Advances in Dynamical Systems and Applications (ADSA).
ISSN 0973-5321, Volume 16, Number 2, (2021) pp. 1075-1096

http://www.ripublication.com/adsa.htm

Exact Analytical Solution of Some Volterra Second Kind Integrodifferential Equations by the Numerical Analysis Methods ADM and VIM

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Abstract

In this paper, we recall and show the convergence of the Adomian algorithm applied to the Volterra second kind integrodifferential equation and compare the results obtained with those obtained with the variational iterative method (VIM).

2010 Mathematics Subject Classification :44Axx, 40C10,45I05

Keywords and phrases : Adomian Decomposition Method (ADM), Variatonnal Iterative Method (VIM), Volterra integrodifferential equation second kind, Algorithm VIM and algorithm ADM modified.

1. INTRODUCTION

In this paper, we are interested in the following

$$\begin{cases} \dfrac{d\varphi\left(x\right)}{dx} = f(x) + \lambda \displaystyle\int_{a}^{x} K(x,t)\varphi^{p}\left(t\right)dt \; ; p \in \mathbb{N}^{*} \\ \hline \varphi(a) = \beta \end{cases}$$

second species Volterra integrodifferential equation. In a first step, we study the convergence and uniqueness of the Adomian algorithm under certain conditions, in a second step, we make some numerical applications and finally we compare the solutions obtained with these two methods.

2. RECALL THE ADOMIAN DECOMPOSITIONAL METHOD (ADM) AND VARIATIONAL ITERATIVE METHOD (VIM)

2.1. The Adomian Decompositional Method

Let's consider the functional equation :

$$F\varphi(t) = f(t) \tag{1}$$

where F is a nonlinear operator of a Hilbert space H in H, comprising a linear term $L - R$, R is the linear remainder and a nonlinear term N, $\varphi(t)$ is the unknown function and $f(t)$ a given function in H.

By posing

$$F = L - R - N \tag{2}$$

hence

$$L\varphi(t) = f(t) + R\varphi(t) + N\varphi(t) \tag{3}$$

then applying L^{-1} to (3), we obtain :

$$\varphi(t) = \theta + L^{-1}f(t) + L^{-1}R\varphi(t) + L^{-1}N\varphi(t)\,. \tag{4}$$

The Adomian method consists in looking for the solution of equation (1) in the form of a series :

$$\varphi(t) = \sum_{n=0}^{+\infty} \varphi_n(t)$$

and then decompose the nonlinear $N\varphi(t)$ term into a series :

$$N\varphi(t) = \sum_{n=0}^{+\infty} A_n(t)$$

Where

$$\begin{cases} A_0(t) = N(\varphi_0(t)) \\ A_n(t) = \dfrac{1}{n!}\dfrac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{+\infty} \lambda^i \varphi_i(t)\right)\right]_{\lambda=0};\, n \geq 0 \end{cases}$$

The terms $A_n(t)$ are Adomian $[1-8]$ polynomials that depend exclusively on : $\varphi_0(t)$,$\varphi_1(t)$, ..., $\varphi_n(t)$.

2.1.1 Convergence study of the Adomian method

Consider the following integrodifferential problem :

$$(P):\begin{cases}\dfrac{d\varphi(x)}{dx}=f(x)+\lambda\displaystyle\int_a^x K(x,t)\,\varphi^p(t)\,dt;\lambda>0\\ \varphi(a)=\beta\end{cases}$$

let us determine the canonical form of Adomian :

$$\varphi(x)=\underbrace{\beta+\int_a^x f(z)\,dz}_{g(x)}+\lambda\int_a^x\left(\int_a^z K(z,t)\,\varphi^p(t)\,dt\right)dz$$

let us determine the canonical form of Adomian :

$$\varphi(x)=g(x)+\lambda\int_a^x\left(\int_a^z K(z,t)\,\varphi^p(t)\,dt\right)dz$$

If $p=1$,then we have :

$$\varphi(x)=g(x)+\lambda\int_a^x\left(\int_a^z K(z,t)\,\varphi(t)\,dt\right)dz$$

we get the Adomian agorithm :

$$\begin{cases}\varphi_0(x)=g(x)\\ \varphi_{n+1}(x)=\lambda\displaystyle\int_a^x\left(\int_a^z K(z,t)\,\varphi_n(t)\,dt\right)dz;n\geq 0\end{cases}\tag{5}$$

2.2. The Variational iteration method

Let's Consider the nonlinear differential equation

$$Lu+Nu=g(x)$$

where L and N are linear and nonlinear operators respectively, and g is a given continuous function.

We can construct a correction functional according to the variational iteration method in the form

$$u_{n+1}=u_n+\int_a^x\lambda(s)\left(Lu_n(s)+N\widetilde{u}_n(s)-g(s)\right)ds$$

where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, u_n is the nth approximate solution and $\widetilde{u}_n$ is a restricted variation, which means $\delta\widetilde{u}_n = 0$.

It is clear that the main steps of the He's variational iteration method $[9-12]$ is to determine the Lagrange multiplier $\lambda(s)$.

2.2.1 Study of the convergence of the algorithm ADM and VIM

Convergence of the algorithm ADM

Theorem 1. *The following linear second Volterra integrodifferential equation*

$$(p_1) : \begin{cases} \dfrac{d\varphi(x)}{dx} = f(x) + \lambda \displaystyle\int_a^x K(x,t)\varphi(t)\,dt; \lambda > 0 \\ \varphi(a) = \beta \end{cases}$$

has a unique solution under the following hypotheses :

- $\Omega = [a, T], a \le t \le x \le T$
- $g \in C(\Omega)$
- $K(x,t) \in C(\Omega^2)$.

