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Intuitionistic Fuzzy Gamma Hypergroup and Intuitionistic Fuzzy Gamma Hypermodule

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Abstract

In this paper, we introduce and study the intuitionistic fuzzy gamma hypergroup and intuitionistic fuzzy gamma hypermodule as a generalization of the usual fuzzy hypergroup and the usual fuzzy gamma hypermodule. Also, we investigate some of the basic properties of intuitionistic fuzzy gamma hypermodule. Examples of intuitionistic fuzzy gamma hypergroup are constructed.

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1. Introduction

The concept of hyperstructure theory was born in 1934, at the eigth congress of Scandinavian Mathematicians. When F. Marty [8] first defined a hypergroup as a set equipped with an associative and reproductive hyperoperations and analysed their properties. Algebraric hyperstructures are a suitable generalization of classical algebraic structures.

The theory of fuzzy sets was proposed by Zadeh [11] in 1965, has provided a useful mathematical tool for describing the two complex or ill defined mathematical analysis by classical methods. In this aspect, the concept of fuzzy groups was defined by Rosenfeld [9] and its structures were investigated. Since, then many papers have been published in the field of fuzzy algebra. Fuzzy hyperstructures are a direct generalization of the concept of fuzzy algebras (Fuzzy groups, Fuzzy rings, Fuzzy modules etc). This approach can be extended to fuzzy hypergroups.

In the year 1986 Atanassov [3] introduced intuitionistic fuzzy set as a generalization of fuzzy set. The study of Intuitionistic fuzzy hyper algebraic structures has started with the introduction of the concepts of intuitionistic fuzzy hypergroups [1].

Leoreanu-Fotea and Davvaz [7] introduced and analyzed the fuzzy hyperring and fuzzy hypermodule notion and obtained a connection between hypermodules and fuzzy hypermodules.

In this paper, our aim is to introduce and study the concept of intuitionistic fuzzy gamma hypergroup and intuitionistic fuzzy gamma hypermodule as a generalization of the usual fuzzy hypergroup and the usual fuzzy gamma hypermodule. Moreover, examples and some of the theorems of intuitionistic fuzzy gamma hypergroup and intuitionistic fuzzy gamma hypermodule are construced and proved.

2. Preliminaries

In this section, basic definitions on Fuzzy Γ hypergroup, Intuitionistic fuzzy hypergroups are summarized.

Definition 2.1. [5] Let X be any non-empty set. The map $\mu : X \to [0, 1]$ is called a fuzzy subset of X.

Definition 2.2. [1] A fuzzy hyperoperation maps the pairs of elements of the cartesian product $H \times H$ to a fuzzy subset of H and is denoted by $F(H)$, then a *fuzzy hyperoperation* is the map $\circ : H \times H \to F(H)$

(1) If $a, b, z \in H$ then, the fuzzy sets are $a \circ b$, $b \circ a$are defined as,

$$(a \circ b)(z) = \vee_{y \in H}[(a \circ y)(z) \wedge b(y)] \text{ same as for } b \circ a$$

(2) If $a, z \in H$, $B \in F(H)$ then, the fuzzy sets are $a \circ B$, $B \circ a$ are defined as

$$(a \circ B)(z) = \vee_{y \in H}[(a \circ y)(z) \wedge B(y)] \text{ and}$$
$$(B \circ a)(z) = \vee_{y \in H}[[(y \circ a)(z) \wedge B(y)]$$

(3) If $a, b, c, z \in H$ then, $((a \circ b) \circ c)(z) = \vee_{y \in H}[(y \circ c)(z) \wedge (a \circ b)(y)]$ and

$$(a \circ (b \circ c))(z) = \vee_{y \in H}[(a \circ y)(z) \wedge (b \circ c)(y)]$$

(4) If $A, B \in F(H)$ then, the fuzzy set $A \circ B$ is defined as,

$$(A \circ B)(z) = \vee_{x,y \in H}[(x \circ y)(z) \wedge A(x) \wedge B(y)]$$

Definition 2.3. [2] Let S and Γ be two nonempty sets. $F^*(S)$ denotes the set of all nonzero fuzzy subsets of S. A *Fuzzy* Γ *Hyperoperation* on S is a map $\circ : S \times \Gamma \times S \to F^*(S)$, which associates a nonzero subset $a \circ \gamma \circ b$ for all $a, b \in S$ and $\gamma \in \Gamma$. $(S, \circ)$ is called a *Fuzzy* Γ *Hypergroupoid*.

Definition 2.4. [2] A *Fuzzy* Γ *Hypergroupoid* $(S, \circ)$ is called a *Fuzzy* Γ *hypersemigroup* if for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$, we have $a \circ \alpha \circ (b \circ \beta \circ c) = (a \circ \alpha \circ b) \circ \beta \circ c$ where for any $\mu \in F^*(S)$,

we have, $(a \circ \gamma \circ \mu)(r) = \vee_{t \in S}((a \circ \gamma \circ t)(r) \wedge \mu(t))$
and $(\mu \circ \gamma \circ a)(r) = \vee_{t \in S}(\mu(t) \wedge (t \circ \gamma \circ a)(r))$ for all $r \in S, \gamma \in \Gamma$.

Definition 2.5. [3] An *Intuitionistic Fuzzy Set* (IFS) A in X is an object of the form $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle | x \in X\}$, where the functions $\mu_A : X \to [0, 1]$ and $\gamma_A : X \to [0, 1]$ denote the degree of membership (namely, $\mu_A(x)$) and the degree of non-membership (namely, $\gamma_A(x)$) of each element $x \in X$ to the set A respectively, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for each $x \in X$.

