillatory.

2. PRELIMINARIES

We present some basic lemmas in this section, which will be used in our main results.

Lemma 2.1. *Consider the difference equation*

$$\Delta_\ell z(\xi) - \frac{\Delta_\ell \phi(\xi)}{\phi(\xi)} z(\xi) + \frac{\Delta_\ell \phi(\xi)}{\phi(\xi)} \Phi(\xi) = 0, \tag{5}$$

where $\{\phi(\xi)\}$, $\{\Phi(\xi)\}$ *are real sequences defined for* $\xi \geq K \in \mathbb{N}_\ell(\xi_0)$ *and* $\phi(\xi) > 0$, $\Delta_\ell \phi(\xi) < 0$ *and* $\lim_{\xi\to\infty} \phi(\xi) = 0$. *Equation (5) has the solution* $\{z(\xi)\}$ *defined for all* $\xi \geq K$ *with the condition* $z(K) = 0$. *Then*

$$\lim_{\xi\to\infty} \Phi(\xi) = \infty \quad \Rightarrow \quad \lim_{\xi\to\infty} z(\xi) = \infty. \tag{6}$$

$$\lim_{\xi\to\infty} \Phi(\xi) = -\infty \quad \Rightarrow \quad \lim_{\xi\to\infty} z(\xi) = -\infty. \tag{7}$$

Proof. From the (5), solution $\{z(\xi)\}$ is referenced as

$$z(\xi) = -\phi(\xi) \sum_{n_0=\xi}^{m_0(\xi)} \frac{\Delta_\ell \phi(\bar{\xi_0} + n_0\ell)}{\phi(\bar{\xi_0} + n_0\ell)\phi(\bar{\xi_0} + n_0\ell + \ell)} \Phi(\bar{\xi_0} + n_0\ell), \tag{8}$$

for $\xi \geq K$. If $\lim_{\xi\to\infty} \Phi(\xi) = \pm\infty$ is true, then it is evident that

$$\lim_{\xi\to\infty} \left(- \sum_{n_0=\xi}^{m_0(\xi)} \frac{\Delta_\ell \phi(\xi_0 + n_0\ell)}{\phi(\xi_0 + n_0\ell)\phi(\xi_0 + n_0\ell + \ell)} \Phi(\xi_0 + n_0\ell) \right) = \pm\infty. \tag{9}$$

Hence by Stolz's theorem,

$$\lim_{\xi\to\infty} z(\xi) = \lim_{\xi\to\infty} \left| \frac{\Delta_\ell \left(-\sum_{n_0=\xi}^{m_0(\xi)} \frac{\Delta_\ell \phi(\xi_0+n_0\ell)}{\phi(\xi_0+n_0\ell)\phi(\xi_0+n_0\ell+\ell)} \Phi(\xi_0+n_0\ell) \right)}{\Delta_\ell \left(\frac{1}{\phi(\xi_0+n_0\ell)} \right)} \right| = \lim_{\xi\to\infty} \Phi(\xi) = \pm\infty. \tag{10}$$

and the lemma is proved. □

Lemma 2.2. *Let $\{\phi(\xi)\}$ and $\{v(\xi)\}$ be real sequences defined for $\xi \geq K \in \mathbb{N}_\ell(\xi_0)$. If $\lim_{\xi\to\infty} (v(\xi) + \phi(\xi)\Delta_\ell v(\xi)) \in \mathbb{R}^*$, then $\lim_{\xi\to\infty} v(\xi) \in \mathbb{R}^*$, where $\mathbb{R}^*$ is the extended real line.*

Proof. Consider $z(\xi) = v(\xi) + \phi(\xi)\Delta_\ell v(\xi)$. Then

$$\begin{aligned}\lim_{\xi\to\infty} z(\xi) &= \lim_{\xi\to\infty} (v(\xi) + \phi(\xi)\Delta_\ell v(\xi)) \\ &= \lim_{\xi\to\infty} v(\xi) + \lim_{\xi\to\infty} \phi(\xi) \lim_{\xi\to\infty} v(\xi+\ell) - \lim_{\xi\to\infty} \phi(\xi) \lim_{\xi\to\infty} v(\xi) \\ &= \lim_{\xi\to\infty} v(\xi)\end{aligned}$$

So, $\lim_{\xi\to\infty} z(\xi)$ exists in $\mathbb{R}^*$ implies $\lim_{\xi\to\infty} v(\xi)$ exists in $\mathbb{R}^*$. □

3. MAIN RESULTS

Theorem 3.1. *Let the first part of the assumption (A_1), $(A_2) - (A_5)$ hold, $y_3(\xi) \equiv 0$ and suppose that each of the following conditions $(C_1) - (C_3)$ are true;*