Proof. We have $g \in C(\Omega)$ and $K(x,t) \in C(\Omega^2)$, then $\exists m > 0$ and $M > 0$ such as $\forall x \in \Omega, |g(x)| \le m$ and $\forall (x,t) \in \Omega^2, |K(x,t)| \le M$,

hence

$$\begin{cases} |\varphi_0(x)| = |g(x)| \le m \\ |\varphi_1(x)| = \lambda \left| \displaystyle\int_a^x \left(\int_a^z K(z,t)\varphi_0(t)\,dt \right) dz \right| \le \dfrac{\left(\sqrt{\dfrac{\lambda M \varepsilon}{T}} |x-a| \right)^2}{2} \\ |\varphi_2(x)| = \lambda \left| \displaystyle\int_a^x \left(\int_a^z K(z,t)\varphi_1(t)\,dt \right) dz \right| \le \dfrac{\left(\sqrt{\dfrac{\lambda M \varepsilon}{T}} |x-a| \right)^4}{4!} \\ \ldots \\ |\varphi_n(x)| = \lambda \left| \displaystyle\int_a^x \left(\int_a^z K(z,t)\varphi_{n-1}(t)\,dt \right) dz \right| \le \dfrac{\left(\sqrt{\dfrac{\lambda M \varepsilon}{T}} |x-a| \right)^{2n}}{(2n)!}; n \ge 1 \end{cases}$$

∎

therefore :

$$\sum_{n=0}^{+\infty}|\varphi_n(x)| \le m + \sum_{n=1}^{+\infty}\frac{\left(\sqrt{\frac{\lambda M\varepsilon}{T}}\,|x-a|\right)^{2n}}{(2n)!} = m - 1 + ch\left(\sqrt{\lambda m M}\,|x-a|\right)$$

so the series

$$\left(\sum_{n=0}^{+\infty}\varphi_n(x)\right)$$

converges absolutely therefore it is convergent. Suppose that the problem (p) admits two different solutions u and φ. For each of the solutions u and φ, we have the following algorithms :

$$\begin{cases} \varphi_0(x) = g(x) \\ \varphi_1(x) = \lambda \int_a^x \left(\int_a^z K(z,t)\,\varphi_0(t)\,dt\right) dz \\ \varphi_2(x) = \lambda \int_a^x \left(\int_a^z K(z,t)\,\varphi_1(t)\,dt\right) dz \\ \ldots \\ \varphi_n(x) = \lambda \int_a^x \left(\int_a^z K(z,t)\,\varphi_{n-1}(t)\,dt\right) dz; n \ge 1 \end{cases}$$

and

$$\begin{cases} u_0(x) = g(x) \\ u_1(x) = \lambda \int_a^x \left(\int_a^z K(z,t)\,u_0(t)\,dt\right) dz \\ u_2(x) = \lambda \int_a^x \left(\int_a^z K(z,t)\,u_1(t)\,dt\right) dz \\ \ldots \\ u_n(x) = \lambda \int_a^x \left(\int_a^z K(z,t)\,u_{n-1}(t)\,dt\right) dz; n \ge 1 \end{cases}$$

Let us consider the difference $w(x) = \varphi(x) - u(x)$. Let's apply the Adomian algorithm to w, we get :

$$\begin{cases} w_0(x) = 0 \\ w_1(x) = \lambda \int_a^x \left(\int_a^z K(z,t)\,(\varphi_0(t) - u_0(t))\,dt\right) dz = 0 \\ w_2(x) = \lambda \int_a^x \left(\int_a^z K(z,t)\,(\varphi_1(t) - u_1(t))\,dt\right) dz = 0 \\ \ldots \\ w_n(x) = \lambda \int_a^x \left(\int_a^z K(z,t)\,(\varphi_{n-1}(t) - u_{n-1}(t))\,dt\right) dz = 0\;; n \ge 1 \end{cases}$$

We get then $\forall x \in \Omega, w(x) = 0$, from where $\forall \in \Omega, \varphi(x) = u(x)$ which contradicts our supposition, therefore $\forall \in \Omega, \varphi(x) = u(x)$ consequently the solution of the problem (p_1) is unique. ■

Theorem 2. *The following nonlinear Volterra second kind integrodifferential equation*

$$(p_2) : \begin{cases} \dfrac{d\varphi(x)}{dx} = f(x) + \lambda \displaystyle\int_a^x K(x,t)\,\varphi^p(t)\,dt; \lambda > 0 \textit{ and } p \geq 2 \\ \varphi(a) = \beta \end{cases}$$

has a unique solution under the following hypotheses :

- $\Omega = [a, T], a \leq t \leq x \leq T < +\infty$
- $g \in C(\Omega)$
- $K(x,t) \in C(\Omega^2)$
- $\forall n \in \mathbb{N}, \forall t \in \Omega, \exists \varepsilon > 0 / a \leq t \leq \varepsilon < T$ and $|A_n(t)| \leq \left(\frac{\varepsilon}{T}\right)^n$.

Proof. We have $g \in C(\Omega)$ and $K(x,t) \in C(\Omega^2)$, then $\exists m > 0$, $M > 0$, $\forall n \in \mathbb{N}, \forall t \in \Omega, \exists \varepsilon > 0 / a \leq t \leq \varepsilon < T$ such as $\forall x \in \Omega, |g(x)| \leq m$, $\forall (x,t) \in \Omega^2, |K(x,t)| \leq M, |A_n(t)| \leq \left(\frac{\varepsilon}{T}\right)^n$. ∎

hence.

Then we have :

$$\varphi(x) = g(x) + \lambda \int_a^x \left(\int_a^z K(z,t)\,\varphi^p(t)\,dt \right) dz$$

and then the following Adomian agorithm :

$$\begin{cases} \varphi_0(x) = g(x) \\ \varphi_{n+1}(x) = \lambda \displaystyle\int_a^x \left(\int_a^z K(x,t)\,A_n(t)\,dt \right) dz; n \geq 0 \end{cases} \tag{6}$$

Hence :

$$\begin{cases} |\varphi_0(x)| = |g(x)| \leq m \\ |\varphi_1(x)| = \lambda \left| \int_a^x \left(\int_a^z K(x,t) A_0(t)\,dt \right) dz \right| \leq \dfrac{\lambda M (x-a)^2}{2} \\ |\varphi_2(x)| = \lambda \left| \int_a^x \left(\int_a^z K(x,t) A_1(t)\,dt \right) dz \right| \leq \lambda M \left(\dfrac{\varepsilon}{T}\right) \dfrac{(x-a)^2}{2} \\ . \\ . \\ . \\ |\varphi_n(x)| = \lambda \left| \int_a^x \left(\int_a^z K(x,t) A_{n-1}(t)\,dt \right) dz \right| \leq \lambda M \left(\dfrac{\varepsilon}{T}\right)^{n-1} \dfrac{(x-a)^2}{2}; n \geq 1 \end{cases}$$

and therefore :

$$\sum_{n=0}^{+\infty} |\varphi_n(x)| \leq m + \lambda M \frac{(x-a)^2}{2} \sum_{n=1}^{+\infty} \left(\frac{\varepsilon}{T}\right)^{n-1} = m + \lambda M \frac{(x-a)^2}{2\left(1-\frac{\varepsilon}{T}\right)}$$

because

$$0 \leq \frac{\varepsilon}{T} < 1$$

thus the series

$$\left(\sum_{n=0}^{+\infty} \varphi_n(x) \right)$$

converges absolutely and thus is convergent.