Definition 2.6. [3] Let A and B be Intuitionistic Fuzzy Sets of the forms $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle | x \in X\}$ and $B = \{\langle x, \mu_B(x), \gamma_B(x)\rangle | x \in X\}$. Then

(a) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$
for all $x \in X$

(b) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$

(c) The complement of A is denoted by $\bar{A}$ and is defined by
$\bar{A} = \{\langle x, \gamma_A(x), \mu_A(x)\rangle | x \in X\}$

(d) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \gamma_A(x) \vee \gamma_B(x)\rangle | x \in X\}$

(e)$A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \gamma_A(x) \wedge \gamma_B(x)\rangle | x \in X\}$

The Intuitionistic Fuzzy Sets $0_\sim = \{\langle x, 0, 1\rangle | x \in X\}$ and $1_\sim = \{\langle x, 1, 0\rangle | x \in X\}$ are represents the empty set and the whole set.

Definition 2.7. [1] An *Intuitionistic Fuzzy Hyperoperation* maps the ordered pairs of elements of the cartesian product $X \times X$ to an intuitionistic fuzzy set of $X(= \mu_X, \gamma_X)$. Thus, if we denote the collection of all Intuitionistic Fuzzy set of X by $IF(X)$, then an Intuitionistic fuzzy hyperoperation is the map $* : X \times X \to IF(X)$. Hence, if $*$ is an Intuitionistic fuzzy hyperoperation, then $a * b$ is an intuitionistic fuzzy set, where $a, b \in X$ and is defined by

$$a * b = \langle x, \mu_{a*b}(x), \gamma_{a*b}(x)\rangle$$

(1) If $a, b, x \in X, B, C \in IF(X)$ then,the intuitionistic fuzzy sets are
$a * B$, $B * a$ are defined as,$a * B = \langle x, \mu_{a*B}(x), \gamma_{a*B}(x)\rangle$
where $\mu_{a*B}(x) = \vee_{y \in X}[\mu_{a*y}(x) \wedge \mu_B(y)]$,
$\gamma_{a*B}(x) = \wedge_{y \in X}[\gamma_{a*y}(x) \vee \gamma_B(y)]$
Similarly, $B * a = \langle x, \mu_{B*a}(x), \gamma_{B*a}(x)\rangle$
where $\mu_{B*a}(x) = \vee_{y \in X}[\mu_{y*a}(x) \wedge \mu_B(y)]$,
$\gamma_{B*a}(x) = \wedge_{y \in X}[\gamma_{y*a}(x) \vee \gamma_B(y)]$

(2) If $a, b, c, x \in X$ then, $(a * b) * c = \langle x, \mu_{(a*b)*c}(x), \gamma_{(a*b)*c}(x)\rangle$
where $\mu_{(a*b)*c}(x) = \vee_{y \in X}[\mu_{y*c}(x) \wedge \mu_{a*b}(y)]$,
$\gamma_{(a*b)*c}(x) = \wedge_{y \in X}[\gamma_{y*c}(x) \vee \gamma_{a*b}(y)]$
and $a * (b * c) = \langle x, \mu_{a*(b*c)}(x), \gamma_{a*(b*c)}(x)\rangle$
where $\mu_{a*(b*c)}(x) = \vee_{y \in X}[\mu_{a*y}(x) \wedge \mu_{b*c}(y)]$,
$\gamma_{a*(b*c)}(x) = \wedge_{y \in X}[\gamma_{a*y}(x) \vee \gamma_{b*c}(y)]$

(3) If $A, B \in IF(X)$, then an intuitionistic fuzzy set $A * B$ is defined as,
$A * B = \langle x, \mu_{A*B}(x), \gamma_{A*B}(x)\rangle$
where $\mu_{A*B}(x) = \vee_{y,z \in X}[\mu_{y*z}(x) \wedge \mu_A(y) \wedge \mu_B(z)]$,

$$\gamma_{A*B}(x) = \wedge_{y,z\in X}[\gamma_{y*z}(x) \vee \gamma_A(y) \vee \gamma_B(z)].$$

Definition 2.8. **[1]** An *Intuitionistic Fuzzy Hypergroup* is a non-empty set X with an Intuitionistic fuzzy hyperoperation, which satisfies the following axioms,

(i) $(a * b) * c = a * (b * c)$ for all $a, b, c \in X$ (Associativity)

(ii) $a * 1_\sim = 1_\sim * a = 1_\sim$ for all $a \in X$ (Reproduction)

where $1 \simeq= \{\langle a, 1, 0\rangle | a \in X\}$.

If only (i) is valid, then an Intuitionistic Fuzzy hyperstructure $(X, *)$ is called an *Intuitionistic fuzzy semi-hypergroup*, while, if only (ii) is valid, then Intuitionistic Fuzzy hyperstructure $(X, *)$ is called an *Intuitionistic fuzzy quasi-hypergroup*.

Definition 2.9. **[2]** Let $(R, \uplus, \circ)$ be a Γ- hyperring and $(\Gamma, *)$ be a canonical hypergroup. We say that $(M, +, .)$ is a *left* Γ*-hypermodule* over R, if $(M, +)$ be a canonical hypergroup and there exists a mapping $. : R \times \Gamma \times M \to P^*(M)$, such that for every $r, s \in R$ and $\alpha, \beta \in F$ and $a, b \in M$,the following conditions are satisfied,

(i) $(r \uplus s).\alpha.a = r.\alpha.a + s.\alpha.a$

(ii) $r.(\alpha * \beta).a = r.\alpha.a + r.\beta.a$

(iii) $r.\alpha.(a + b) = r.\alpha.a + r.\alpha.b$

(iv) $(r \circ \alpha \circ s) = r.\alpha.(s.\beta.a)$.

