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On the Conformal Transformation of Douglas Space of Second Kind with Generalized (α, β)-metric

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Abstract

The scope of this paper, we have showed that the second kind Douglas space with generalized form of (α, β)-metric F, is conformally transformed to a second kind Douglas space. We have used the notations of Douglas space and Conformal transformation. Further, we obtained certain results which prove that the second kind Douglas space with different (α, β)-metrics such as Randers metric, generalized Kropina metric and some generalized metrics are invariant under conformal change.

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1. INTRODUCTION

A Finsler space with (α, β)-metric is said to be a Douglas space of second kind, if the Douglas tensor D^h_{ijk} vanishes identically [8]. S. Bacso and M. Matsumoto [3] proposed Douglas space as a generalisation of Berwald space from the perspective of geodesic equations. They also consider the concept of Landsberg space as a generalisation of Berwald space. S. Bacso and B. Szilagyi in 2022 [4], introduced the concept of weakly-Berwald space as another generalisation of Berwald space. I. Y. Lee [7] recently examined Douglas space of second and discovered a criteria for a Finsler space with Matsumoto metric to be a second kind Douglas space.

The theory of sprays and Finsler space was studied in [1]. The authors have made significant contributions to the development of Finsler geometry by developing theories of Finsler space and Berwald space with an (α, β)-metric in ([8], [11]). M. S. Kneblman [6] introduced the conformal theory of Finsler spaces in 1929, and M. Hashiguchi [5] investigated it in depth. In [13], they demonstrated that a second kind Douglas space with generalised Kropina metric and Matsumoto metric is a conformal second kind Douglas space. Recently in ([2], [12]), they proved that the second kind Douglas space with special (α, β)-metric is conformally changed to second kind Douglas space. And also, they have obtained certain results which prove that the second kind Douglas space with some (α, β)-metrics are invariant under conformal change.
The main goal of this paper is to prove that a second kind Douglas space with generalized (α, β)-metric given by $F = \mu_1\alpha + \mu_2\beta + \mu_3\dfrac{\alpha^{t+1}}{\beta^t}$, where μ_1, μ_2 and μ_3 are constants, is conformally changed to a second kind Douglas space. As a result, we have obtained certain results which prove that the second kind Douglas space with some (α, β)-metrics such as Randers metric, generalized Kropina metric and two generalized form of (α, β)-metrics is conformally changed to a second kind Douglas space. Firstly, we gave a introduction for Douglas space in section one. In section two, we go through some basic concepts of Douglas space and conformal transformation. In section three, we have discussed the criteria of second kind Douglas space with (α, β)-metric. Also, we have discussed the conformal transformation of second kind Douglas space with (α, β)-metric in section four. Finally, in section five we have discussed certain results which prove that the conformal transformations of second kind Douglas space with generalized (α, β)-metric.

2. PRELIMINARIES

Let $F^n = (M^n, F(\alpha, \beta))$ be a Finsler space is known to be with an (α, β)-metric, if positively homogeneous function $F(\alpha, \beta)$ in α and β of degree one, whereas $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is Riemannian metric and β is 1-form given by $\beta = b_i(x)y^i$. The Riemannian space $R^n = (M, \alpha)$ associated with F^n. We will employ the following symbols

$$\begin{aligned} &b^2 = a^{rs}b_r b_s, \quad b^i = a^{ir}b_r, \\ &2s_{ij} = b_{i;j} - b_{j;i}, \quad 2r_{ij} = b_{j;i} + b_{i;j}, \\ &s_j = b_r s^r_j, \quad s^i_j = a^{ir}s_{rj}. \end{aligned}$$

Let $B\Gamma = \{G^i_{jk}(x, y), G^i_j\}$ be the Berwald connection of Finsler space F^n plays a significant role in the present work. B^i_{jk} represents the difference tensor of G^i_{jk} and χ^i_{jk}

as well as

$$G^i_{jk}(x,y) = B^i_{jk}(x,y) + \chi^i_{jk}(x), \tag{1}$$

with the subscript 0 and contracting by y^i, we get

$$G^i_j = B^i_j + \chi^i_j \quad and \quad 2G^i = 2B^i + \chi^i_{00} \tag{2}$$

and then $B^i_{jk} = \dot{\partial}_k B^i_j$ and $B^i_j = \dot{\partial}_j B^i$.