Suppose that the problem (p) admits two different solutions u and φ. For each of the solutions u and φ, we have the following algorithms :

$$\begin{cases} \varphi_0(x) = g(x) \\ \varphi_1(x) = \lambda \int_a^x \left(\int_a^z K(z,t) \varphi_0(t)\,dt \right) dz \\ \varphi_2(x) = \lambda \int_a^x \left(\int_a^z K(z,t) \varphi_1(t)\,dt \right) dz \\ \ldots \\ \varphi_n(x) = \lambda \int_a^x \left(\int_a^z K(z,t) \varphi_{n-1}(t)\,dt \right) dz; n \geq 1 \end{cases}$$

and

$$\begin{cases} u_0(x) = g(x) \\ u_1(x) = \lambda \int_a^x \left(\int_a^z K(z,t)\, u_0(t)\, dt \right) dz \\ u_2(x) = \lambda \int_a^x \left(\int_a^z K(z,t)\, u_1(t)\, dt \right) dz \\ \ldots \\ u_n(x) = \lambda \int_a^x \left(\int_a^z K(z,t)\, u_{n-1}(t)\, dt \right) dz; n \geq 1 \end{cases}$$

Let us consider the difference $w(x) = \varphi(x) - u(x)$. Let's apply the Adomian algorithm to w, we get :

$$\begin{cases} w_0(x) = 0 \\ w_1(x) = \lambda \int_a^x \left(\int_a^z K(z,t)\,(\varphi_0(t) - u_0(t))\, dt \right) dz = 0 \\ w_2(x) = \lambda \int_a^x \left(\int_a^z K(z,t)\,(\varphi_1(t) - u_1(t))\, dt \right) dz = 0 \\ \ldots \\ w_n(x) = \lambda \int_a^x \left(\int_a^z K(z,t)\,(\varphi_{n-1}(t) - u_{n-1}(t))\, dt \right) dz = 0\,; n \geq 1 \end{cases}$$

We get then $\forall x \in \Omega, w(x) = 0$, from where $\forall \in \Omega, \varphi(x) = u(x)$ which contradicts our supposition, therefore $\forall \in \Omega, \varphi(x) = u(x)$ consequently the solution of the problem (p_1) is unique ■

2.2.2 Convergence of the algoritm VIM

Consider the following integrodifferential problem :

$$(P) : \begin{cases} \dfrac{d\varphi(x)}{dx} = f(x) + \lambda \int_a^x K(x,t)\, \varphi^p(t)\, dt; \lambda > 0 \\ \varphi(a) = \beta \end{cases}$$

We construct a correction functional

$$\varphi_{n+1}(x) = \varphi_n(x) + \int_0^x \lambda(s) \left(\varphi_n'(s) - f(s) - \lambda \int_0^s K(s,t)\, \varphi^p(t)\, dt \right) ds$$

where is a restricted variation, which means $\delta\widetilde{\varphi}_n = 0$.
After identifying the multiplier, we have

$$\varphi_{n+1}(x) = \varphi_n(x) - \int_0^x \left(\varphi_n'(s) - f(s) - \lambda \int_0^s K(s,t)\, \varphi^p(t)\, dt \right) ds \tag{7}$$

If $p = 1$,then we have :

$$\varphi_{n+1}(x) = \varphi_n(x) - \int_0^x \left(\varphi'_n(s) - f(s) - \lambda \int_0^s K(s,t)\varphi(t)\,dt \right) ds$$

we get the VIM agorithm :

$$\begin{cases} \varphi_0(x) = \\ \varphi_{n+1}(x) = \varphi_n(x) - \int_0^x \left(\varphi'_n(s) - f(s) - \lambda \int_0^s K(s,t)\varphi(t)\,dt \right) ds \end{cases} \tag{8}$$

2.2.3 Study of the convergence of the algorithm

Theorem 3. *if $f \in C([0,T])$ and $K \in C(\Omega)$ with $\Omega = [0,T] \times [0,T]$, the sequence $(\varphi_n(x))$ converges to the solution $\varphi(x)$ of the problem*

$$\begin{cases} \dfrac{d\varphi(x)}{dx} = f(x) + \lambda \int_0^x K(x,t)\varphi(t)\,dt \\ \varphi(0) = \beta \end{cases}$$

Proof. By subtracting $\varphi(x)$ from both sides of (8), the equation can be rewritten as :

$$\begin{cases} \varphi_{n+1}(x) - \varphi(x) &= \varphi_n(x) - \varphi(x) - \int_0^x \left(\varphi'_n(s) - f(s) - \lambda \int_0^s K(s,t)\varphi(t)\,dt \right) ds \\ &= \varphi_n(x) - \varphi(x) - \int_0^x \varphi'_n(s) - \varphi'(s) - f(s) + \varphi'(s) - \\ & \lambda \int_0^s K(s,t)(\varphi_n(t) - \varphi(t))\,dt - \lambda \int_0^s K(s,t)\varphi(t)\,dtds \\ &= \varphi_n(x) - \varphi(x) - \int_0^x (\varphi'_n(s) - \varphi'(s))\,ds - \int_0^x \varphi'(s) - f(s)\,dt - \\ & \lambda \int_0^s K(s,t)(\varphi_n(t) - \varphi(t)) - \lambda \int_0^s K(s,t)\varphi(t)\,dtds \\ &= \varphi_n(x) - \varphi(x) - (\varphi_n(x) - \varphi(x) - (\varphi_n(0) - \varphi(0))) \\ & - \int_0^x \varphi'(s) - f(s) - \\ & \lambda \int_0^s K(s,t)\varphi(t)\,dt - \lambda \int_0^s K(s,t)(\varphi_n(t) - \varphi(t))\,dtds \\ &= -\int_0^x \left(-\lambda \int_0^s K(s,t)(\varphi_n(t) - \varphi(t))\,dt \right) ds \end{cases}$$

∎

because $\varphi'(s) - f(s) - \lambda \int_0^s K(s,t)\varphi(t)\,dt = 0$.

by posing $E_n(x) = \varphi_n(x) - \varphi(x)$, we have

$$E_{n+1}(x) = \int_0^x \left(\lambda \int_0^s K(s,t)(\varphi_n(t) - \varphi(t))\, dt \right) ds$$

then

$$|E_{n+1}(x)| = \lambda \left| \int_0^x \left(\int_0^s K(s,t) E_n(t)\, dt \right) ds \right|$$