Definition 2.10. **[2]** Let R, Γ be two nonempty sets and $\boxplus, \boxdot$ be two fuzzy hyperoperations on R and $\otimes$ be a fuzzy hyperoperation on Γ. Let $(R, \boxplus)$ and $(\Gamma, \otimes)$ be two canonical fuzzy hypergroups. R is called a *fuzzy* Γ*-hyperring* if there exists the mapping, $\boxdot : R \times \Gamma \times R \to F^*(R)$ such that for all $r, s, t \in R, \alpha, \beta \in \Gamma$, the following conditions are satisfied,

(i) $r \boxdot \alpha \boxdot (s \boxplus t) = (r \boxdot \alpha \boxdot s) \boxplus (r \boxdot \alpha \boxdot t)$,

(ii) $r \boxdot (\alpha \otimes \beta) \boxdot s = (r \boxdot \alpha \boxdot s) \boxplus (r \boxdot \beta \boxdot s)$,

(iii) $(r \boxplus s) \boxdot \alpha \boxdot t = (r \boxdot \alpha \boxdot t) \boxplus (s \boxdot \alpha \boxdot t)$,

(iv) $r \boxdot \alpha \boxdot (s \boxdot \beta \boxdot t) = (r \boxdot \alpha \boxdot s) \boxdot \beta \boxdot t$.

Definition 2.11. **[2]** Let $(\Gamma, \otimes)$ be a canonical fuzzy hypergroups. Let $(R, \boxplus, \boxdot)$ be a fuzzy Γ-hyperring. A nonempty set M together with two fuzzy Γ-hyperoperation $\oplus, \odot$ is called a *left fuzzy* Γ*-hypermodule* over $(R, \boxplus, \boxdot)$ if the following conditions hold,

(1) $(M, \oplus)$ is a canonical fuzzy Γ-hypergroup,

(2) $\odot : R \times \Gamma \times M \to F^*(M)$ is such that for all $a, b \in M, r, s \in R, \alpha, \beta \in \Gamma$ we have,

(i) $r \odot \alpha \odot (a \oplus b) = (r \odot \alpha \odot a) \oplus (r \odot \alpha \odot b)$,

(ii) $(r \boxplus s) \odot \alpha \odot a = (r \odot \alpha \odot a) \oplus (s \odot \alpha \odot a)$,

(iii) $r \odot (\alpha \otimes \beta) \odot a = (r \odot \alpha \odot a) \oplus (r \odot \beta \odot a)$,

(iv) $r \odot \alpha \odot (s \odot \beta \odot a) = (r \odot \alpha \odot s) \odot \beta \odot a$.

Definition 2.12. **[2]** Let R be a Γ-ring. A (left) gamma module over R is an additive abelian group M together with a mapping $. : R \times \Gamma \times M \to M$, such that for all $m, m_1, m_2 \in M$ and $\gamma, \gamma_1, \gamma_2 \in \Gamma$ and $r, r_1, r_2 \in R$, the following conditions are satisfied,

(i) $r.\gamma.(m_1 + m_2) = r.\gamma.m_1 + r.\gamma.m_2$,

(ii) $(r_1 + r_2).\gamma.m = r_1.\gamma.m + r_2.\gamma.m$,

(iii) $r.(\gamma_1 + \gamma_2).m = r.\gamma_1.m + r.\gamma_2.m$,

(iv) $r_1.\gamma_1.(r_2.\gamma_2.m) = (r_1.\gamma_1.r_2).\gamma_2.m$. In this case, we say that M is a left (or right) R_Γ-module.

3. Intuitionistic Fuzzy Γ-Hypergroup

In this section, we introduce the notion of Intuitionistic Fuzzy Γ-hypergroup as a generalization of Fuzzy hypergroup and their properties of them are investigated.

Definition 3.1. Let I, Γ be two nonempty sets. $F^*(I)$ denotes the non-zero intuitionistic fuzzy set in I. An *Intuitionistic Fuzzy* Γ*-Hyperoperation* on I is a map, $\circ : I \times \Gamma \times I \to F^*(I)$ and $(I, \circ)$ is called an intuitionistic fuzzy Γ-hypergroupoid.

(i) If $\circ$ is an intuitionistic fuzzy Γ-hyperoperation, then $a \circ \alpha \circ b$ is a non-zero intuitionistic fuzzy set for all $a, b \in I, \alpha \in \Gamma$ and is
defined as:

$$a \circ \alpha \circ b = \langle x, \mu_{a \circ \alpha \circ b}(x), \gamma_{a \circ \alpha \circ b}(x) \rangle.$$

(ii) If B is a nonempty subset of I and $x \in I$, then for all
$y \in I, \alpha \in \Gamma$, then the non-zero intuitionistic fuzzy set is
defined as:

$$x \circ \alpha \circ B = \langle y, \mu_{x \circ \alpha \circ B}(y), \gamma_{x \circ \alpha \circ B}(y) \rangle$$

where $\mu_{x \circ \alpha \circ B}(y) = \vee_{b \in B}[\mu_{x \circ \alpha \circ b}(y)]$,
and $\gamma_{x \circ \alpha \circ B}(y) = \wedge_{b \in B}[\gamma_{x \circ \alpha \circ b}(y)]$.

(ii). Let A, B be two non-zero intuitionistic fuzzy set of an
intuitionistic fuzzy Γ-hypergroupoid $(I, \circ)$, we define
$A \circ \alpha \circ B = \langle x, \mu_{A \circ \alpha \circ B}(x), \gamma_{A \circ \alpha \circ B}(x) \rangle$ for all $x \in I, \alpha \in \Gamma$
where $\mu_{A \circ \alpha \circ B}(x) = \vee_{p,q \in I}[\mu_A(p) \wedge \mu_{p \circ \alpha \circ q}(x) \wedge \mu_B(q)]$,
and $\gamma_{A \circ \alpha \circ B}(x) = \wedge_{p,q \in I}[\gamma_A(p) \vee \gamma_{p \circ \alpha \circ q}(x) \vee \gamma_B(q)]$.

Definition 3.2. An intuitionistic fuzzy Γ-hypergroupoid $(I, \circ)$ is called an *intutitionistic fuzzy* Γ*-hypersemigroup* if for all $a, b, c \in I, \alpha, \beta \in \Gamma$, we have, $a \circ \alpha \circ (b \circ \beta \circ c) = (a \circ \alpha \circ b) \circ \beta \circ c$.