Let $F^n = (M^n, F)$ be a Finsler space, the geodesics of an n-dimensional F^n are given by

$$\frac{d^2x^i}{dt^2}y^j - \frac{d^2x^i}{dt^2}y^i = 2\left(G^j y^i - G^i y^j\right); \quad y^i = \frac{dx^i}{dt}. \tag{3}$$

By [3] we know that F^n be a Finsler space becomes a Douglas space if and only if Douglas tensor

$$D^h_{ijk} = G^h_{ijk} - \frac{1}{n+1}\left(G_{ij}\delta^h_k + G_{jk}\delta^h_i + G_{ki}\delta^h_j + G_{ijk}y^h\right), \tag{4}$$

vanishes identically, here $G^h_{ijk} = \partial_k G^h_{ij}$ is $h\nu$-curvature tensor of $B\Gamma$.
Further, let F^n be Finsler space of n-dimensional is known as Douglas space[10], if

$$D^{ij} = -G^i(x,y)y^i + G^i(x,y)y^j \tag{5}$$

is second degree homogeneous polynomials in (y^i).

Now, differentiating (5) by y^m, we get the following definitions as a result

Definition 2.1. *[10] Let F^n be a Finsler space is called a second kind Douglas space if $D^{im}_m = (n+1)G^i - G^{im}_m y^i$ is second degree homogeneous polynomials in (y^i).*

In contrast, a Finsler space with (α, β)-metric is a second kind Douglas-space if and only if

$$B^{im}_m = -B^m_m y^i + B^i(n+1) \tag{6}$$

is second degree homogeneous polynomial in (y^i), where B^m_m is given in [10].
Moreover, differentiating (4) by y^h, y^j and y^k, we get

$$B^{im}_{hjkm} = B^i_{hjk} = 0. \tag{7}$$

Definition 2.2. *Let F^n be a Finsler space with (α, β)-metric is called a second kind Douglas space if $B^{im}_m = -B^m_m y^i + (n+1)B^i$ is second degree homogeneous polynomial in (y^i).*

3. DOUGLAS SPACE OF SECOND KIND WITH (α, β)-METRIC.

In the present section, we will look into the requirements for a Finsler space with a (α, β)-metric being a second kind Douglas space.

Let us consider the spray co-efficients $G^i(x, y)$ of F^n be a Finsler space with an (α, β)-metric. As stated by [9], $G^i(x, y)$ of Finsler space F^n can be written as

$$2G^i = \chi^i_{00} + 2B^i, \tag{8}$$

$$B^i = C^* \left[\frac{\beta F_\beta}{\alpha F_\alpha} y^i - \frac{\alpha F_{\alpha\alpha}}{F_\alpha} \left(\frac{y^i}{\alpha} - \frac{\alpha b^i}{\beta} \right) \right] + \frac{\alpha F_\beta}{F_\alpha} s^i_0, \tag{9}$$

where

$$\begin{aligned} C^* &= \frac{1}{2} \frac{\alpha\beta \left(2\alpha s_0 F_\beta - r_{00} F_\alpha\right)}{\left(\alpha\chi^2 F_{\alpha\alpha} + \beta^2 F_\alpha\right)}, \\ \chi^2 &= -\beta^2 + b^2\alpha^2. \end{aligned} \tag{10}$$

Considering $\chi^i_{00} = \chi^i_{jk}(x) y^i y^k$ is $hp(2)$, then (7) becomes

$$B^{ij} = \frac{\alpha^2 F_{\alpha\alpha}}{\beta F_\alpha} C^* (b^i y^j - b^j y^i) + \frac{\alpha F_\beta}{F_\alpha} (s^i_0 - s^j_0 y^i). \tag{11}$$

we derive the following lemma [10] using (5) and (11)