We have $K(x,t) \in C(\Omega)$, then $\exists M > 0$ such as $\forall (x,t) \in \Omega, |K(x,t)| \leq M$. So

$$\begin{aligned} |E_{n+1}(x)| &\leq \lambda M \left| \int_0^x \left(\int_0^s E_n(t)\, dt \right) ds \right| \\ &\leq \lambda M \int_0^x \left(\int_0^s |E_n(t)|\, dt \right) ds \end{aligned}$$

We have successively

$$\begin{aligned} |E_1(x)| &\leq \lambda M \int_0^x \left(\int_0^s |E_0(t)|\, dt \right) ds \\ &\leq \lambda M \int_0^x \left(\|E_0(t)\|_\infty \int_0^s dt \right) ds \\ &= \lambda M \int_0^x (\|E_0(t)\|_\infty s)\, ds = \lambda M \|E_0(t)\|_\infty \frac{1}{2} x^2 \end{aligned}$$

$$|E_2(x)| \leq (\lambda M)^2 \|E_0(t)\|_\infty \int_0^x \left(\int_0^s \frac{1}{2} t^2 dt \right) ds = (\lambda M)^2 \|E_0(t)\|_\infty \frac{1}{4!} x^4$$

$$|E_3(x)| \leq (\lambda M)^3 \|E_0(t)\|_\infty \int_0^x \left(\int_0^s \frac{1}{24} t^4 dt \right) ds = (\lambda M)^3 \|E_0(t)\|_\infty \frac{1}{6!} x^6$$

...

$$|E_n(x)| \leq (\lambda M)^n \|E_0(t)\|_\infty \frac{1}{(2n)!} x^{2n}$$

it exist K such as $\|E_0(t)\|_\infty \leq K$. So

$$|E_n(x)| \leq K (\lambda M)^n \frac{1}{(2n)!} x^{2n}$$

The sequence $\left(K (\lambda M)^n \frac{1}{(2n)!} x^{2n} \right)$ converges uniformly to 0 and thus it follows that $|E_n(x)| \to 0$, which means $(\varphi_n(x))$ converges to $\varphi(x)$ ■

Theorem 4. *if* $f \in C([0,T])$ *and* $K \in C(\Omega)$ *woth* $\Omega = [0,T] \times [0,T]$, *the sequence* $(\varphi_n(x))$ *converges to the solution* $\varphi(x)$ *of the problem*

$$\begin{cases} \dfrac{d\varphi(x)}{dx} = f(x) + \lambda \displaystyle\int_0^x K(x,t)\varphi^p(t)\,dt \\ \varphi(0) = \beta \end{cases}$$

By subtracting $\varphi(x)$ from both sides of (7), the equation can be rewritten as

$$\begin{aligned}
\varphi_{n+1}(x) - \varphi(x) &= \varphi_n(x) - \varphi(x) - \int_0^x \left(\varphi_n'(s) - f(s) - \lambda \int_0^s K(s,t)\varphi^p(t)\,dt \right) ds \\
&= \varphi_n(x) - \varphi(x) - \int_0^x \big(\varphi_n'(s) - \varphi'(s) - f(s) + \varphi'(s) \\
&\quad -\lambda \int_0^s K(s,t)(\varphi_n^p(t) - \varphi^p(t))\,dt \\
&\quad -\lambda \int_0^s K(s,t)\varphi^p(t)\,dt \Big) ds \\
&= \varphi_n(x) - \varphi(x) - \int_0^x (\varphi_n'(s) - \varphi'(s))\,ds - \\
&\quad \int_0^x \Big(\varphi'(s) - f(s) - \lambda \int_0^s K(s,t)(\varphi_n^p(t) - \varphi^p(t))\,dt \\
&\quad -\lambda \int_0^s K(s,t)\varphi^p(t)\,dt \Big) ds \\
&= \varphi_n(x) - \varphi(x) - (\varphi_n(x) - \varphi(x) - (\varphi_n(0) - \varphi(0))) \\
&\quad - \int_0^x \Big(\varphi'(s) - f(s) - \lambda \int_0^s K(s,t)\varphi^p(t)\,dt \\
&\quad -\lambda \int_0^s K(s,t)(\varphi_n^p(t) - \varphi^p(t))\,dt \Big) ds \\
&= -\int_0^x \left(-\lambda \int_0^s K(s,t)(\varphi_n^p(t) - \varphi^p(t))\,dt \right) ds
\end{aligned}$$

cause $\varphi'(s) - f(s) - \lambda \displaystyle\int_0^s K(s,t)\varphi^p(t)\,dt = 0$.

By posing $E_n(x) = \varphi_n(x) - \varphi(x)$, we have

$$E_{n+1}(x) = \int_0^x \left(\lambda \int_0^s K(s,t)(\varphi_n^p(t) - \varphi^p(t))\,dt \right) ds$$

then

$$\begin{aligned}
|E_{n+1}(x)| &= \lambda \left| \int_0^x \left(\int_0^s K(s,t)(\varphi_n^p(t) - \varphi^p(t))\,dt \right) ds \right| \\
&\le \lambda M \left| \int_0^x \left(\int_0^s (\varphi_n^p(t) - \varphi^p(t))\,dt \right) ds \right|
\end{aligned}$$

φ being continuous on $[0,T]$, then φ is bounded and φ^k is too for any $2 \le k \le p$. There is therefore Q_k such us $\left|\varphi^k(x)\right| \le Q_k$ for all $x \in [0,T]$.

The same, it exists M_k such us $\left|\varphi_n^k(x)\right| \le M_k$ for all $x \in [0,T]$.
We have

$$\begin{aligned}
\varphi_n^p(t) - \varphi^p(t) &= (\varphi_n(t) - \varphi(t))\left(\varphi_n^{p-1}(t) + \varphi_n^{p-2}(t)\varphi(t) + \cdots + \varphi_n(t)\varphi^{p-2}(t) + \varphi^{p-1}(t)\right) \\
&\Rightarrow \left|\varphi_n^p(t) - \varphi^p(t)\right| = \left|\varphi_n(t) - \varphi(t)\right| \times \\
&\quad \left|\varphi_n^{p-1}(t) + \varphi_n^{p-2}(t)\varphi(t) + \cdots + \varphi_n(t)\varphi^{p-2}(t) + \varphi^{p-1}(t)\right| \\
&\le \left|\varphi_n(t) - \varphi(t)\right| \times \left(\left|\varphi_n^{p-1}(t)\right| + \left|\varphi_n^{p-2}(t)\varphi(t)\right| + \cdots + \right. \\
&\quad \left.\left|\varphi_n(t)\varphi^{p-2}(t)\right| + \left|\varphi^{p-1}(t)\right|\right)
\end{aligned}$$