(i.e). $a \circ \alpha \circ (b \circ \beta \circ c) = \langle x, \mu_{a\circ\alpha\circ(b\circ\beta\circ c)}(x), \gamma_{a\circ\alpha\circ(b\circ\beta\circ c)}(x)\rangle$
where $\mu_{a\circ\alpha\circ(b\circ\beta\circ c)}(x) = \vee_{t\in I}[\mu_{a\circ\alpha\circ t}(x) \wedge \mu_{b\circ\beta\circ c}(t)]$
and $\gamma_{a\circ\alpha\circ(b\circ\beta\circ c)}(x) = \wedge_{t\in I}[\gamma_{a\circ\alpha\circ t}(x) \vee \gamma_{b\circ\beta\circ c}(t)]$ for all $x \in I$.

Definition 3.3. An intuitionistic fuzzy Γ-hypergroupoid $(I, \circ)$ is called an *intutitionistic fuzzy* Γ*-hypergroup* if for all $a \in I, \alpha \in \Gamma$, we have,
$a \circ \alpha \circ I = I \circ \alpha \circ a = 1_{\sim}$.

Example 3.4. If $I = \{0, 1\}$, then the four Intuitionistic fuzzy set of I defines an intuitionistic fuzzy Γ-hypersemigroup structure on I as follows,

$A = 0 \circ \alpha \circ 0 = \langle x, \mu_A(x), \gamma_A(x)\rangle$
where $\mu_A(x) = \dfrac{a_1}{0} + \dfrac{a_2}{1}, \quad \gamma_A(x) = \dfrac{b_1}{0} + \dfrac{b_2}{1}$
with $0 \leq a_1 + b_1 \leq 1$ and $0 \leq a_2 + b_2 \leq 1$

$B = 0 \circ \alpha \circ 1 = \langle x, \mu_B(x), \gamma_B(x)\rangle$
where $\mu_B(x) = \dfrac{a_3}{0} + \dfrac{a_4}{1}, \quad \gamma_B(x) = \dfrac{b_3}{0} + \dfrac{b_4}{1}$
with $0 \leq a_3 + b_3 \leq 1$ and $0 \leq a_4 + b_4 \leq 1$

$C = 1 \circ \alpha \circ 0 = \langle x, \mu_C(x), \gamma_C(x)\rangle$
where $\mu_C(x) = \dfrac{a_5}{0} + \dfrac{a_6}{1}, \quad \gamma_C(x) = \dfrac{b_5}{0} + \dfrac{b_6}{1}$
with $0 \leq a_5 + b_5 \leq 1$ and $0 \leq a_6 + b_6 \leq 1$

$D = 1 \circ \alpha \circ 1 = \langle x, \mu_D(x), \gamma_D(x)\rangle$
where $\mu_D(x) = \dfrac{a_7}{0} + \dfrac{a_8}{1}, \quad \gamma_D(x) = \dfrac{b_7}{0} + \dfrac{b_8}{1}$
with $0 \leq a_7 + b_7 \leq 1$ and $0 \leq a_8 + b_8 \leq 1$

To verify the Associative axiom:

$0 \circ \alpha \circ D = B \circ \beta \circ 1$

L.H.S = $0 \circ \alpha \circ D = \langle x, \mu_{0\circ\alpha\circ D}(x), \gamma_{0\circ\alpha\circ D}(x)\rangle$
where $\mu_{0\circ\alpha\circ D}(x) = \dfrac{a_3}{0} + \dfrac{a_4}{1}, \quad \gamma_{0\circ\alpha\circ D}(x) = \dfrac{b_3}{0} + \dfrac{b_4}{1}$

R.H.S = $B \circ \beta \circ 1 = \langle x, \mu_{B\circ\beta\circ 1}(x), \gamma_{B\circ\beta\circ 1}(x)\rangle$
where $\mu_{B\circ\beta\circ 1}(x) = \dfrac{a_4}{0} + \dfrac{a_4}{1}, \quad \gamma_{B\circ\beta\circ 1}(x) = \dfrac{b_4}{0} + \dfrac{b_3}{1}$

If $a_4 = a_3$ and $b_4 = b_3$ then, $\circ$ is associative for all $0, 1 \in I$.

Hence, an Intuitionistic Fuzzy Γ-hyperstructure $(I, \circ)$ is an *Intuitionistic fuzzy* Γ*-semihypergroup*.

If not, then $\circ$ is not associative for all $0, 1 \in I$ and hence $(I, \circ)$ is not an *Intuitionistic fuzzy* Γ*- semihypergroup*.

Corollary 3.5. If $I = \{0, 1\}$, then the intuitionistic fuzzy set defines an *intuitionistic fuzzy* Γ*-hypergroup* on I iff $A = 1_{\sim}$.

Definition 3.6. Let $(I, \circ)$ be an intuitionistic fuzzy hypergroup together with atleast an identity. An element $a^{'} \in I$ is called an *inverse* of $a \in I$ if there exists an identity $e \in I$ such that $\mu_{a\circ a'}(e) > 0$ and $\gamma_{a\circ a'}(e) > 0$. where $\mu_{a\circ a'}(e) + \gamma_{a\circ a'}(e) \leq 1$. Similarly for $a^{'} \circ a$.

Definition 3.7. An intuitionistic fuzzy hypergroup is said to *regular* if it has atleast one identity and each element has atleast one inverse.

Definition 3.8. A regular intuitionistic fuzzy hypergroup $(I, \circ)$ is called *reversible* if for any $x, y, a \in I$, it satisfies the following conditions,

if $\mu_{a\circ x}(y) > 0$, then there exists an inverse $a^{'}$ of a, such that $\mu_{a'\circ y}(x) > 0$ and $\gamma_{a\circ x}(y) > 0$, then there exists an inverse $a^{'}$ of a, such that $\gamma_{a'\circ y}(x) > 0$. where $\mu_{a\circ x}(y) + \gamma_{a\circ x}(y) \leq 1$ also, $\mu_{a'\circ y}(x) + \gamma_{a'\circ y}(x) \leq 1$. Similarly for $x \circ a$.