(C_1) $\sum_{n_0=0}^{\infty} \left(\sum_{n_1=0}^{m_0(n_0)} a_3(\bar{\xi_0}+n_1\ell) \left(\sum_{n_2=0}^{m_0(n_1)} a_2(\bar{\xi_0}+n_2\ell) \left(\sum_{n_3=0}^{m_0(n_2)} a_1(\bar{\xi_0}+n_2\ell) \right)\right)\right) \sum_{t=c}^{d} q(\bar{\xi_0}+n_0\ell, t) = \infty.$

(C_2) *If* $\sum_{n_0=0}^{\infty} \sum_{t=c}^{d} q(\bar{\xi_0}+n_0\ell, t) < \infty$ *and* $\sum_{n_0=0}^{\infty} a_3(\bar{\xi_0}+n_1\ell) \sum_{n_1=n_0}^{\infty} \sum_{t=c}^{d} q(\bar{\xi_0}+n_0\ell, t) < \infty,$
then $\sum_{n_0=0}^{\infty} a_2(\bar{\xi_0}+n_0\ell) \left(\sum_{n_1=n_0}^{\infty} a_3(\bar{\xi_0}+n_1\ell) \left(\sum_{n_2=n_1}^{\infty} \sum_{t=c}^{d} q(\bar{\xi_0}+n_2\ell, t) \right)\right) = \infty.$

(C_3) $\sum_{n_0=0}^{\infty} \left(\sum_{n_1=0}^{m_0(n_0)} a_1(\bar{\xi_0}+n_1\ell) \left(\sum_{n_2=0}^{m_0(n_1)} a_2(\bar{\xi_0}+n_2\ell) \right)\right) \sum_{t=c}^{d} q(\bar{\xi_0}+n_0\ell, t) = \infty.$

Then every solution of (1) is oscillatory.

Proof. Let (1) has a non-oscillatory solution $\{x(\xi)\}$. Since, $\{x(\xi)\}$ is a non-oscillatory solution, it can be assumed that $\{y_1(\xi)\} > 0$. However, the proof is similar when $\{y_1(\xi)\} < 0$. Therefore, an integer $\xi_0 \in \mathbb{N}_\ell$ such that $y_1(\xi) > 0$ for all $\xi \geq \xi_0$. Let

$$E_i(\xi) = \begin{cases} y_1(\xi) & i = 0; \\ a_i(\xi)\Delta_\ell E_{i-1}(\xi) & i = 1, 2, 3. \end{cases} \tag{11}$$

Now the system

$$\Delta_\ell E_{i-1}(\xi) = \begin{cases} \frac{E_i(\xi)}{a_i(\xi)} & i = 1, 2, 3; \\ -y_2(\xi) & i = 4. \end{cases} \tag{12}$$

is satisfied. Clearly, $E_3(\xi)$ is non-increasing and is either positive or negative. If there is an integer $\xi_1 \geq \xi_0 \in \mathbb{N}_\ell$ such that $E_3(\xi_1) < 0$, then

$$E_{i-1}(\xi) = E_{i-1}(\bar{\xi_0}) + \sum_{n_0=0}^{m_0(\xi)} \frac{E_i(\bar{\xi_0} + n_0\ell)}{a_i(\bar{\xi_0} + n_0\ell)}, \qquad i = 1, 2, 3. \tag{13}$$

Letting $\xi \to \infty$, and using the assumption (A_1), we have that $E_{i-1}(\xi) \to -\infty$ for $i = 1, 2, 3$, which is a contradiction. Thus, $E_3(\xi) \geq 0$ for all $\xi_0 \leq \xi$, so $\lim_{\xi\to\infty} E_3(\xi) = E_3(\infty)$ exists and $0 \leq E_3(\infty)$. Also, $0 < E_3(\xi_1)$ if $\xi_0 < \xi_1$. Then, $E_3(\xi) = 0$ whenever $\xi_1 < \xi$. Thus, from (12), $\Delta_\ell E_3(\xi) = 0$ and $q(\xi, t) = 0$ whenever $\xi_1 < \xi$. But, this contradicts (C_1), so $0 < E_3(\xi)$ for $\xi_0 < \xi$. Thus $\{E_3(\xi)\}$ is increasing for $\xi_0 \leq \xi$.

Let us consider the other cases.

