Advances in Fuzzy Mathematics (AFM).
ISSN 0974-0201 Volume 11, Number 1 (2016), pp. 67-77

http://www.ripublication.com/afm.htm

Contraction and convexity in Fuzzy metric spaces

P. Thirunavukarasu

P.G. and Research Department of Mathematics,
Periyar E.V.R. College (Autonomous),
Tiruchirappalli-620 023,
TamilNadu, India.

U. Palaniyappa

Senior Principal,
Alpha Plus Matric Higher Secondary School,
K.K.Nagar, Tiruchirappalli-620 021,
TamilNadu, India.

K.K. Viswanathan

Department of Mathematical Sciences,
Faculty of Science,
Universiti Teknologi Malaysia,
81310 Johor Bahru, Johor, Malaysia.

Abstract

In this paper, we establish some new generalized fixed point theorems in complete fuzzy metric spaces and proved the existence of fixed points of non-expansive compact self-mapping defined on a closed subset having a contractive jointly continuous family when the underlying space is a Fuzzy metric space.

AMS Subject Classification: 47H10, 54H25.
Keywords: Fuzzy metric space, Convex metric spaces, set-valued mappings, convex structure, p-star shaped.

1. Introduction and Preliminaries

The study of fixed point theorems of maps satisfying contractive type conditions in fuzzy metric spaces has been a very active field of research activity recently. George and

Veeramani [5] introduced the concept of fuzzy metric spaces in different ways. Grabiec [3] obtained the fuzzy version of Banach contraction principle. Mishra and sharma et al. [8] proved common fixed point theorems for compatible maps on fuzzy metric spaces. In this paper we introduce the concept of convex structure in fuzzy metric space and obtain fixed point and common fixed point theorems for a pair of non-expansive compact self mappings under sufficient contractive type conditions.

2. Preliminaries

Definition 2.1. A fuzzy set A in X is a function with domain X and values in [0, 1].

Definition 2.2. A triangular norm $(t - norm)*$ is a binary operation on the unit interval [0, 1] such that for all $a, b, c, d \in [0, 1]$ the following conditions are satisfied:

1. $a * 1 = a$
2. $a * b = b * a$
3. $a * b \leq c * d$, whenever $a \leq c$ and $b \leq d$
4. $a * (b * c) = (a * b) * c$.

Definition 2.3. The 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary set, $*$ is a continuous $t - norm$ and M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions: for all $x, y, z \in X$ and $t_1, t_2 > 0$

1. $M(x, y, 0) = 0$.
2. $M(x, y, t) = 1, t > 0$ if and only if $x = y$.
3. $M(x, y, t) = M(y, x, t)$
4. $M(x, z, t_1 + t_2) \geq M(x, y, t_1) * M(y, z, t_2)$
5. $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous.
6. $\lim_{t \to \infty} {}'M(x, y, t) = 1$.

Example 2.4. Let (X, d) be a metric space. Define $a * b = ab$ (or) $(a.b = min\{a, b\})$ and for all $x, y \in X$ and $t > 0$,

$$M(x, y, t) = \frac{t}{t + d(x, y, z)}.$$

Then $(X, M, *)$ is a fuzzy metric space. We call this fuzzy metric M induced by the metric d as standard fuzzy metric.

Definition 2.5. The 3-tuple $(X, M, *)$ is said to be a fuzzy metric space in the sense of George and Veeramani if X is an arbitrary set, T is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions:

(i) $M(x, y, t) > 0, x, y \in X, t > 0,$

(ii) $M(x, y, t) = 1, t > 0, x = y,$

(iii) $M(x, y, t) = M(y, x, t), x, y \in X, t > 0,$

(iv) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s),$ where $x, y, z \in X, s, t > 0.$

(v) $M(x, y, \cdot) : X^2 \times (0, \infty) \to [0, 1]$ is continuous for every $x, y, z \in X.$

If (iv) is replaced by condition $M(x, z, t) \geq M(x, y, t) * M(y, z, t)$, where $x, y, z \in X$, $t > 0$. then $(X, M, *)$ is called a strong fuzzy metric space.

Throughout the paper by $\varphi(x, y, t)$ is denote $\left(\dfrac{1}{M(x, y, t)} - 1\right)$.

Definition 2.6. Let $(X, M, *)$ be a fuzzy metric space. Then a sequence $\{x_n\}$ in X is said to be a Cauchy sequence if

$$\lim_{t\to\infty} \varphi(x_n, x_{n+p}, t) = 0 \text{ for all } t > 0 \text{ and } n, p \in N.$$

Definition 2.7. Let $(X, M, *)$ be a fuzzy metric space. Then a sequence $\{x_n\}$ in X is said to be convergent to a point x ∈ X if

$$\lim_{t\to\infty} \varphi(x_n, x, t) = 0 \text{ for all } t > 0.$$

Definition 2.8. A fuzzy metric space X is said to be complete if every Cauchy sequence in X converges to some point in X.