Definition 3.9. An intuitionistic fuzzy hypergroup I is said to be an *intuitionistic fuzzy canonical*, if

(i) It is commutative

(ii) It has an scalar identity

(iii) Every element has a unique inverse

(iv) It is reversible.

Definition 3.10. Let I, Γ be two non empty sets and I together with an intuitionistic fuzzy Γ-hyperoperation $\circ$ for all $a, b \in I, \alpha \in \Gamma$ and $p \in [0, 1]$. Consider the p-cuts as,

$(a \circ \alpha \circ b)_p = a \circ_p \alpha \circ_p b = \langle x, \mu_{a\circ\alpha\circ b}(x), \gamma_{a\circ\alpha\circ b}(x)\rangle$

where $\mu_{a\circ\alpha\circ b}(x) \geq p$ and $\gamma_{a\circ\alpha\circ b}(x) \leq 1 - p$ for all $x \in I$.

Proposition 3.11. Let $(I, \circ)$ be an intuitionistic fuzzy Γ-hyperoperation. For all $a, b, c, x \in I$ and $\alpha, \beta \in \Gamma$, $p \in [0, 1]$, the following equivalence holds,

$a \circ \alpha \circ (b \circ \beta \circ c) \geq p \Longleftrightarrow x \in a \circ_p \alpha \circ_p (b \circ_p \beta \circ_p c)$ and

$(a \circ \alpha \circ b) \circ \beta \circ c \geq p \Longleftrightarrow x \in (a \circ_p \alpha \circ_p b) \circ_p \beta \circ_p c$.

Proof. Let $(I, \circ)$ be an intuitionistic fuzzy Γ-hyperoperation.

Let us assume that $a \circ \alpha \circ (b \circ \beta \circ c) \geq p$

where $\mu_{a\circ\alpha\circ(b\circ\beta\circ c)}(x) = \vee_{t\in I}[\mu_{a\circ\alpha\circ t}(x) \wedge \mu_{b\circ\beta\circ c}(t)] \geq p$ and

$\gamma_{a\circ\alpha\circ(b\circ\beta\circ c)}(x) = \wedge_{t\in I}[\gamma_{a\circ\alpha\circ t}(x) \vee \gamma_{b\circ\beta\circ c}(t)] \leq 1 - p$.

Now, if and only if there exists $t_0 \in I$, such that

$\mu_{a\circ\alpha\circ t_0}(x) \geq p, \mu_{b\circ\beta\circ c}(t_0)] \geq p$ and

$\gamma_{a\circ\alpha\circ t_0}(x) \leq 1 - p, \gamma_{b\circ\beta\circ c}(t_0)] \leq 1 - p$.

Thus, $x \in a \circ_p \alpha \circ_p t_0$ and $t_0 \in b \circ_p \beta \circ_p c$

Hence, $x \in a \circ_p \alpha \circ_p (b \circ_p \beta \circ_p c)$. ∎

4. Intuitionistic Fuzzy Γ-HyperModule

In this section, we introduce the notion of Intuitionistic Fuzzy Γ-hypermodule as a generalization of Fuzzy Γ-hypermodule and some of their propositions are investigated.

Definition 4.1. Let R, Γ be two nonempty sets and $+$, . be two intuitionistic fuzzy hyperoperations on R and $\times$ be an intuitionistic fuzzy hyperoperation on Γ. Let $(R, +)$ and $(\Gamma, \times)$ be two intuitionistic fuzzy canonical hypergroups. R is called an *intuitionistic fuzzy* Γ*-hyperring* if there exists the mapping, . : $R \times \Gamma \times R \to F^*(R)$ such that for all $r, s, t \in R, \alpha, \beta \in \Gamma$, the following conditions are satisfied,

(i) $r.\alpha.(s + t) = (r.\alpha.s) + (r.\alpha.t)$,

(ii) $r.(\alpha \times \beta).s = (r.\alpha.s) + (r.\beta.s)$,

(iii) $(r + s).\alpha.t = (r.\alpha.t) + (s.\alpha.t)$,

(iv) $r.\alpha.(s.\beta.t) = (r.\alpha.s).\beta.t$.

Example 4.2. Let $R = \{0, 1, 2\}$ and $\Gamma = \{1, 2\}$. R has the following two intuitionistic fuzzy hyperoperations $+$, . and Γ has an intuitionistic fuzzy hyperoperation $\times$ and they are defined as follows,

$+$	0	1	2
0	0	1	2
1	1	2	$\{1, 2\}$
2	2	$\{1, 2\}$	R

.	0	1	2
0	0	0	0
1	0	1	2
2	0	2	$\{0, 1\}$

$\times$	1	2
1	1	$\{1, 2\}$
2	$\{1, 2\}$	2

Now, the non-empty sets R, Γ satisfies the above 4 conditions and thus R is an *intuitionistic fuzzy* Γ*-hyperring*.

Definition 4.3. Let $(\Gamma, \times)$ be an intuitionistic fuzzy canonical hypergroups. Let $(R, +, .)$ be an intuitionistic fuzzy Γ-hyperring. A nonempty set M together with two intuitionistic fuzzy Γ-hyperoperation $\oplus, *$ is called an *intuitionistic fuzzy left* Γ*-hypermodule* over $(R, +, .)$ if the following conditions hold,

(1). $(M, \oplus)$ is an intuitionistic fuzzy canonical Γ-hypergroup,
(2). $* : R \times \Gamma \times M \to F^*(M)$ is such that for all $a, b \in M, r, s \in R$, $\alpha, \beta \in \Gamma$ we have,

(i) $r * \alpha * (a \oplus b) = (r * \alpha * a) \oplus (r * \alpha * b)$,
(ii) $(r + s) * \alpha * a = (r * \alpha * a) \oplus (s * \alpha * a)$,
(iii) $r * (\alpha \times \beta) * a = (r * \alpha * a) \oplus (r * \beta * a)$,
(iv) $r * \alpha * (s * \beta * a) = (r * \alpha * s) * \beta * a$.