Let $Q = \max\limits_{1\le k\le p}\left(\varphi_n^{p-k}(t)\varphi^{k-1}(t)\right)$ then

$$\left|\varphi_n^p(t) - \varphi^p(t)\right| \le pQ\left|\varphi_n(t) - \varphi(t)\right|$$

We have $K(x,t) \in C(\Omega)$, then $\exists M > 0$ such as $\forall (x,t) \in \Omega, |K(x,t)| \le M$. therefore

$$\begin{aligned}
|E_{n+1}(x)| &= \lambda\left|\int_0^x\left(\int_0^s K(s,t)E_n(t)\,dt\right)ds\right| \\
&\le \lambda MpQ\left|\int_0^x\left(\int_0^s E_n(t)\,dt\right)ds\right| \\
&\le \lambda MpQ\int_0^x\left(\int_0^s |E_n(t)|\,dt\right)ds
\end{aligned}$$

Let $A = \lambda MpQ$, we have successively

$$\begin{aligned}
|E_1(x)| &\le A\int_0^x\left(\int_0^s |E_0(t)|\,dt\right)ds \le A\int_0^x\left(\|E_0(t)\|_\infty \int_0^s dt\right)ds \\
&= A\int_0^x\left(\|E_0(t)\|_\infty\, s\right)ds = AE_0(t)_\infty \frac{1}{2}x^2
\end{aligned}$$

$$|E_2(x)| \le (A)^2\|E_0(t)\|_\infty \int_0^x\left(\int_0^s \frac{1}{2}t^2dt\right)ds = (A)^2\|E_0(t)\|_\infty \frac{1}{4!}x^4$$

$$|E_3(x)| \le (A)^3\|E_0(t)\|_\infty \int_0^x\left(\int_0^s \frac{1}{24}t^4dt\right)ds = (A)^3\|E_0(t)\|_\infty \frac{1}{6!}x^6$$

...

$$|E_n(x)| \le (A)^n\|E_0(t)\|_\infty \frac{1}{(2n)!}x^{2n}$$

It exists K such us $\|E_0(t)\|_\infty \leq K$. So

$$|E_n(x)| \leq K(A)^n \frac{1}{(2n)!}x^{2n}$$

The sequence $\left(K(A)^n \frac{1}{(2n)!}x^{2n}\right)$ converges uniformly to 0 and thus it follows that $|E_n(x)| \to 0$, which means $(\varphi_n(x))$ converges to $\varphi(x)$ ■

3. APPLICATIONS

3.0.1 Applications 1: algorithm ADM modified

Exemple 1 Let's consider the following integrodifferential equation :

$$(h_1):\begin{cases} \dfrac{d\varphi(x)}{dx} = e^x - \dfrac{1}{6}e^x\left(e^{2x}-1\right) + \dfrac{1}{3}\displaystyle\int_0^x e^{x-t}\varphi^3(t)\,dt \\ \varphi(0) = 1 \end{cases}$$

we obtain the canonical Adomian form associated to the above problem (h_1) :

$$\varphi(x) = \varphi(0) - \frac{1}{18}e^{3x} + \frac{7}{6}e^x - \frac{10}{9} + \frac{1}{3}\int_0^x\left(\int_0^z e^{z-t}\varphi^3(t)\,dt\right)dz$$

or else :

$$\varphi(x) = e^x - \frac{1}{18}e^{3x} + \frac{1}{6}e^x - \frac{1}{9} + \frac{1}{3}\int_0^x\left(\int_0^z e^{z-t}\varphi^3(t)\,dt\right)dz$$

By using the modified Adomian algorithm :

$$\begin{cases} \varphi_0(x) = e^x \\ \varphi_1(x) = -\frac{1}{18}e^{3x} + \frac{1}{6}e^x - \frac{1}{9} + \dfrac{1}{3}\displaystyle\int_0^x\left(\int_0^z e^{z-t}A_0(t)\,dt\right)dz \\ \varphi_2(x) = \dfrac{1}{3}\displaystyle\int_0^x\left(\int_0^z e^{z-t}A_1(t)\,dt\right)dz \\ \varphi_3(x) = \dfrac{1}{3}\displaystyle\int_0^x\left(\int_0^z e^{z-t}A_2(t)\,dt\right)dz \\ . \\ . \\ . \\ \varphi_{n+1}(x) = \dfrac{1}{3}\displaystyle\int_0^x\left(\int_0^z e^{z-t}A_n(t)\,dt\right)dz\ ;\ n \geq 1 \end{cases}$$

Let's calculate the Adomian polynomials $A_0(x), A_1(x), A_2(x), A_3(x), ...$ and solutions : $\varphi_0(x), \varphi_1(x), \varphi_2(x), \varphi_3(x), ...$

$$\begin{cases} \varphi_0(x) = e^x \Rightarrow A_0(x) = \varphi_0^3(x) \\ \varphi_1(x) = 0 \Rightarrow A_1(x) = 3\varphi_1(x)\varphi_0^2(x) = 0 \\ \varphi_2(x) = 0 \Rightarrow A_2(x) = 3\varphi_0^2(x)\varphi_2(x) + 3\varphi_0(x)\varphi_1^2(x) = 0 \\ \varphi_3(x) = 0 \Rightarrow A_3(x) = 3\varphi_0^2(x)\varphi_3(x) + 6\varphi_0(x)\varphi_1(x)\varphi_2(x) + \varphi_1^3(x) = 0 \\ ... \\ \varphi_n(x) = 0 \Rightarrow A_n(x) = 0, n \geq 1 \end{cases}$$

The exact solution of the problem (h_1) is therefore :

$$\varphi(x) = \varphi_0(x) + \varphi_1(x) + \varphi_2(x) + ... = e^x.$$

Exemple 2 Let's consider the following problem (h_2):

$$(h_2) : \begin{cases} \dfrac{d\varphi(x)}{dx} = 1 - 2x - \frac{1}{90}x^4\left(2x^2 - 6x + 5\right) + \dfrac{2}{3}\displaystyle\int_0^x (x-t)\varphi^2(t)\,dt \\ \varphi(0) = 0 \end{cases}$$

Let's determine the canonical form of Adomian, we get :

$$\varphi(x) = \varphi(0) - \frac{1}{315}x^7 + \frac{1}{90}x^6 - \frac{1}{90}x^5 - x^2 + x + \frac{2}{3}\int_0^x \left(\int_0^z (z-t)\varphi^2(t)\,dt\right)dz$$

or

$$\varphi(x) = -x^2 + x - \frac{1}{315}x^7 + \frac{1}{90}x^6 - \frac{1}{90}x^5 + \frac{2}{3}\int_0^x \left(\int_0^z (z-t)\varphi^2(t)\,dt\right)dz.$$