Suppose $E_2(\xi) < 0$ for $\xi_0 \leq \xi$. Now, $E_2(\infty) \leq 0$; and if $E_2(\infty) < 0$, then, (13) is a contradiction again and so $E_2(\infty) = 0$. Now, for $\xi_0 \leq \xi$, $E_1(\xi)$ is decreasing, and $E_1(\infty) < 0$ is impossible and hence $E_1(\infty) \geq 0$. If $\xi_0 \leq \xi \leq K$, then

$$E_3(K) - E_3(\bar{\xi}) = -\sum_{n_1=0}^{\frac{\xi-\bar{\xi}-\ell}{\ell}} y_2(\bar{\xi} + n_1\ell). \tag{14}$$

So,

$$E_3(\infty) - E_3(\bar{\xi}) = -\sum_{n_1=0}^{\infty} y_2(\bar{\xi} + n_1\ell), \tag{15}$$

which yields

$$E_3(\bar{\xi}) \geq \sum_{n_1=0}^{\infty} \sum_{t=c}^{d} q(\bar{\xi} + n_1\ell, t) f(x(\bar{\xi} + n_1\ell + t\ell - \sigma\ell)). \tag{16}$$

Since $E_1(\xi) > 0$, $E_1(\xi)$ is increasing, so $E_3(\xi) \geq f(x(\bar{\xi_0}+c\ell-\sigma\ell)) \sum_{n_1=0}^{\infty}\sum_{t=c}^{d} q(\bar{\xi}+n_1\ell, t)$ for $\xi_0 \leq \xi$. If $\sum_{n_1=0}^{\infty}\sum_{t=c}^{d} q(\bar{\xi}+n_1\ell, t) < \infty$ fails, we obtained a contradiction. Hence assume $\sum_{n_1=0}^{\infty}\sum_{t=c}^{d} q(\bar{\xi}+n_1\ell, t) < \infty$ holds. Since $E_2(\infty) = 0$, we have

$$E_2(\xi) = -\sum_{n_0=0}^{\infty} \frac{E_3(\bar{\xi}+n_0\ell)}{a_3(\bar{\xi_0}+n_0\ell)}, \qquad \text{for } \xi \geq \xi_0. \tag{17}$$

However, the inequality in the above conclusion shows that if

$$\sum_{n_2=0}^{\infty} \frac{1}{a_3^{\frac{1}{\beta_3}}(\bar{\xi}+n_2\ell)} \sum_{n_1=n_2}^{\infty}\sum_{t=c}^{d} q(\bar{\xi}+n_1\ell, t) < \infty \tag{18}$$

thus fails, which is a contradiction. Therefore, assume

$$\sum_{n_2=0}^{\infty} \frac{1}{a_3^{\frac{1}{\beta_3}}(\bar{\xi}+n_2\ell)} \sum_{n_1=n_2}^{\infty}\sum_{t=c}^{d} q(\bar{\xi}+n_1\ell, t) < \infty \tag{19}$$

holds. If $\xi \geq \xi_0$, then,

$$E_1(\xi) - E_1(\bar{\xi_0}) = \sum_{n_3=0}^{m_0(\xi)} \frac{E_2(\bar{\xi_0}+n_3\ell)}{a_2(\bar{\xi_0}+n_3\ell)} = -\sum_{n_3=0}^{m_0(\xi)} \frac{1}{a_2(\bar{\xi_0}+n_3\ell)} \left(\sum_{n_2=n_3}^{\infty} \frac{E_3(\bar{\xi}+n_2\ell)}{a_3(\bar{\xi_0}+n_2\ell)}\right), \tag{20}$$

and so

$$\begin{aligned}
-E_1(\bar{\xi_0}) &\leq -\sum_{n_3=0}^{m_0(\xi)} \frac{1}{a_2(\bar{\xi_0}+n_3\ell)} \left(\sum_{n_2=n_3}^{\infty} \frac{E_3(\bar{\xi}+n_2\ell)}{a_3(\bar{\xi_0}+n_2\ell)}\right) \\
E_1(\bar{\xi_0}) &\geq \sum_{n_3=0}^{m_0(\xi)} \frac{1}{a_2(\bar{\xi_0}+n_3\ell)} \left(\sum_{n_2=n_3}^{\infty} \frac{E_3(\bar{\xi}+n_2\ell)}{a_3(\bar{\xi_0}+n_2\ell)}\right) \\
&\geq f(E_0(\bar{\xi_0}+c\ell-\sigma\ell)) \sum_{n_3=0}^{m_0(\xi)} \frac{1}{a_2(\bar{\xi_0}+n_3\ell)} \sum_{n_2=n_3}^{\infty} \frac{1}{a_3(\bar{\xi_0}+n_2\ell)} \sum_{n_1=n_2}^{\infty}\sum_{t=c}^{d} q(\bar{\xi}+n_1\ell, t).
\end{aligned}$$

However, this contradicts condition (C_2), and we get through the case $E_2(\xi) < 0$ for $\xi \geq \xi_0$.