Example 4.4. Let $R = \{0, 1, 2, 3\}, M = \{0, 1, 2, 3\}$ and $\Gamma = \{2, 3\}$. M has the following two intuitionistic fuzzy hyperoperations $\oplus, *$. Γ has an intuitionistic fuzzy hyperoperation $\times$ and R has $+, .$ and they are defined as follows,

$+$	0	1	2	3
0	0	1	2	3
1	1	$\{2, 3\}$	$\{1, 3\}$	$\{0, 1\}$
2	2	3	$\{1, 3\}$	$\{0, 2\}$
3	3	$\{0, 1\}$	$\{0, 2\}$	$\{1, 2\}$

$\times$	2	3
2	3	$\{2, 3\}$
3	$\{2, 3\}$	2

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	2	3	$\{1, 3\}$
2	2	3	$\{1, 3\}$	$\{2, 3\}$
3	3	$\{1, 3\}$	$\{2, 3\}$	$\{1, 2\}$

$*$	0	1	2	3
0	0	0	$\{1, 2\}$	$\{2, 3\}$
1	0	1	2	3
2	$\{1, 2\}$	2	$\{1, 3\}$	$\{2, 3\}$
3	$\{2, 3\}$	3	$\{2, 3\}$	$\{1, 2\}$

Now, the non-empty sets R, M and Γ satisfies the above 2 conditions and thus M is an *intuitionistic fuzzy* Γ*-hypermodule*.

Proposition 4.5. Let $(M, \oplus, *)$ be an intuitionistic fuzzy Γ-hypermodule over an intuitionistic fuzzy Γ hyperring $(R, +, .)$. Then, for all $a \in M, r \in R$ and $\alpha \in \Gamma$, the following conditions are equivalence,
1). $a \oplus M = 1_{\sim} \Longleftrightarrow$ for all $p \in [0, 1], a \oplus_p M = M$.
2). $r * \alpha * M = 1_{\sim} \Longleftrightarrow$for all $p \in [0, 1], r *_p \alpha *_p M = M$.

Proof. Let $(M, \oplus, *)$ be an intuitionistic fuzzy Γ-hypermodule over an intuitionistic fuzzy Γ hyperring $(R, +, .)$.

1). Let us assume that $a \oplus M = 1_{\sim} = \langle x, 1, 0\rangle$

Then, for all $t \in M$, $p \in [0, 1]$ we have $a \oplus M = \langle x, \mu_{a\oplus M}(x), \gamma_{a\oplus M}(x)\rangle$

where $\mu_{a\oplus M}(x) = \vee_{y\in M}[\mu_{a\oplus y}(x)] = 1 \geq p$ and

$\gamma_{a\oplus M}(x) = \wedge_{y\in M}[\gamma_{a\oplus y}(x)] = 0 \leq 1 - p$, then there exists $y_0 \in M$ such that $\mu_{a\oplus y_0}(x) \geq p$ and $\gamma_{a\oplus y_0}(x) \leq 1 - p$.

Therefore, $x \in a \oplus_p y_0$ for all $p \in [0, 1]$.

Thus, $a \oplus_p M = M$.

Conversely, Let us assume that $a \oplus_1 M = M$ for $p = 1$.

For all $x \in M$, there exists $u \in M$ such that $x \in a \oplus_1 M$

(ie) $\mu_{a\oplus_1 M}(x) = 1$ and $\gamma_{a\oplus_1 M}(x) = 1 - p = 0$

Thus, $a \oplus M = 1_{\sim}$.

Similarly, we can able to prove the (2) equivalence condition. ■

Proposition 4.6. Let $(M, +, .)$ be a Γ- hypermodule over a Γ-hyperring. Let $(R, \uplus, \circ)$ be a canonical hypergroup $(\Gamma, \times)$. Then, $(M, \oplus, *)$ is an intuitionistic fuzzy Γ-hypermodule over an intuitionistic fuzzy Γ-hyperring $(R, +, .)$ and intuitionistic fuzzy canonical hypergroup $(\Gamma, \otimes)$.

Proof. Let $(M, +, .)$ be a Γ- hypermodule over a Γ-hyperring. $(M, \oplus)$ is a commutative intuitionistic fuzzy Γ-hypergroup.

To Prove:$(M, \oplus)$ is an intuitionistic fuzzy canonical Γ-hypergroup.

We have, $(M, +)$ is a canonical Γ-hypergroup,

Identity: Then there exists $e \in M$, for all $a \in M$ such that

$a = e + a = a + e \Rightarrow e \oplus a = \langle x, \mu_{e\oplus a}(x), \gamma_{e\oplus a}(x)\rangle$

where $\mu_{e\oplus a}(x) = \mu_{e+a}(x) > 0$ and $\gamma_{e\oplus a}(x) = \gamma_{e+a}(x) > 0$

Similarly for $a \oplus e$

Inverse: There exists $b \in M$ such that $e \in a + b \cap b + a$

$\Rightarrow a \oplus b = \langle x, \mu_{a\oplus b}(x), \gamma_{a\oplus b}(x)\rangle$

where $\mu_{a\oplus b}(x) = \mu_{a+b}(x) > 0$ and $\gamma_{a\oplus b}(x) = \gamma_{a+b}(x) > 0$

Similarly for $b \oplus a$. Let $a \oplus x = \langle y, \mu_{a\oplus x}(y), \gamma_{a\oplus x}(y)\rangle$

where $\mu_{a\oplus x}(y) = \mu_{a+x}(y) > 0 \Rightarrow y \in a + x$,

there exists b such that $x \in b + y \Rightarrow \mu_{b+y}(x) = \mu_{b\oplus y}(x) > 0$.