Using the modified Adomian algoritm, we obtain the following Algorithm :

$$\begin{cases} \varphi_0(x) = x - x^2 \\ \varphi_1(x) = -\dfrac{1}{315}x^7 + \dfrac{1}{90}x^6 - \dfrac{1}{90}x^5 + \dfrac{2}{3}\displaystyle\int_0^x \left(\int_0^z (z-t)A_0(t)\,dt\right)dz \\ \varphi_2(x) = \dfrac{2}{3}\displaystyle\int_0^x \left(\int_0^z (z-t)A_1(t)\,dt\right)dz \\ ... \\ \varphi_{n+1}(x) = \dfrac{2}{3}\displaystyle\int_0^x \left(\int_0^z (z-t)A_n(t)\,dt\right)dz \; ; n \geq 1 \end{cases}$$

we obtain :

$$\begin{cases} \varphi_0(x) = x - x^2 \Rightarrow A_0(x) = \varphi_0^2(x) = (x - x^2)^2 \\ \varphi_1(x) = 0 \Rightarrow A_1(x) = 2\varphi_1(x)\varphi_0(x) = 0 \\ \varphi_2(x) = 0 \Rightarrow A_2(x) = 2\varphi_0(x)\varphi_2(x) + \varphi_i^2(x) = 0 \\ \varphi_3(x) = 0 \Rightarrow A_3(x) = 2\varphi_1(x)\varphi_2(x) + 2\varphi_0(x)\varphi_3(x) = 0 \\ ... \\ \varphi_n(x) = 0 \Rightarrow A_n(x) = 0; n \geq 1 \end{cases}$$

The exact solution to the problem (h_2) is :

$$\varphi(x) = \sum_{n=0}^{+\infty} \varphi_n(x) = \varphi_0(x) + \varphi_1(x) + \varphi_2(x) + ... = \varphi_0(x) = x - x^2.$$

3.0.2 Exemple 3

Let's consider the following problem (h_3) :

$$(h_3) : \begin{cases} \dfrac{d\varphi(x)}{dx} = -\sin x - \frac{1}{5}x \sin x + \dfrac{2}{5}\displaystyle\int_0^x \sin(x-t)\varphi(t)dt \\ \varphi(0) = 1 \end{cases}$$

Let's determine the canonical form of Adomian, we get :

$$\varphi(x) = 1 - \int_0^x \left(\sin z + \frac{1}{5} z \sin z \right) dz + \frac{2}{5} \int_0^x \left(\int_0^z \sin(z-t)\varphi(t)dt \right) dz$$

or

$$\varphi(x) = \cos x - \frac{1}{5}\sin x + \frac{1}{5} x \cos x + \frac{2}{5} \int_0^x \left(\int_0^z \sin(z-t)\varphi(t)\,dt \right) dz.$$

Using the modified Adomian algoritm, we obtain the following Algorithm :

$$\begin{cases} \varphi_0(x) = \cos x \\ \varphi_1(x) = -\dfrac{1}{5}\sin x + \frac{1}{5} x\cos x + \dfrac{2}{5}\displaystyle\int_0^x \left(\int_0^z \sin(z-t)\varphi_0(t)\,dt \right) dz = 0 \\ \varphi_2(x) = \dfrac{2}{5}\displaystyle\int_0^x \left(\int_0^z \sin(z-t)\varphi_1(t)\,dt \right) dz = 0 \\ ... \\ \varphi_{n+1}(x) = \dfrac{2}{5}\displaystyle\int_0^x \left(\int_0^z (z-t)\varphi_n(t)\,dt \right) dz = 0\,;\, n \geq 1 \end{cases}$$

The exact solution to the problem (h_3) is :

$$\varphi(x) = \sum_{n=0}^{+\infty} \varphi_n(x) = \varphi_0(x) + \varphi_1(x) + \varphi_2(x) + ... = \varphi_0(x) = \cos x.$$

3.0.3 Exemple 4

Let's consider the following nonlinear problem (h_4) :

$$(h_4) : \begin{cases} \dfrac{d\varphi(x)}{dx} = \cos x - \dfrac{1}{6}\sin x + \dfrac{1}{12}\sin 2x + \dfrac{1}{4}\displaystyle\int_0^x \cos(x-t)\varphi^2(t)dt \\ \varphi(0) = 0 \end{cases}$$

Let's determine the canonical form of Adomian, we get :

$$\varphi(x) = \frac{1}{6}\cos x + \sin x - \frac{1}{24}\cos 2x - \frac{1}{8} + \frac{1}{4}\int_0^x \left(\int_0^z \cos(z-t)\varphi^2(t)dt\right)dz$$

Using the modified Adomian algoritm, we obtain the following Algorithm :

$$\begin{cases} \varphi_0(x) = \sin x \\ \varphi_1(x) = \dfrac{1}{6}\cos x - \dfrac{1}{24}\cos 2x - \dfrac{1}{8} + \dfrac{1}{4}\displaystyle\int_0^x \left(\int_0^z \cos(z-t)A_0(t)dt\right)dz \\ \varphi_2(x) = \dfrac{1}{4}\displaystyle\int_0^x \left(\int_0^z (z-t)A_1(t)\,dt\right)dz \\ ... \\ \varphi_{n+1}(x) = \dfrac{1}{4}\displaystyle\int_0^x \left(\int_0^z (z-t)A_n(t)\,dt\right)dz \ ; n \geq 1 \end{cases}$$

we obtain :

$$\begin{cases} \varphi_0(x) = \sin x \Rightarrow A_0(x) = \varphi_0^2(x) = \sin^2 x \\ \varphi_1(x) = 0 \Rightarrow A_1(x) = 2\varphi_1(x)\varphi_0(x) = 0 \\ \varphi_2(x) = 0 \Rightarrow A_2(x) = 2\varphi_0(x)\varphi_2(x) + \varphi_i^2(x) = 0 \\ \varphi_3(x) = 0 \Rightarrow A_3(x) = 2\varphi_1(x)\varphi_2(x) + 2\varphi_0(x)\varphi_3(x) = 0 \\ ... \\ \varphi_n(x) = 0 \Rightarrow A_n(x) = 0; n \geq 1 \end{cases}$$

The exact solution to the problem (h_4) is :

$$\varphi(x) = \sum_{n=0}^{+\infty} \varphi_n(x) = \varphi_0(x) + \varphi_1(x) + \varphi_2(x) + ... = \varphi_0(x) = \sin x.$$