Since, $\{E_2(\xi)\}$ is ascending and $E_2(\xi) < 0$ is false, confirms that there is an integer $\xi_1 \in \mathbb{N}_\ell(\xi_0)$ such that $\xi_1 \geq \xi_0$ and $E_2(\xi) > 0$ for all $\xi \geq \xi_1$. Now $\{E_1(\xi)\}$ is increasing for all $\xi \geq \xi_1$. If $E_1(\xi) \leq 0$ for all $\xi \geq \xi_1$, then $\{E_1(\xi)\}$ is bounded. But by condition (C_1) and the Theorem 2 in [19] we can conclude that any bounded solution of (1) is oscillatory. Therefore, there exists an integer $\xi_2 \geq \xi_1$ such that $E_1(\xi) > 0$ for all $\xi \geq \xi_2$. Now if $\xi \geq \xi_2$, then

$$\begin{aligned} E_0(\xi) &= E_0(\bar{\xi_2}) + \sum_{n_0=0}^{m_2(\xi)} \frac{E_1(\bar{\xi_2}+n_0\ell)}{a_1(\bar{\xi_2}+n_0\ell)} \\ &\geq \sum_{n_0=0}^{m_2(\xi)} \frac{E_1(\bar{\xi_2}+n_0\ell)}{a_1(\bar{\xi_2}+n_0\ell)} \\ &= \sum_{n_0=0}^{m_2(\xi)} \frac{1}{a_1(\bar{\xi_2}+n_0\ell)} \left(E_1(\bar{\xi_2}) + \sum_{n_1=n_0}^{m_2(\xi)} \frac{E_2(\bar{\xi_2}+n_1\ell)}{a_2(\bar{\xi_2}+n_1\ell)} \right) \\ &\geq \sum_{n_0=0}^{m_2(\xi)} \frac{1}{a_1(\bar{\xi_2}+n_0\ell)} \left(\sum_{n_1=n_0}^{m_2(\xi)} \frac{E_2(\bar{\xi_2}+n_1\ell)}{a_2(\bar{\xi_2}+n_1\ell)} \right) \\ &\geq E_2(\bar{\xi_2}) \sum_{n_0=0}^{m_2(\xi)} \frac{1}{a_1(\bar{\xi_2}+n_0\ell)} \left(\sum_{n_1=n_0}^{m_2(\xi)} \frac{1}{a_2(\bar{\xi_2}+n_1\ell)} \right). \end{aligned}$$

Also

$$0 < E_3(\xi) = E_3(\bar{\xi_2}) + \sum_{n_0=0}^{m_2(\xi)} \Delta_\ell E_3(\bar{\xi_2}+n_0\ell) = E_3(\bar{\xi_2}) - \sum_{n_0=0}^{m_2(\xi)} y_2(\bar{\xi_2}+n_0\ell). \tag{21}$$

$$\begin{aligned} E_3(\bar{\xi_2}) &\geq \sum_{n_0=0}^{m_2(\xi)} \sum_{t=c}^{d} q(\bar{\xi_2}+n_0\ell, t) f(E_0(\bar{\xi_2}+n_0\ell+t\ell-\sigma\ell)) \\ &\geq f(E_0(\xi)) \sum_{n_0=0}^{m_2(\xi)} \sum_{t=c}^{d} q(\bar{\xi_2}+n_0\ell, t)) \\ &\geq M_0 E_0(\xi) \sum_{n_0=0}^{m_2(\xi)} \sum_{t=c}^{d} q(\bar{\xi_2}+n_0\ell, t)) \\ &\geq M_0 E_2(\bar{\xi_2}) \sum_{n_0=0}^{m_2(\xi)} \sum_{t=c}^{d} q(\bar{\xi_2}+n_0\ell, t)) \left(\sum_{n_0=0}^{m_2(\xi)} \frac{1}{a_1(\bar{\xi_2}+n_0\ell)} \left(\sum_{n_1=n_0}^{m_2(\xi)} \frac{1}{a_2(\bar{\xi_2}+n_1\ell)} \right) \right). \end{aligned} \tag{22}$$