Lemma 3.1. *Let F^n be a Finsler space with an (α, β)-metric is a Douglas space if and only if $B^{ij} = B^i y^j - B^j y^i$ are $hp(3)$.*

Differentiating (11) by y^h, y^k, y^p and y^q, we have $D^{ij}_{hkpq} = 0$, which is related to $D^{im}_{hkpm} = (n+1) D^i_{hkp} = 0$. Thus, F^n be a Finsler space that satisfies the criteria $D^{ij}_{hkpq} = 0$ is known as Douglas-space. Further, differentiating (11) by y^m and transvecting m and j in the resultant equation, yields

$$\begin{aligned} B^{im}_m = & \frac{\alpha \left\{(n+1)\alpha^2 \Omega F_{\alpha\alpha} b^i + \beta\gamma^2 A y^i\right\} r_{00}}{2\Omega^2} + \frac{(n+1)\alpha F_\beta s^i_0}{F_\alpha} \\ & - \frac{\alpha^2 \left\{(n+1)\alpha^2 \Omega F_\beta F_{\alpha\alpha} b^i + B y^i\right\} s_0}{F_\alpha \Omega^2} - \frac{\alpha^3 F_{\alpha\alpha} y^i r_0}{\Omega}, \end{aligned} \tag{12}$$

where

$$\begin{aligned} & A = 3F_\alpha F_{\alpha\alpha} + \alpha F_\alpha F_{\alpha\alpha\alpha} - 3\alpha \left(F_{\alpha\alpha}\right)^2 \\ & \Omega = \left(\alpha\chi^2 F_{\alpha\alpha} + \beta^2 F_\alpha\right), \text{ provided that } \Omega \neq 0, \text{and} \\ & B = \Omega F F_{\alpha\alpha} + \alpha\beta\chi^2 F_\alpha F_\beta F_{\alpha\alpha\alpha} + \beta \left\{\left(3\chi^2 - \beta^2\right) F_\alpha - 4\alpha\chi^2 F_{\alpha\alpha}\right\} F_\beta F_{\alpha\alpha}. \end{aligned} \tag{13}$$

We will make use of the following result in the next section.

Theorem 3.1. *Let F^n be a Finsler space is a second kind Douglas space if and only if B^{im}_m is homogeneous polynomials of degree two in (y^m), whereas B^{im}_m is provided by (12) and (13), given that $\Omega \neq 0$.*

4. CONFORMAL TRANSFORMATION OF SECOND KIND DOUGLAS SPACE WITH (α, β)-METRIC

Under this section, we discuss the conditions for second kind Douglas space with (α, β)-metric is invariant under conformal change.

Let us consider two Finsler spaces $F^n = (M^n, F)$ and $\bar{F}^n = (M^n, \bar{F})$ defined on manifold M^n. If we have $\sigma(x)$ be a function in each co-ordinate neighbourhood of M^n such that $\bar{F} = e^{\sigma(x)} F(x, y)$, then F^n is known as conformal to $\bar{F}^n$ and $F \to \bar{F}$ is known as conformal transformation.

A conformal transformation of a (α, β)-metric is denoted as $(\alpha, \beta) \to (\bar{\alpha}, \bar{\beta})$, here $\bar{\beta} = e^\sigma \beta$ and $\bar{\alpha} = e^\sigma \alpha$. Then we have

$$\bar{b}_i = e^\sigma b_i, \quad \bar{a}_{ij} = e^{2\sigma} a_{ij}, \tag{14}$$

$$\bar{b}^i = e^{-\sigma} b^i, \quad \bar{a}^{ij} = e^{-2\sigma} a^{ij}, \tag{15}$$

and $b^2 = a^{ij} b_i b_j = \bar{a}^{ij} \bar{b}_i \bar{b}_j$.