4. APPLICATIONS 2: VIM

Example 1 Let's consider the following integrodifferential equation :

$$(h_1):\begin{cases} \dfrac{d\varphi(x)}{dx} = e^x - \dfrac{1}{6}e^x\left(e^{2x}-1\right) + \dfrac{1}{3}\displaystyle\int_0^x e^{x-t}\varphi^3(t)\,dt \\ \varphi(0) = 1 \end{cases}$$

The correction functional is:

$$\varphi_{n+1}(x) = \varphi_n(x) + \int_0^x \lambda(s)\left(\frac{d\varphi_n(s)}{ds} - e^s + \frac{1}{6}e^s\left(e^{2s}-1\right) - \frac{1}{3}\int_0^s e^{s-t}\varphi_n^3(t)\,dt\right)ds$$

After calculation, we find $\lambda = -1$. So we get the following variational iteration formula:

$$\begin{cases} \varphi_{n+1}(x) = \varphi_n(x) - \displaystyle\int_0^x \left(\frac{d\varphi_n(s)}{ds} - e^s + \tfrac{1}{6}e^s\left(e^{2s}-1\right) - \frac{1}{3}\int_0^s e^{s-t}\varphi_n^3(t)\,dt\right)ds \\ \varphi_0(x) = \varphi(0) = 1 \end{cases}$$

And we obtain :

$$\begin{cases} \varphi_0(x) &=& 1 \\ \varphi_1(x) &=& \varphi_0(x) - \displaystyle\int_0^x \left(\varphi_0'(s) - e^s + \tfrac{1}{6}e^s\left(e^{2s}-1\right) - \frac{1}{3}\int_0^s e^{s-t}\varphi_0^3(t)\,dt\right)ds = e^x \\ \varphi_2(x) &=& \varphi_1(x) - \displaystyle\int_0^x \left(\varphi_0'(s) - e^s + \tfrac{1}{6}e^s\left(e^{2s}-1\right) - \frac{1}{3}\int_0^s e^{s-t}\varphi_1^3(t)\,dt\right)ds = e^x \\ . & . & . \\ \varphi_n(x) &=& \varphi_{n-1}(x) - \displaystyle\int_0^x \left(\varphi_{n-1}'(s) - e^s + \tfrac{1}{6}e^s\left(e^{2s}-1\right) - \frac{1}{3}\int_0^s e^{s-t}\varphi_{n-1}^3(t)\,dt\right)ds = e^x \end{cases}$$

The exact solution of the problem is $\varphi(x) = \lim\limits_{n\to+\infty}\varphi_n(x) = e^x$.

Example 2 Let's consider the following problem (h_2):

$$(h_2):\begin{cases} \dfrac{d\varphi(x)}{dx} = 1 - 2x - \tfrac{1}{90}x^4\left(2x^2-6x+5\right) + \dfrac{2}{3}\displaystyle\int_0^x (x-t)\,\varphi^2(t)\,dt \\ \varphi(0) = 0 \end{cases}$$

The correction functional is:

$$\varphi_{n+1}(x) = \varphi_n(x)$$
$$+\int_0^x \lambda(s)\left(\frac{d\varphi_n(s)}{ds} - 1 + 2s + \frac{1}{90}s^4\left(2s^2-6s+5\right) - \frac{2}{3}\int_0^x (s-t)\,\varphi_n^2(t)\,dt\right)ds$$

After calculation, we find $\lambda = -1$ and we have the algorithm:

$$\begin{cases} \varphi_{n+1} = \varphi_n - \displaystyle\int_0^x \left(\frac{d\varphi_n(s)}{ds} - 1 + 2s + \tfrac{1}{90}s^4\left(2s^2 - 6s + 5\right) - \frac{2}{3}\int_0^s (s-t)\,\varphi_n^2(t)\,dt \right) ds \\ \varphi_0(x) = \varphi(0) = 0 \end{cases}$$

And we obtain :

$$\begin{cases} \varphi_0(x) &= 0 \\ \varphi_1(x) &= \varphi_0(x) - \displaystyle\int_0^x \left(\varphi_0'(s) - 1 + 2s + \tfrac{1}{90}s^4\left(2s^2 - 6s + 5\right) - \frac{2}{3}\int_0^s (s-t)\,\varphi_0^2(t)\,dt \right) ds = x - x^2 \\ \varphi_2(x) &= \varphi_1(x) - \displaystyle\int_0^x \left(\varphi_1'(s) - 1 + 2s + \tfrac{1}{90}s^4\left(2s^2 - 6s + 5\right) - \frac{2}{3}\int_0^s (s-t)\,\varphi_1^2(t)\,dt \right) ds = x - x^2 \\ . &\ . &\ . \\ \varphi_n(x) &= \varphi_{n-1}(x) - \displaystyle\int_0^x \left(\varphi_{n-1}'(s) - 1 + 2s + \tfrac{1}{90}s^4\left(2s^2 - 6s + 5\right) - \frac{2}{3}\int_0^s (s-t)\,\varphi_{n-1}^2(t)\,dt \right) ds = x - x^2 \end{cases}$$

The exact solution of the problem is $\varphi(x) = \lim_{n\to+\infty} \varphi_n(x) = x - x^2$.

4.0.1 Example 3

Let's consider the following problem (h_3) :

$$(h_3) : \begin{cases} \dfrac{d\varphi(x)}{dx} = -\sin x - \tfrac{1}{5}x\sin x + \dfrac{2}{5}\displaystyle\int_0^x \sin(x-t)\varphi(t)dt \\ \varphi(0) = 1 \end{cases}$$

The correction functional is:

$$\varphi_{n+1}(x) = \varphi_n(x) + \int_0^x \lambda(s) \left(\frac{d\varphi(s)}{ds} + \sin s + \frac{1}{5}s\sin s - \frac{2}{5}\int_0^x \sin(s-t)\,\varphi(t)\,dt \right) ds$$

After calculation, we find $\lambda = -1$. So we get the following variational iteration formula:

$$\begin{cases} \varphi_{n+1}(x) = \varphi_n(x) - \displaystyle\int_0^x \left(\frac{d\varphi(s)}{ds} + \sin s + \tfrac{1}{5}s\sin s - \frac{2}{5}\int_0^x \sin(s-t)\,\varphi(t)\,dt \right) ds \\ \varphi_0(x) \end{cases}$$

The approximate solutions $\varphi_n(x)$ are obtained iteratively by substituting $\varphi_0(x) = \cos x$ which satisfies the initial condition.