As per Stolz's Theorem [4], we have

$$\lim_{\xi_2\to\infty} \frac{\sum_{n_0=0}^{m_2(\xi)} \frac{1}{a_1(\bar{\xi_2}+n_0\ell)} \left(\sum_{n_1=n_0}^{m_2(\xi)} \frac{1}{a_2(\bar{\xi_2}+n_1\ell)} \right)}{\sum_{n_0=0}^{m_2(\xi)} \frac{1}{a_1(\bar{\xi_0}+n_0\ell)} \left(\sum_{n_1=n_0}^{m_2(\xi)} \frac{1}{a_2(\bar{\xi_0}+n_1\ell)} \right)} = 1 \tag{23}$$

and so condition (C_3) implies the divergence of the summations in (22) as ξ tends to ∞. This contradiction confirms the theorem. $\square$

Theorem 3.2. *Let the first part of the assumption (A_1), $(A_2)-(A_5)$ hold, $y_3(\xi)\equiv 0$. Assume condition (C_3) holds and*

$$\sum_{n_0=0}^{m_2(\xi)}\left(\left(\sum_{t=c}^{d} q(\bar{\xi_2}+n_0\ell,t)\right)\left(\sum_{n_1=0}^{m_1(n_0)}\frac{1}{a_3(\bar{\xi_1}+n_1\ell)}\sum_{n_2=0}^{m_0(n_1)}\frac{1}{a_2(\bar{\xi_0}+n_2\ell)}\right)\right)=\infty, \tag{24}$$

for $0<\beta_3<1$. Then every solution of equation (1) is oscillatory.

Proof. Let (1) has a non-oscillatory solution $\{x(\xi)\}$. We may assume that $\{x(\xi)\}$ is eventually positive. If condition (C_2) holds and $\xi\in\mathbb{N}_\ell(\xi_0)$, then applying summation by parts twice, it yields

$$\begin{aligned}
&\sum_{n_0=0}^{\infty}\frac{1}{a_2(\bar{\xi_0}+n_0\ell)}\left(\sum_{n_1=n_0}^{\infty}\frac{1}{a_3(\bar{\xi_0}+n_1\ell)}\left(\sum_{n_2=n_1}^{\infty}\sum_{t=c}^{d}q(\bar{\xi_0}+n_2\ell,t)\right)\right)\\
&=\left(\sum_{n_0=0}^{m_0(\xi)}\frac{1}{a_2(\bar{\xi_0}+n_0\ell)}\right)\left(\sum_{n_0=m_0(\xi)+\ell}^{\infty}\frac{1}{a_3(\bar{\xi_0}+n_0\ell)}\left(\sum_{n_1=n_0}^{\infty}\sum_{t=c}^{d}q(\bar{\xi_0}+n_1\ell,t)\right)\right)\\
&\quad+\sum_{n_0=0}^{m_0(\xi)}\frac{1}{a_3(\bar{\xi_0}+n_0\ell)}\left(\sum_{n_1=0}^{n_0}\frac{1}{a_2(\bar{\xi_0}+n_1\ell)}\left(\sum_{n_2=n_0}^{\infty}\sum_{t=c}^{d}q(\bar{\xi_0}+n_2\ell,t)\right)\right)\\
&\geq\sum_{n_0=0}^{m_0(\xi)}\frac{1}{a_3(\bar{\xi_0}+n_0\ell)}\left(\sum_{n_1=0}^{n_0}\frac{1}{a_2(\bar{\xi_0}+n_1\ell)}\sum_{n_2=n_0}^{\infty}\left(\sum_{t=c}^{d}q(\bar{\xi_0}+n_2\ell,t)\right)\right)\\
&=\sum_{n_0=0}^{m_0(\xi)}\frac{1}{a_3(\bar{\xi_0}+n_0\ell)}\left(\sum_{n_1=0}^{n_0}\frac{1}{a_2(\bar{\xi_0}+n_1\ell)}\sum_{n_2=n_0}^{\infty}\left(\sum_{t=c}^{d}q(\bar{\xi_0}+n_2\ell,t)\right)\right)\\
&\quad+\sum_{n_0=0}^{m_0(\xi)}\left(\left(\sum_{t=c}^{d}q(\bar{\xi_0}+n_0\ell,t)\right)\left(\sum_{n_1=0}^{n_0}\frac{1}{a_3(\bar{\xi_0}+n_1\ell)}\sum_{n_2=0}^{n_1}\frac{1}{a_2(\bar{\xi_0}+n_2\ell)}\right)\right)\\
&\geq\sum_{n_0=0}^{m_2(\xi)}\left(\left(\sum_{t=c}^{d}q(\bar{\xi_2}+n_0\ell,t)\right)\left(\sum_{n_1=0}^{m_1(n_0)}\frac{1}{a_3(\bar{\xi_1}+n_1\ell)}\sum_{n_2=0}^{m_0(n_1)}\frac{1}{a_2(\bar{\xi_0}+n_2\ell)}\right)\right),
\end{aligned}$$