From (15), the conformal transformation of Christoffel symbols is provided by

$$\bar{\chi}^i_{jk} = \chi^i_{jk} + \delta^i_j \sigma_k + \delta^i_k \sigma_j - \sigma^i a_{jk}, \tag{16}$$

where $\sigma^i = a^{ij} \sigma_j$ and $\sigma_j = \partial_j \sigma$.

Using (15) and (16), we receive the following axioms

$$\begin{aligned} \bar{\nabla}_j \bar{b}_i &= e^\sigma \left(\rho a_{ij} - \sigma_i b_j + \nabla_j b_i \right), \\ \bar{r}_{ij} &= e^\sigma \left[\rho a_{ij} + r_{ij} - \frac{1}{2} \left(b_j \sigma_i + b_i \sigma_j \right) \right], \\ \bar{s}_{ij} &= e^\sigma \left[\frac{1}{2} \left(b_i \sigma_j - b_j \sigma_i \right) + s_{ij} \right], \\ \bar{s}^i_j &= e^{-\sigma} \left[\frac{1}{2} \left(b^i \sigma_j - b_j \sigma^i \right) + s^i_j \right], \end{aligned}$$

$$\bar{s}_j = s_j + \frac{1}{2} \left(b^2 \sigma_j - \rho b_j \right), \tag{17}$$

where $\rho = \sigma_r b^r$.
Using (16) and (17), we may easily obtain

$$\bar{\chi}^i_{00} = \chi^i_{00} + 2\sigma_0 y^i - \alpha^2 \sigma_j, \tag{18}$$

$$\bar{r}_{00} = e^\sigma \left(\rho \alpha^2 + r_{00} - \sigma_0 \beta \right), \tag{19}$$

$$\bar{s}_0^i = e^{-\sigma}\left[\frac{1}{2}\left(\sigma s_0 b^i - \beta\sigma^i\right) + s_0^i\right], \tag{20}$$

$$\bar{s}_0 = \frac{1}{2}\left(\sigma_0 b^i - \rho\beta\right) + s_0. \tag{21}$$

Next, we get the conformal change of B^{ij} which is given by (11). We have $\bar{F}(\alpha,\beta) := e^{\sigma}F(\alpha,\beta)$ then

$$\bar{F}_{\bar{\beta}} = F_\beta, \quad \bar{F}_{\bar{\alpha}} = F_\alpha, \quad \bar{F}_{\bar{\alpha}\bar{\alpha}} = e^{-\sigma}F_{\alpha\alpha}, \quad \bar{\chi}^2 = e^{2\alpha}\chi^2. \tag{22}$$

By using (10), (21), (22) and Theorem 3.1, we get

$$\bar{C}^* = e^{\sigma}\left(D^* + C^*\right), \tag{23}$$

where

$$D^* = \frac{\alpha\beta\left[\alpha\left(\rho\beta - b^2\sigma_0\right)F_\beta - \left(\sigma_0\beta - \beta\alpha^2\right)F_\alpha\right]}{2\left(\alpha\chi^2 F_{\alpha\alpha} + \beta^2 F_\alpha\right)}. \tag{24}$$

Hence, the conformal change of B^{ij} can be expressed as

$$\begin{aligned}\bar{B}^{ij} &= \frac{\alpha^2 F_{a\alpha}}{\beta F_\alpha}C^*\left(b^i y^j - b^j y^i\right) + \frac{\alpha F_\beta}{F_\alpha}\left(s_0^i y^j - s_0^j y^i\right)\\ &+ \left(\frac{\alpha\sigma_0 F_\beta}{F_\alpha} + \frac{\alpha^2 F_{a\alpha}}{\beta F_a}D^*\right)\left(b^i y^j - b^j y^j\right) - \frac{\alpha\beta F_\beta}{2F_\alpha}\left(\sigma^j y^j - \sigma^j y^i\right)\\ \bar{B}^{ij} &= B^{ij} + C^{ij},\end{aligned}$$

where

$$C^{ij} = \left(\frac{\alpha^2 F_{a\alpha}}{\beta F_\alpha}D^* + \frac{\alpha\sigma_0 F_\beta}{F_\alpha}\right)\left(b^i y^j - b^j y^i\right) - \frac{\alpha\beta F_\beta}{2F_\alpha}\left(\sigma^i y^j - \sigma^j y^i\right).$$