Definition 2.9. Let $(X, M, *)$ be a fuzzy metric space. We will say the mapping $T : X \to X$ is fuzzy contractive if there exists $k \in (0, 1)$ such that

$$\varphi(Tx, Ty, t) \leq k\varphi(x, y, t)$$

for each $x, y \in X$ and $t > 0$. (k is called the contractive constant of T.)

Lemma 2.10. Let $\{x_n\}$ is a sequence in a fuzzy metric space X and if

$$\varphi(x_n, x_{n+1}, t) \leq k^n \varphi(x_0, x_1, t)$$

where

$$0 < k < 1, \quad n \in N.$$

Then $\{x_n\}$ is a Cauchy sequence in X.

Proof. Suppose that $\varphi(x_n, x_{n+1}, t) \leq k^n \varphi(x_0, x_1, t)$ where $0 < k < 1$, and $t \geq 0$.

Let m, n be two positive integers with $m \geq n$, say $m = n + p$, $p > 0$. Then we have

$$\begin{aligned}\varphi(x_n, x_{n+p}, t) &\leq \varphi(x_n, x_{n+1}, t) + \varphi(x_{n+1}, x_{n+2}, t) + \cdots + \varphi(x_{n+p-1}, x_{n+p}, t) \\ &\leq k^n \varphi(x_0, x_1, t) + k^{n+1}\varphi(x_0, x_1, t) + \cdots + k^{n+p-1}\varphi(x_0, x_1, t)\end{aligned}$$

Taking limit as $n \to \infty$ on both sides, we get

$$\lim_{n\to\infty} \varphi(x_n, x_{n+p}, t) = 0.$$

Hence $\{x_n\}$ is a Cauchy sequence in X. ■

3. Main results

Theorem 3.1. Let $(X, M, *)$ be a complete fuzzy metric space and $T : X \to X$ be a contraction mapping. Then T has a fixed point in X.

Proof. Let $x_0 \in X$ and $x_1 = Tx_0$ then there exists $x_2 = Tx_1$ such that

$$\varphi(x_2, x_1, t) \leq \varphi(Tx_1, Tx_0, t) + \mu \leq \mu\varphi(x_1, x_0, t) + \mu$$

Where $\mu \in (0, 1)$. similarly, there is $x_3 = Tx_2$ such that

$$\begin{aligned}\varphi(x_3, x_2, t) &\leq \varphi(Tx_2, Tx_1, t) + \mu^2 \\ &\leq \mu\varphi(x_2, x_1, t) + \mu^2 \\ &\leq \mu^2\varphi(x_1, x_0, t) + 2\mu^2.\end{aligned}$$

By induction, we get a sequence $\{x_n\}$ such that $x_{n+1} = Tx_n$ and

$$\varphi(x_{n+1}, x_n, t) \leq \mu^n\varphi(x_1, x_0, t) + n\mu^n.$$

Thus, $\{x_n\}$ is a cauchy sequence. Since X is complete, let $x_n \to x$. Now

$$\begin{aligned}\varphi(x, Tx, t) &\leq \varphi(x, x_n, t) + \varphi(x_n, Tx, t) \\ &\leq \varphi(x, x_n, t) + \varphi(Tx_{n-1}, Tx, t) \\ &\leq \varphi(x, x_n, t) + \mu\varphi(x_{n-1}, x, t) \to 0.\end{aligned}$$

Hence $x = Tx$. ■

Theorem 3.2. Let $(X, M, *)$ be a complete fuzzy metric space and let mapping $T_1, T_2 :$ $\mathbf{X} \to \mathbf{X}$ are contraction and satisfying the following conditions: For each $x \in X$, $T_1(x), T_2(x) \in X$ and

$$\begin{aligned}\varphi(T_1(x), T_2(y), t) &\leq \alpha_1\varphi(x, T_1(x), t) + \alpha_2\varphi(y, T_2(y), t) \\ &\quad + \alpha_3[\varphi(y, T_1(x), t) + \varphi(x, T_2(y), t)] + \alpha_4\varphi(x, y, t).\end{aligned}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4,$ are positive real numbers and $\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 < 1$. Then T_1 and T_2 has a common fixed point in X.