for $\xi_0<\xi_1<\xi_2\leq\xi$. Thus, (24) implies condition (C_3). Now, assumption (A_1), and two successive implementations of Stolz's Theorem, it follows that

$$\lim_{\xi\to\infty}\frac{\sum\limits_{n_1=0}^{m_1(\xi)}\frac{1}{a_3(\bar{\xi_1}+n_1\ell)}\sum\limits_{n_2=0}^{m_0(n_1)}\frac{1}{a_2(\bar{\xi_0}+n_2\ell)}}{\sum\limits_{n_1=0}^{m_2(\xi)}\frac{1}{a_3(\bar{\xi_2}+n_1\ell)}\sum\limits_{n_2=0}^{m_1(n_1)}\frac{1}{a_2(\bar{\xi_1}+n_2\ell)}\sum\limits_{n_3=0}^{m_0(n_2)}\frac{1}{a_1(\bar{\xi_0}+n_3\ell)}}=0. \tag{25}$$

Hence, there exists an integer $K \in \mathbb{N}_\ell$ such that

$$\sum_{n_1=0}^{m_2(\xi)} \frac{1}{a_3(\bar{\xi_2}+n_1\ell)} \sum_{n_2=0}^{m_1(n_1)} \frac{1}{a_2(\bar{\xi_1}+n_2\ell)} \sum_{n_3=0}^{m_0(n_2)} \frac{1}{a_1(\bar{\xi_0}+n_3\ell)} \geq \sum_{n_1=0}^{m_1(\xi)} \frac{1}{a_3(\bar{\xi_1}+n_1\ell)} \sum_{n_2=0}^{m_0(n_1)} \frac{1}{a_2(\bar{\xi_0}+n_2\ell)} \tag{26}$$

whenever $\xi \geq K$, and we conclude that (24) implies condition (C_1), which proves the theorem. □

Remark 3.3. If $a_3(\xi) \equiv a_1(\xi)$, then (24) is identical with (C_3), implies that every solution of (1) is oscillatory.

Theorem 3.4. *Assuming that the second part of (A_1) and (A_6) hold, let us suppose $\lim_{x\to\infty} f(x) > 0$ and $\limsup_{x\to\infty} f(x) < 0$. If*

$$\sum_{n_0=\xi}^{\infty} \phi(\bar{\xi_0}+n_0\ell) \sum_{t=c}^{d} q(\bar{\xi_0}+n_0\ell, t) = \infty, \tag{27}$$

and

$$\sum_{n_0=\xi}^{\infty} \phi(\bar{\xi_0}+n_0\ell) \sum_{t=g}^{h} |r(\bar{\xi_0}+n_0\ell, t)| < \infty, \tag{28}$$

then, as ξ increases, all non-oscillatory solutions of (1) are bounded and tend to zero as $\xi \to \infty$.

Proof. Let (1) has a non-oscillatory solution $\{x(\xi)\}$ and $x(\xi) > 0$ for $\xi \geq \xi_1 \in \mathbb{N}$. Define

$$z_i(\xi) = \sum_{n_0=0}^{m_1(\xi)} \phi_{3-i}(\bar{\xi_1}+n_0\ell)\Delta_\ell E_{3-i}(\bar{\xi_1}+n_0\ell), \tag{29}$$

for $i = 0, 1, 2, 3.$ we begin by illustrating that $\{x(\xi)\}$ is bounded above. From (1) we get

$$E_3(\xi) - E_3(\xi_1) + \sum_{n_0=0}^{m_1(\xi)} y_2(\bar{\xi_1}+n_0\ell) = \sum_{n_0=0}^{m_1(\xi)} y_3(\bar{\xi_1}+n_0\ell). \tag{30}$$

Since the left-hand side sum is positive and the right-hand side sum is bounded by (28), in this case, there exists a constant M_2 such that

$$E_3(\xi) = a_3(\xi)\left(\Delta_\ell E_2(\xi)\right) \leq M_2, \quad \text{for } \xi \geq \xi_1. \tag{31}$$

Dividing the above inequality by $a_3(\xi)$ and summing from ξ_1 to $\xi - \ell$, we obtain

$$E_2(\xi) - E_2(\xi_1) \leq M_2 \sum_{n_0=0}^{m_1(\xi)} \frac{1}{a_3(\bar{\xi_1}+n_0\ell)}, \quad \text{for } \xi \geq \xi_1. \tag{32}$$

By assumption (A_6) there exists M_3, a constant such that

$$E_2(\xi) = a_2(\xi)\Delta_\ell E_1(\xi) \leq M_3, \quad \text{for } \xi \geq \xi_2 \geq \xi_1. \tag{33}$$