From (13), we have

$$\bar{A} = e^{-\sigma}A, \quad \bar{\Omega} = e^{2\alpha}\Omega \quad \text{and} \quad \bar{B} = e^{2\alpha}B. \tag{25}$$

Next, we apply conformal change to B_m^{im} and we get

$$\bar{B}_m^{im} = B_m^{im} + K_m^{im} \tag{26}$$

where

$$\begin{aligned}2K_m^{im} =& \frac{\alpha(n+1)F_\beta}{F_\alpha}\left(\sigma_0 b^i - \beta\sigma^i\right) + \alpha\left\{\frac{\alpha^2(n+1)\Omega F_{\alpha a} b^i + \beta\chi^2 A y^i}{\Omega^2}\right\}\left(\rho\alpha^2 - \sigma_0\beta\right)\\ &- \left[\frac{\alpha^2\left\{\alpha^2(n+1)\Omega F_\beta F_{\alpha\alpha} b^i + B y^i\right\}}{F_\alpha\Omega^2} - \frac{\alpha^3 F_{\alpha\alpha} y^i}{\Omega}\right]\left(b^2\sigma_0 - \rho\beta\right).\end{aligned} \tag{27}$$

Hence, we have the following result

Theorem 4.1. *A second kind Douglas space is invariant under conformal transformation if and only if K_m^{im} is second degree homogeneous polynomial in (y^i).*

5. CONFORMAL TRANSFORMATION OF SECOND KIND DOUGLAS SPACE WITH GENERALIZED (α, β)-METRIC $F = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\alpha^{T+1}}{\beta^T}$

Let us define a Finsler space with generalized (α, β)-metric is given by

$$F = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\alpha^{t+1}}{\beta^t}, \tag{28}$$

where μ_1, μ_2 and μ_3 are constants.
Then from (28), we obtain

$$\begin{aligned}
F_\alpha &= \mu_1 + \mu_3(t+1)\frac{\alpha^t}{\beta^t},\\
F_{\alpha\alpha} &= \mu_3 t(t+1)\frac{\alpha^{t-1}}{\beta^t},\\
F_{\alpha\alpha\alpha} &= \mu_3 t(t+1)(t-1)\frac{\alpha^{t-2}}{\beta^t},\\
F_\beta &= \mu_2 - \mu_3 t\frac{\alpha^{t+1}}{\beta^{t+1}}.
\end{aligned} \tag{29}$$

Hence, using (13) we obtain

$$\begin{aligned}
\Omega &= \mu_3\beta^2 + \frac{\mu_3\alpha^t(t+1)(\beta^2 + t\chi^2)}{\beta^t},\\
A &= \frac{\mu_3 t(t+1)\alpha^t\left(2\mu_1\beta^t + 2\mu_3\alpha^t + \mu_1 t\beta^t - 2\mu_3 t^2\alpha^t\right)}{\alpha\beta^{2t}},\\
B &= B_1 + B_2,
\end{aligned} \tag{30}$$

where

$$\begin{aligned}
B_1 = {} & \frac{\mu_3 t(t+1)\alpha^{t-1}}{\beta^{3t}}\left[\mu_3^2\chi^2 t\alpha^{2t+1}\left(3t^2 + 2t - 1\right) + \mu_1\beta^{2t+1}\left(t\mu_2\chi^2 + \mu_1\alpha + 2\mu_2\chi^2\right)\right.\\
& \left. - \mu_1\mu_3\chi^2 t\alpha^{t+1}\beta^t\right],\\
B_2 = {} & \frac{\mu_3 t(t+1)\alpha^{t-1}}{\beta^{3t}}\left[\mu_3^2\alpha^{2t+1}\beta^2(t+1)^2 + 2\mu_2\mu_3\chi^2\alpha^t\beta^{t+1}(1-t^2) + 2\mu_1\mu_3\alpha^{t+1}\right.\\
& \left.\beta^{t+2}(1+t)\right].
\end{aligned} \tag{31}$$