Thus, $(M, \oplus)$ is an intuitionistic fuzzy canonical Γ-hypergroup.

Claim: $(M, \oplus, *)$ is an intuitionistic fuzzy Γ-hypermodule.

Let $r, s \in R$, $\alpha, \beta \in \Gamma$, $a \in M$

Now, L.H.S= $r * \alpha * (s * \beta * a) = \langle x, \mu_{r*\alpha*(s*\beta*a)}(x), \gamma_{r*\alpha*(s*\beta*a)}(x)\rangle$

$$\begin{aligned} \text{where } \mu_{r*\alpha*(s*\beta*a)}(x) &= \vee_{t\in M}[\mu_{r*\alpha*t}(x) \wedge \mu_{s*\beta*a}(t)] \\ &= \vee_{t\in M}[\mu_{r.\alpha.t}(x) \wedge \mu_{s.\beta.a}(t)] \\ &= \begin{cases} 1 & \text{if } t \in r.\alpha.(s.\beta.a) \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

and $\gamma_{r*\alpha*(s*\beta*a)}(x) = \wedge_{t\in M}[\gamma_{r*\alpha*t}(x) \vee \gamma_{s*\beta*a}(t)]$

$= \wedge_{t\in M}[\gamma_{r.\alpha.t}(x) \vee \gamma_{s.\beta.a}(t)]$

$= \begin{cases} 0 & \text{if } t \in r.\alpha.(s.\beta.a) \\ 1 & \text{otherwise.} \end{cases}$

Similarly, for R.H.S hence, $r * \alpha * (s * \beta * a) = (r * \alpha * s) * \beta * a$. ■

Proposition 4.7. Let $(M, \oplus, *)$ be an intuitionistic fuzzy Γ-hypermodule over an intuitionistic fuzzy Γ hyperring $(R, +, .)$ and intuitionistic fuzzy canonical hypergroup $(\Gamma, \otimes)$, then $(M, +, .)$ is a Γ-hypermodule over the Γ-hyperring $(R, \uplus, \circ)$ and canonical hypergroup $(\Gamma, \times)$.

Proof. Let $(M, \oplus, *)$ be an intuitionistic fuzzy Γ-hypermodule over an intuitionistic fuzzy Γ hyperring $(R, +, .)$ and intuitionistic fuzzy canonical hypergroup $(\Gamma, \otimes)$.

To Prove: $(M, +)$, $(R, \uplus)$, $(\Gamma, \times)$ and $(M, .)$ are canonical hypergroups. For that, we claim that $(M, .)$ is a Γ-hypermodule. By definition 2.9,

for all $r, s \in R, \alpha \in \Gamma, a \in M$, $(r \uplus s).\alpha.a = r.\alpha.a + s.\alpha.a$

Let $x \in (r \uplus s).\alpha.a = \cup_{y\in r\uplus s}(y * \alpha * a)$

$\Rightarrow \mu_{r+s}(y) > 0$ and $\gamma_{r+s}(y) > 0$

also, $\mu_{y*\alpha*a}(x) > 0$ and $\gamma_{y*\alpha*a}(x) > 0$ for all $y \in r \uplus s, x \in M$.

Thus, $\vee_{y\in M}[\mu_{r+s}(y) \wedge \mu_{y*\alpha*a}(x)] > 0$

and $\wedge_{y\in M}[\gamma_{r+s}(y) \vee \gamma_{y*\alpha*a}(x)] > 0$.

(ie) $((r + s) * \alpha * a)(x) > 0$

$\Rightarrow \mu_{(r*\alpha*a)\oplus(s*\alpha*a)}(x) > 0$ and $\gamma_{(r*\alpha*a)\oplus(s*\alpha*a)}(x) > 0$.

Thus, there exists $z, t \in M$ such that $\mu_{z\oplus t}(x) > 0$ and $\gamma_{z\oplus t}(x) > 0$.

(ie) $x \in z + t \Rightarrow x \in r.\alpha.a + s.\alpha.a$.

Thus, $(r \uplus s).\alpha.a \subseteq r.\alpha.a + s.\alpha.a$. Similarly, $r.\alpha.a + s.\alpha.a \subseteq (r \uplus s).\alpha.a$.

Hence, $(r \uplus s).\alpha.a = r.\alpha.a + s.\alpha.a$.

Likewise, the rest of the remaining three conditions shall be proved.

$\Rightarrow (M, .)$ is a Γ-hypermodule. ■

Proposition 4.8. Let M be an R_Γ-module and A be an intuitionistic fuzzy Γ-module of M. Then, the set M will be an intuitionistic fuzzy Γ-hypermodule.

Proof. Let $(\Gamma, *)$ be an abelian group and $(M, +, .)$ be a Γ-module over a Γ-ring $(R, \uplus, \circ)$. We define an intuitionistic fuzzy Γ-hyperoperation on M as follows,

1) $a \oplus b = \langle x, \mu_{a\oplus b}(x), \gamma_{a\oplus b)}(x)\rangle$,

where $\#\mu_{a\oplus b}(x) = \mu_{a+b}(x)$ and $\gamma_{a\oplus b}(x) = \gamma_{a+b}(x)$.

2) $r * \alpha * a = \langle x, \mu_{r*\alpha*a}(x), \gamma_{r*\alpha*a}(x)\rangle$,

where $\mu_{r*\alpha*a}(x) = \mu(r.\alpha.a - x)$ and $\gamma_{r*\alpha*a}(x) = \gamma(r.\alpha.a - x)$.

To Prove: $(M, \oplus, *)$ is an intuitionistic fuzzy Γ-hypermodule.