We get the following result by repeatedly applying the above arguments.

$$E_1(\xi) \leq M_4 \quad \text{and} \quad E_0(\xi) \leq M_5, \text{for } \xi \geq \xi_3 \geq \xi_2, \tag{34}$$

where M_4 and M_5 are constants. which implies that $\{x(\xi)\}$ is bounded above for $\xi \geq \xi_1$. Taking summation by parts in (29) we obtain

$$\begin{aligned}
z_{i-1}(\xi) &= \sum_{n_0=0}^{m_1(\xi)} \phi_{4-i}(\bar{\xi_1} + n_0\ell)\Delta_\ell E_{4-i}(\bar{\xi_1} + n_0\ell) \\
&= \phi_{4-i}(\xi)E_{4-i}(\xi) - \phi_{4-i}(\bar{\xi_1})E_{4-i}(\bar{\xi_1}) - \sum_{n_0=0}^{m_1(\xi)} \Delta_\ell\phi_{4-i}(\bar{\xi_1} + n_0\ell)E_{4-i}(\bar{\xi_1} + n_0\ell) \\
&= \frac{\phi_{4-i}(\xi)\phi_{3-i}(\xi)}{\phi_{3-i}(\xi)} a_{4-i}(\xi)\Delta_\ell E_{3-i}(\xi) - \phi_{4-i}(\bar{\xi_1})E_{4-i}(\bar{\xi_1}) \\
&\qquad + \sum_{n_0=0}^{m_1(\xi)} \frac{\phi_{3-i}(\bar{\xi_1} + n_0\ell)}{a_{4-i}(\bar{\xi_1} + n_0\ell)} E_{4-i}(\bar{\xi_1} + n_0\ell) \\
&= -\frac{\phi_{4-i}(\xi)}{\Delta_\ell\phi_{4-i}(\xi)}\Delta_\ell z_i(\xi) + z_i(\xi) - \frac{\phi_{4-i}(\xi)}{\Delta_\ell\phi_{4-i}(\xi)}\phi_{3-i}(\bar{\xi})E_{3-i}(\bar{\xi_1}) - \phi_{4-i}(\bar{\xi_1})E_{4-i}(\bar{\xi_1}).
\end{aligned}$$

This shows that $\{z_i(\xi)\}$ satisfies the difference equation

$$\frac{\phi_{4-i}(\xi)}{\Delta_\ell\phi_{4-i}(\xi)}\Delta_\ell z_i(\xi) + z_i(\xi) + \Phi_i(\xi) = 0, \tag{35}$$

or equivalently,

$$\Delta_\ell z_i(\xi) - \frac{\Delta_\ell\phi_{4-i}(\xi)}{\phi_{4-i}(\xi)} z_i(\xi) + \frac{\Delta_\ell\phi_{4-i}(\xi)}{\phi_{4-i}(\xi)}\Phi_i(\xi) = 0, \tag{36}$$

where $\Phi_i(\xi) = z_{i-1}(\xi) + \phi_{4-i}(\bar{\xi_1})E_{4-i}(\bar{\xi_1}) + \frac{\phi_{4-i}(\xi)}{\Delta_\ell\phi_{4-i}(\xi)}\phi_{3-i}(\bar{\xi})E_{3-i}(\bar{\xi_1})$. Since $z_i(\xi_0) = 0$ by (29) and since $\phi_{4-i}(\xi) > 0$, $\Delta_\ell\phi_{4-i}(\xi) < 0$ and $\phi_{4-i}(\xi)$ is equal to zero as $\xi \to \infty$, by assumption (A_7) we apply Lemma 2.1 to (36) which concludes $z_{i-1}(\xi)$ is equal to $\pm\infty$ as $\xi \to \infty$ implies $z_i(\xi)$ is equal to $\pm\infty$ as $\xi \to \infty$. Henceforth, applying Lemma 2.2 to (35), we conclude that $\lim_{\xi\to\infty} z_i(\xi) \in \mathbb{R}$ whenever $\lim_{\xi\to\infty} z_{i-1}(\xi) \in \mathbb{R}$.
On multiplying either sides of (1) by $\phi_3(\xi)$, and summing from ξ_1 to $\xi - \ell$ we obtain

$$\sum_{n_0=0}^{m_1(\xi)} \phi_3(\bar{\xi_1}+n_0\ell)\delta_\ell E_3(\bar{\xi_1}+n_0\ell) + \sum_{n_0=0}^{m_1(\xi)} \phi_3(\bar{\xi_1}+n_0\ell)y_2(\bar{\xi_1}+n_0\ell) = \sum_{n_0=0}^{m_1(\xi)} \phi_3(\bar{\xi_1}+n_0\ell)y_3(\bar{\xi_1}+n_0\ell). \tag{37}$$