Thus, using (29) , K^{im}_m can be written as

$$2K^{im}_m = \frac{\alpha(n+1)\left(\mu_2\beta^{t+1} - \mu_3 t\alpha^{t+1}\right)\left(\sigma_0 b^i - \beta\sigma^i\right)}{\beta\left(\mu_1\beta^t + \mu_3(t+1)\alpha^t\right)} + T_1 + T_2 + T_3 + T_4 + T_5, \tag{32}$$

where

$$T_1 = \frac{\mu_3(n+1)t(t+1)\alpha^{t+1}b^i\left(\rho\alpha^2 - \sigma_0\beta\right)}{\left(\mu_3\alpha^t(1+t)\left(\beta^2+t\chi^2\right)+\mu_1\beta^{t+2}\right)},$$

$$T_2 = \frac{\mu_3\chi^2 t(t+1)\beta y^i\left(\mu_1\beta^t(2+t)+2\mu_3\alpha^t(1-t^2)\right)\left(\rho\alpha^2-\sigma_0\beta\right)}{\left(\mu_3\alpha^t(1+t)\left(\beta^2+t\chi^2\right)+\mu_1\beta^{t+2}\right)^2},$$

$$T_3 = \frac{-\mu_3(n+1)(t+1)b^i\alpha^{t+1}\left(\mu_2\beta^{t+1}-\mu_3 t\alpha^{t+1}\right)\left(\sigma_0 b^2-\rho\beta\right)}{\beta\left(\mu_1\beta^t+\mu_3(t+1)\alpha^t\right)\left(\mu_3\alpha^t(1+t)\left(\beta^2+t\chi^2\right)+\mu_1\beta^{t+2}\right)},$$

$$T_4 = \frac{-\mu_3 t(t+1)\alpha^{t+1}y^i\left(\sigma_0 b^2-\rho\beta\right)}{\left(\mu_1\beta^t+\mu_3(t+1)\alpha^t\right)\left(\mu_3\alpha^t(1+t)\left(\beta^2+t\chi^2\right)+\mu_1\beta^{t+2}\right)^2}\left[\mu_3^2\alpha^{2t+1}(\beta^2+2\chi^2t^2+3\chi^2t^3\right.$$
$$-\chi^2 t+2t\beta^2+t^2\beta^2)+\mu_1\beta^{2t+2}\left(\mu_1\alpha+2\mu_2\chi^2+\mu_2 t\chi^2\right)+2\mu_1\mu_3\alpha^{t+1}\beta^{t+2}(t+1)+2\mu_2\mu_3$$
$$\left.t\chi^2\alpha^t\beta^{t+1}(1-t^2)-\mu_1\mu_3\chi^2 t\alpha^{t+1}\beta^t\right],$$

$$T_5 = \frac{\mu_3 t(t+1)\alpha^{t+2}y^i\left(\sigma_0 b^2-\rho\beta\right)}{\left(\mu_3\alpha^t(1+t)\left(\beta^2+t\chi^2\right)+\mu_1\beta^{t+2}\right)}.$$

This shows that K^{im}_m is a second degree homogeneous polynomial in (y^i).
Using this we have the following

Theorem 5.1. *A second kind Douglas space with generalized* (α,β)*-metric* $F = \mu_1\alpha+\mu_2\beta+\mu_3\dfrac{\alpha^{t+1}}{\beta^t}$*, where* μ_1*,* μ_2 *and* μ_3 *are constants, is conformally changed to a second kind Douglas space.*

From the theorem(5.1), we can show that a second kind Douglas-space with a Finsler space of some (α,β)-metric is conformally changed to a second kind Douglas-space. Using this there are several cases that might occur, as follows.