For that, Let $r, s \in R, a, b \in M$. By definition 4.3

Now, (i). L.H.S=$r * \alpha * (a \oplus b) = \langle x, \mu_{r*\alpha*(a\oplus b)}(x), \gamma_{r*\alpha*(a\oplus b)}(x)\rangle$

where $\mu_{r*\alpha*(a\oplus b)}(x) = \vee_{p\in M}[\mu_{r*\alpha*p}(x) \wedge \mu_{a\oplus b}(p)]$

$= \vee_{p\in M}[\mu(r.\alpha.p - x) \wedge \mu_{a+b}(p)]$
$= \mu(r.\alpha.(a+b) - x)$
and $\gamma_{r*\alpha*(a\oplus b)}(x) = \wedge_{p\in M}[\gamma_{r*\alpha*p}(x) \vee \gamma_{a\oplus b}(p)]$
$= \wedge_{p\in M}[\gamma(r.\alpha.p - x) \vee \gamma_{a+b}(p)]$
$= \gamma(r.\alpha.(a+b) - x)$
R.H.S= $(r*\alpha*a) \oplus (r*\alpha*b) = \langle x, \mu_{(r*\alpha*a)\oplus(r*\alpha*b)}(x), \gamma_{(r*\alpha*a)\oplus(r*\alpha*b)}(x)\rangle$
where $\mu_{(r*\alpha*a)\oplus(r*\alpha*b)}(x) = \vee_{p,q\in M}[\mu_{r*\alpha*a}(p) \wedge \mu_{p\oplus q}(x) \wedge \mu_{r*\alpha*b}(q)]$
$=\vee_{p,q\in M}[\mu(r.\alpha.a - p) \wedge \mu_{p+q}(x) \wedge \mu(r.\alpha.b - q)]$
$\mu_{(r*\alpha*a)\oplus(r*\alpha*b)}(x) = \vee_{x=p+q}[\mu(r.\alpha.a - p) \wedge \mu(r.\alpha.b - q)]$
$\leq \mu(r.\alpha.a - p + r.\alpha.b - q)$
$= \mu(r.\alpha.(a+b) - (p+q))$
Thus, $\mu_{(r*\alpha*a)\oplus(r*\alpha*b)}(x) \leq \mu(r.\alpha.(a+b) - x) ----(1)$
If $q = r.\alpha.b$, $p = x - r.\alpha.b$, then
$\vee_{p,q\in M,x=p+q}[\mu(r.\alpha.a - p) \wedge \mu(r.\alpha.b - q) \geq \vee_{p\in M}[\mu(r.\alpha.a - p)]$
$= \mu(r.\alpha.a - (x - r.\alpha.b)$
$\mu_{(r*\alpha*a)\oplus(r*\alpha*b)}(x) \geq \mu(r.\alpha.(a+b) - x) ----(2)$
From (1) and (2), we get $\mu_{(r*\alpha*a)\oplus(r*\alpha*b)}(x) = \mu(r.\alpha.(a+b) - x)$
and $\gamma_{(r*\alpha*a)\oplus(r*\alpha*b)}(x) = \wedge_{p,q\in M}[\gamma_{r*\alpha*a}(p) \vee \gamma_{p\oplus q}(x) \vee \gamma_{r*\alpha*b}(q)]$
$=\wedge_{p,q\in M}[\gamma(r.\alpha.a - p) \vee \gamma_{p+q}(x) \vee \gamma(r.\alpha.b - q)]$
$=\wedge_{x=p+q}[\gamma(r.\alpha.a - p) \vee \gamma(r.\alpha.b - q)]$
$\geq \gamma(r.\alpha.a - p + r.\alpha.b - q) = \gamma(r.\alpha.(a+b) - (p+q))$
Thus, $\gamma_{(r*\alpha*a)\oplus(r*\alpha*b)}(x) \geq \gamma(r.\alpha.(a+b) - x) ----(3)$
If $q = r.\alpha.b$, $p = x - r.\alpha.b$, then
$\wedge_{p,q\in M,x=p+q}[\gamma(r.\alpha.a - p) \vee \gamma(r.\alpha.b - q) \leq \wedge_{p\in M}[\gamma(r.\alpha.a - p)]$
$= \gamma(r.\alpha.a - (x - r.\alpha.b)$
$\gamma_{(r*\alpha*a)\oplus(r*\alpha*b)}(x) \leq \gamma(r.\alpha.(a+b) - x) ----(4)$
From (3) and (4), we get $\gamma_{(r*\alpha*a)\oplus(r*\alpha*b)}(x) = \gamma(r.\alpha.(a+b) - x)$
Hence,L.H.S=$r*\alpha*(a\oplus b) = (r*\alpha*a) \oplus (r*\alpha*b)$=R.H.S
Similarly, we can be able to prove the remaining three conditions.
$\Rightarrow (M, \oplus, *)$ is an intuitionistic fuzzy Γ-hypermodule. ■

Definition 4.9. Let $(M, \oplus, *)$ be an intuitionistic fuzzy Γ-hypermodule over an intuitionistic fuzzy Γ-hyperring $(R, +, .)$. A nonempty subset N of M is called an intuitionistic subfuzzy Γ-hypermodule, if for all $x, y \in N, r \in R, \alpha \in \Gamma$, the following conditions are hold,

(i) $x \oplus y = \langle t, \mu_{x\oplus y}(t), \gamma_{x\oplus y)}(t)\rangle$
where $\mu_{x\oplus y}(t) > 0$ and $\gamma_{x\oplus y}(t) > 0 \Rightarrow t \in N$.

(ii) $x \oplus N = \langle t, \mu_{x\oplus N}(t), \gamma_{x\oplus N)}(t)\rangle$
where $\mu_{x\oplus N}(t) = \begin{cases} 1 & if\ t \in N \\ 0 & if\ t \notin N. \end{cases}$ and $\gamma_{x\oplus N}(t) = \begin{cases} 0 & if\ t \in N \\ 1 & if\ t \notin N. \end{cases}$

(iii) $r*\alpha*x = \langle t, \mu_{r*\alpha*x}(t), \gamma_{r*\alpha*x}(t)\rangle$
where $\mu_{r*\alpha*x}(t) > 0$ and $\gamma_{r*\alpha*x}(t) > 0 \Rightarrow t \in N$.

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