We consider the two cases

$$\sum_{n_0=0}^{m_1(\xi)} \phi_3(\bar{\xi_1}+n_0\ell)y_2(\bar{\xi_1}+n_0\ell) = \pm\infty. \tag{38}$$

Assume that (38) is true. Bacause of (28) the right hand side of (37) tends to a finite limit as $\xi \to \infty$, and thus we can see from (37) that $\lim_{\xi\to\infty} z_0(\xi) = -\infty$. Hence, by Lemma 2.1 applied to (36) with $i = 1$ we have $\lim_{\xi\to\infty} z_1(\xi) = -\infty$. Applying Lemma 2.1 again to (36) with $i = 2$ we find $z_2(\xi)$ is equal to $-\infty$ as $\xi \to \infty$. Again, we conclude from the same argument that $z_3(\xi)$ is equal to $-\infty$ as $\xi \to \infty$ implying that $z_n(\xi)$ is equal to $-\infty$ as $\xi \to \infty$. However, the positivity of z_n contradicts. Hence, (38) is impossible.

We now see in $\xi \to \infty$ as (37) and with (38) we see $z_0(\xi)$ is finite as ξ tends to ∞. As Lemma 2.2 is applied to (35) with $i = 1$, we derive $z_1(\xi)$ exists in $\mathbb{R}^*$ as $\xi \to \infty$. This limit must be finite since $\lim_{\xi\to\infty} z_1(\xi) = -\infty$ would imply $\lim_{\xi\to\infty} z_i(\xi) = -\infty$ which is a contradiction, and $\lim_{\xi\to\infty} z_1(\xi) = \infty$ would imply $\lim_{\xi\to\infty} z_i(\xi) = \infty$ which is a contradiction to the boundedness of z_n. Hence, $\lim_{\xi\to\infty} z_i(\xi)$ exists in $\mathbb{R}^*$. On the other hand, from (27) and (38) we see that $z_i(\xi) = 0$ as $\xi \to \infty$. Thus, we sum up that z_i tends to zero as ξ tends to infinity. □

4. EXAMPLES

Example 4.1. *Consider the generalized fourth-order neutral difference equation with distributed delay*

$$\Delta_\ell\left(\frac{1}{\xi^2}\Delta_\ell^3\left(x(\xi)+\sum_{t=1}^{3} tx(\xi+t\ell-2\ell)\right)\right)+\sum_{t=0}^{2}\frac{4(2\xi^2+2\xi\ell+\ell^2}{\xi^2(\xi+\ell)^2}x(\xi+t\ell-\ell)=0. \tag{39}$$

Here $a_1(\xi) = a_2(\xi) = 1$, $a_3(\xi) = \xi^{-2}$, $p(\xi,t) = t$, $q(\xi,t) = \frac{4(2\xi^2+2\xi\ell+\ell^2)}{\xi^2(\xi+\ell)^2}$, $\tau = 2$, $\sigma = 1$ *and* $f(x(\xi)) = x(\xi)$. *It is easy to verify that all the conditions of Theorem 3.1 are satisfied and hence every solution of equation (39) is oscillatory. In fact* $\{x(\xi)\} = \{(-1)^{[\frac{\xi}{\ell}]}\}$ *is one such oscillatory solution of equation (39).*

Example 4.2. *Consider the generalized fourth-order neutral difference equation with distributed delay*

$$\Delta_\ell\left((\xi+\ell)\Delta_\ell\left(\frac{1}{\xi}\Delta_\ell\left(\xi\Delta_\ell\left(x(\xi)+\sum_{t=1}^{2}\frac{x(\xi+t\ell-\ell)}{t}\right)\right)\right)\right)$$
$$+\frac{4\xi^3+12\xi^2\ell+10\xi\ell^2+\ell^3}{\xi(\xi+\ell)}x(\xi)\left(2+(x(\xi))^2\right)=0. \tag{40}$$

Here $a_1(\xi) = \xi$, $a_2(\xi) = \frac{1}{\xi}$, $a_3(\xi) = \xi + \ell$, $p(\xi, t) = \frac{1}{t}$, $q(\xi, s) = \frac{4\xi^3+12\xi^2\ell+10\xi\ell^2+\ell^3}{\xi(\xi+\ell)}$, $\tau = 1$, $\sigma = 1$ *and* $f(x(\xi)) = x(\xi)(1 + (x(\xi))^2)$. *All conditions of Theorem 3.2 are satisfied and hence every solution of equation (40) is oscillatory.*

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