Case(i). If $\mu_1 = 1$, $\mu_2 = 1$ and $\mu_3 = 0$, then generalized (α,β)-metric becomes $F := \alpha+\beta$, this is known as Randers metric. In this instance, $2K^{im}_m$ obtained as

$$2K^{im}_m = \alpha(n+1)(\sigma_0 b^i - \beta\sigma^i). \tag{33}$$

This shows that K^{im}_m is a second degree homogeneous polynomial in (y^i).

Noting that $T_1 = T_2 = T_3 = T_4 = T_5 = 0$ in this case. Hence, we have the following

Corollary 5.1.1. *A second kind Douglas space with Randers metric $F := \alpha + \beta$, is invariant under conformal transformation.*

Case(ii). If $\mu_1 = 0$, $\mu_2 = 0$ and $\mu_3 = 1$, then generalized (α, β)-metric becomes $F = \dfrac{\alpha^{t+1}}{\beta^t}$, which is known as generalized Kropina metric. In this instance, $2K^{im}_m$ can be written as

$$2K^{im}_m = \frac{-t(n+1)\alpha^2(\sigma_0 b^i - \beta\sigma^i)}{(t+1)\beta} + S_1 + S_2 + S_3 + S_4 + S_5, \tag{34}$$

where

$$S_1 = \frac{(n+1)t\alpha^2 b^i\left(\rho\alpha^2 - \sigma_0\beta\right)}{(\beta^2 + t\chi^2)},$$

$$S_2 = \frac{2\chi^2 t(1-t)\beta y^i\left(\rho\alpha^2 - \sigma_0\beta\right)}{(\beta^2 + t\chi^2)^2},$$

$$S_3 = \frac{(n+1)t^2 b^i\alpha^2\left(\sigma_0 b^2 - \rho\beta\right)}{\beta(1+t)\left(\beta^2 + t\chi^2\right)},$$

$$S_4 = \frac{ty^i\left(3\chi^2 t^2 - \chi^2 t + t\beta^2 + \beta^2\right)\left(\sigma_0 b^2 - \rho\beta\right)}{(1+t)\left(\beta^2 + t\chi^2\right)^2},$$

$$S_5 = \frac{t\alpha^2 y^i\left(\sigma_0 b^2 - \rho\beta\right)}{(\beta^2 + t\chi^2)}.$$

This proves that K^{im}_m is s second degree homogeneous polynomial in (y^i). Hence, we have the following

Corollary 5.1.2. *A second kind Douglas space with generalized Kropina metric $F = \dfrac{\alpha^{t+1}}{\beta^t}$, is invariant under conformal transformation.*

Case(iii). If $\mu_1 = 1$, $\mu_2 = 1$ and $\mu_3 = 1$, then generalized (α, β)-metric reduces to $F = \alpha + \beta + \dfrac{\alpha^{t+1}}{\beta^t}$. In this instance, $2K^{im}_m$ can be written as

$$2K^{im}_m = \frac{\alpha(n+1)(\beta^{t+1} - t\alpha^{t+1})(\sigma_0 b^i - \beta\sigma^i)}{\beta(t\alpha^t + \alpha^t + \beta^t)} + W_1 + W_2 + W_3 + W_4 + W_5, \tag{35}$$

where

$$W_1 = \frac{(n+1)t(t+1)\alpha^{t+1}b^i\left(\rho\alpha^2-\sigma_0\beta\right)}{\left(\alpha^t(1+t)\left(\beta^2+t\chi^2\right)+\beta^{t+2}\right)},$$

$$W_2 = \frac{\chi^2 t(t+1)\beta y^i\left(\beta^t(2+t)+2\alpha^t(1-t^2)\right)\left(\rho\alpha^2-\sigma_0\beta\right)}{\left(\alpha^t(1+t)\left(\beta^2+t\chi^2\right)+\beta^{t+2}\right)^2},$$

$$W_3 = \frac{-(n+1)(t+1)b^i\alpha^{t+1}\left(\beta^{t+1}-t\alpha^{t+1}\right)\left(\sigma_0 b^2-\rho\beta\right)}{\beta\left(\beta^t+(t+1)\alpha^t\right)\left(\alpha^t(1+t)\left(\beta^2+t\chi^2\right)+\beta^{t+2}\right)},$$