Some approximate solutions are listed below,

$$\begin{cases} \varphi_0(x) & = & \cos x \\ \varphi_1(x) & = & \varphi_0(x) - \int_0^x \left(\varphi_0'(s) + \sin s + \frac{1}{5} s \sin s - \frac{2}{5} \int_0^x \sin(s-t)\,\varphi_0(t)\,dt \right) ds = \cos x \\ \varphi_2(x) & = & \varphi_1(x) - \int_0^x \left(\varphi_1'(s) + \sin s + \frac{1}{5} s \sin s - \frac{2}{5} \int_0^x \sin(s-t)\,\varphi_1(t)\,dt \right) ds = \cos x \\ . & . & . \\ \varphi_n(x) & = & \varphi_{n-1}(x) - \int_0^x \left(\varphi_{n-1}'(s) + \sin s + \frac{1}{5} s \sin s - \frac{2}{5} \int_0^x \sin(s-t)\,\varphi_{n-1}(t)\,dt \right) ds = \cos x \end{cases}$$

The exact solution of the problem is $\varphi(x) = \lim_{n \to +\infty} \varphi_n(x) = \cos x$.

4.0.2 Example 4

Let's consider the following nonlinear problem (h_4) :

$$(h_4) : \begin{cases} \dfrac{d\varphi(x)}{dx} = \cos x - \dfrac{1}{6}\sin x + \dfrac{1}{12}\sin 2x + \dfrac{1}{4}\displaystyle\int_0^x \cos(x-t)\varphi^2(t)dt \\ \varphi(0) = 0 \end{cases}$$

The correction functional is:

$$\varphi_{n+1}(x) = \varphi_n(x) + \int_0^x \lambda(s) \left(\frac{d\varphi(s)}{ds} - \cos s + \frac{1}{6}\sin s - \frac{1}{12}\sin 2s - \frac{1}{4}\int_0^s \cos(x-t)\varphi_n^2(t)dt \right) ds$$

After calculation, we find $\lambda = -1$. So we get the following variational iteration formula:

$$\begin{cases} \varphi_{n+1}(x) = \varphi_n(x) - \displaystyle\int_0^x \left(\frac{d\varphi(s)}{ds} - \cos s + \frac{1}{6}\sin s - \frac{1}{12}\sin 2s - \frac{1}{4}\int_0^s \cos(s-t)\varphi_n^2(t)dt \right) ds \\ \varphi_0(x) \end{cases}$$

The approximate solutions $\varphi_n(x)$ are obtained iteratively by substituting $\varphi_0(x) = \sin x$ which satisfies the initial condition.

Some approximate solutions are listed below,

$$\begin{cases} \varphi_0(x) & = & \sin x \\ \varphi_1(x) & = & \varphi_0(x) - \int_0^x \left(\varphi_0'(s) - \cos s + \frac{1}{6}\sin s - \frac{1}{12}\sin 2s - \frac{1}{4}\int_0^s \cos(s-t)\varphi_0^2(t)dt \right) ds = \sin x \\ \varphi_2(x) & = & \varphi_1(x) - \int_0^x \left(\varphi_1'(s) - \cos s + \frac{1}{6}\sin s - \frac{1}{12}\sin 2s - \frac{1}{4}\int_0^s \cos(s-t)\varphi_1^2(t)dt \right) ds = \sin x \\ . & . & . \\ \varphi_n(x) & = & \varphi_{n-1}(x) - \int_0^x \left(\varphi_{n-1}'(s) - \cos s + \frac{1}{6}\sin s - \frac{1}{12}\sin 2s - \frac{1}{4}\int_0^s \cos(s-t)\varphi_{n-1}^2(t)dt \right) ds = \sin x \end{cases}$$

The exact solution of the problem is $\varphi(x) = \lim_{n \to +\infty} \varphi_n(x) = \sin x$.

5. SUMMARY SYNTHESIS

Equation	$\begin{cases} \dfrac{d\varphi(x)}{dx} = f(x) + \lambda \displaystyle\int_a^x K(x,t)\,\varphi(t)\,dt; \lambda > 0 \\ \varphi(a) = \beta \end{cases}$
Equation	$(h_3): \begin{cases} \dfrac{d\varphi(x)}{dx} = -\sin x - \frac{1}{5}x\sin x + \dfrac{2}{5}\displaystyle\int_0^x \sin(x-t)\varphi(t)dt \\ \varphi(0) = 1 \end{cases}$
solution	$\begin{cases} \text{by VIM: } \varphi(x) = \cos x \\ \text{by ADM: } \varphi(x) = \cos x \end{cases}$

Equation	$\begin{cases} \dfrac{d\varphi(x)}{dx} = f(x) + \lambda \displaystyle\int_a^x K(x,t)\,\varphi^p(t)\,dt; \lambda > 0 \\ \varphi(a) = \beta \end{cases}$
Equation	$(h_1): \begin{cases} \dfrac{d\varphi(x)}{dx} = e^x - \dfrac{1}{6}e^x\left(e^{2x} - 1\right) + \dfrac{1}{3}\displaystyle\int_0^x e^{x-t}\varphi^3(t)\,dt \\ \varphi(0) = 1 \end{cases}$
Solution	$\begin{cases} \text{by VIM: } \varphi(x) = e^x \\ \text{by ADM: } \varphi(x) = e^x \end{cases}$
Equaion	$(h_2): \begin{cases} \dfrac{d\varphi(x)}{dx} = 1 - 2x - \dfrac{1}{90}x^4\left(2x^2 - 6x + 5\right) + \dfrac{2}{3}\displaystyle\int_0^x (x-t)\,\varphi^2(t)\,dt \\ \varphi(0) = 0 \end{cases}$
Solution	$\begin{cases} \text{by VIM: } \varphi(x) = x - x^2 \\ \text{by ADM: } \varphi(x) = x - x^2 \end{cases}$
Equation	$(h_4): \begin{cases} \dfrac{d\varphi(x)}{dx} = \cos x - \dfrac{1}{6}\sin x + \dfrac{1}{12}\sin 2x + \dfrac{1}{4}\displaystyle\int_0^x \cos(x-t)\varphi^2(t)dt \\ \varphi(0) = 0 \end{cases}$
solution	$\begin{cases} \text{by VIM: } \varphi(x) = \sin x \\ \text{by ADM: } \varphi(x) = \sin x \end{cases}$

In all these applications, the same solutions are obtained by both the modified **ADM** and the **VIM** method.

5.1. Conclusion

In this paper, we have successfully solved some linear and nonlinear integrodifferential equations. To do so, we first performed a convergence study of the VIM and ADM algorithms and then showed the uniqueness of the solution of this type of problem.

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