$$W_4 = \frac{-t(t+1)\alpha^{t+1}y^i\left(\sigma_0 b^2-\rho\beta\right)}{\left(\beta^t+(t+1)\alpha^t\right)\left(\alpha^t(1+t)\left(\beta^2+t\chi^2\right)+\beta^{t+2}\right)^2}\left[\alpha^{2t+1}(\beta^2+2\chi^2t^2+3\chi^2t^3-\chi^2t+2t\beta^2\right.$$
$$\left.+t^2\beta^2)+\beta^{2t+2}\left(\alpha+2\chi^2+t\chi^2\right)+2\alpha^{t+1}\beta^{t+2}(t+1)+2t\chi^2\alpha^t\beta^{t+1}(1-t^2)-\chi^2t\alpha^{t+1}\beta^t\right],$$

$$W_5 = \frac{t(t+1)\alpha^{t+2}y^i\left(\sigma_0 b^2-\rho\beta\right)}{\left(\alpha^t(1+t)\left(\beta^2+t\chi^2\right)+\beta^{t+2}\right)}.$$

This proves that K^{im}_m is s second degree homogeneous polynomial in (y^i). Hence, we have the following

Corollary 5.1.3. *A second kind Douglas space with* (α,β)*-metric* $F=\alpha+\beta+\dfrac{\alpha^{t+1}}{\beta^t}$, *is conformally changed to a second kind Douglas space.*

Case(iv). If $\mu_1=1$, $\mu_2=1$, $\mu_3=1$ and $t=1$, then generalized (α,β)-metric reduces to $F=\alpha+\beta+\dfrac{\alpha^2}{\beta}$. In this instance, $2K^{im}_m$ can be written as

$$2K^{im}_m = \frac{\alpha(n+1)(\beta^2-\alpha^2)(\sigma_0 b^i-\beta\sigma^i)}{\beta(2\alpha+\beta)}+V_1+V_2+V_3+V_4+V_5, \tag{36}$$

where

$$V_1 = \frac{2(n+1)\alpha^2 b^i\left(\rho\alpha^2-\sigma_0\beta\right)}{\left(2\alpha\left(\beta^2+\chi^2\right)+\beta^3\right)},$$

$$V_2 = \frac{6\chi^2\beta^2 y^i\left(\rho\alpha^2-\sigma_0\beta\right)}{\left(2\alpha\left(\beta^2+\chi^2\right)+\beta^3\right)^2},$$

$$V_3 = \frac{-2(n+1)b^i\alpha^2(\beta^2-\alpha^2)(\sigma_0 b^2-\rho\beta)}{\beta(\beta+2\alpha)(2\alpha(\beta^2+\chi^2)+\beta^3)},$$

$$V_4 = \frac{-\alpha^2 y^i(\sigma_0 b^2-\rho\beta)}{(\beta+2\alpha)(2\alpha(\beta^2+\chi^2)+\beta^3)^2}\left[\alpha^3(4\beta^2+4\chi^2)+\beta^3\left(\alpha+3\chi^2\right)+4\alpha^2\beta^3-\chi^2\alpha^2\beta\right],$$

$$V_5 = \frac{2\alpha^3 y^i(\sigma_0 b^2-\rho\beta)}{(2\alpha(\beta^2+\chi^2)+\beta^3)}.$$

This proves that K^{im}_m is s second degree homogeneous polynomial in (y^i). Hence, we have the following

Corollary 5.1.4. *A second kind Douglas space with* (α,β)*-metric* $F=\alpha+\beta+\dfrac{\alpha^2}{\beta}$, *is invariant under conformal transformation.*

6. CONCLUSION

In this current work, we have been proved that the second kind Douglas space of with generalized (α,β)-metric F, is invariant under conformal transformation. And also, we have obtained certain results which prove that the second kind Douglas space with some (α,β)-metrics are conformally invariant.

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