Proof. Let $x_0 \in X$ and $x_1 = T_1(x_0)$ then there exists $x_2 = T_2(x_1)$ such that

$$\varphi(x_1, x_2, t) \leq \varphi(T_1(x_0), T_2(x_1), t) + \mu$$

where

$$\mu = \max\left(\frac{\alpha_1 + \alpha_3 + \alpha_4}{1 - \alpha_2 - \alpha_3}, \frac{\alpha_2 + \alpha_3 + \alpha_4}{1 - \alpha_1 - \alpha_3}\right)$$

and hence $\mu \in (0, 1)$. Then we have

$$\begin{aligned}
\varphi(x_1, x_2, t) &\leq \varphi(T_1(x_0), T_2(x_1), t) + \mu \\
&\leq \alpha_1\varphi(x_0, T_1(x_0), t) + \alpha_2\varphi(x_1, T_2(x_1), t) \\
&+ \alpha_3[\varphi(x_0, T_2(x_1), t) + \varphi(x_1, T_1(x_0), t)] + \alpha_4\varphi(x_0, x_1, t) + \mu. \\
&\leq \alpha_1\varphi(x_0, x_1, t) + \alpha_2\varphi(x_1, x_2, t) \\
&+ \alpha_3[\varphi(x_0, x_2, t) + \varphi(x_1, x_1, t)] + \alpha_4\varphi(x_0, x_1, t) + \mu. \\
&\leq \alpha_1\varphi(x_0, x_1, t) + \alpha_2\varphi(x_1, x_2, t) + \alpha_3\varphi(x_0, x_2, t) + \alpha_4\varphi(x_0, x_1, t) + \mu. \\
&\leq \alpha_1\varphi(x_0, x_1, t) + \alpha_2\varphi(x_1, x_2, t) + \alpha_3\varphi(x_0, x_1, t) + \alpha_3\varphi(x_1, x_2, t) \\
&+ \alpha_4\varphi(x_0, x_1, t) + \mu.
\end{aligned}$$

$$\begin{aligned}
(1 - \alpha_2 - \alpha_3)\varphi(x_1, x_2, t) &\leq (\alpha_1 + \alpha_3 + \alpha_4)\varphi(x_0, x_1, t) + \mu. \\
\varphi(x_1, x_2, t) &\leq \frac{\alpha_1 + \alpha_3 + \alpha_4}{1 - \alpha_2 - \alpha_3}\varphi(x_0, x_1, t) + \mu. \\
\varphi(x_1, x_2, t) &\leq \mu\, \varphi(x_0, x_1, t) + \mu.
\end{aligned}$$

Now, there exists $x_3 = T_1(x_2)$ such that

$$\begin{aligned}
\varphi(x_2, x_3, t) &\leq \varphi(T_1(x_2), T_2(x_1), t) + \mu^2 \\
&\leq \alpha_1\varphi(x_2, T_1(x_2), t) + \alpha_2\varphi(x_1, T_2(x_1), t) \\
&+ \alpha_3[\varphi(x_2, T_2(x_1), t) + \varphi(x_1, T_1(x_2), t)] + \alpha_4\varphi(x_1, x_2, t) + \mu^2. \\
&\leq \alpha_1\varphi(x_2, x_3, t) + \alpha_2\varphi(x_1, x_2, t) \\
&+ \alpha_3[\varphi(x_2, x_2, t) + \varphi(x_1, x_3, t)] + \alpha_4\varphi(x_1, x_2, t) + \mu^2. \\
&\leq \alpha_1\varphi(x_2, x_3, t) + \alpha_2\varphi(x_1, x_2, t) + \alpha_3\varphi(x_1, x_2, t) \\
&+ \alpha_3\varphi(x_2, x_3, t) + \alpha_4\varphi(x_1, x_2, t) + \mu^2.
\end{aligned}$$

$$\begin{aligned}
(1 - \alpha_1 - \alpha_3)\varphi(x_2, x_3, T) &\leq (\alpha_2 + \alpha_3 + \alpha_4)\varphi(x_1, x_2, t) + \mu^2. \\
\varphi(x_2, x_3, t) &\leq \frac{\alpha_2 + \alpha_3 + \alpha_4}{1 - \alpha_1 - \alpha_3}\varphi(x_1, x_2, t) + \mu^2. \\
&\leq \mu\{\mu\varphi(x_0, x_1, t) + \mu\} + \mu^2. \\
&\leq \mu^2\varphi(x_0, x_1, t) + 2\mu^2.
\end{aligned}$$

Continuing this process, we get a sequence $\{x_n\}$ such that $x_{n+1} = T_2(x_n)$ or $x_{n+1} = T_1(x_n)$ and

$$\varphi(x_{n+1}, x_n, t) \leq \mu^n \varphi(x_0, x_1, t) + n\mu^n.$$

Since $\mu < 1$, it follows from Cauchy's root test that $\sum n\mu^n$ is convergent and hence $\{x_n\}$ is a Cauchy sequence in X. Then there exists $x \in X$ such that $x_n \to x$ as $n \to \infty$.

$$\begin{aligned}
\varphi(T_1(x), x, t) &\leq \varphi(x, x_n, t) + \varphi(x_n, T_1(x), t) \\
&\leq \varphi(x, x_n, t) + \varphi(T_2(x_{n-1}), T_1(x), t) \\
&\leq \varphi(x, x_n, t) + \alpha_1\varphi(x, T_1(x), t) + \alpha_2\varphi(x_{n-1}, T_2(x_{n-1}), t) \\
&+ \alpha_3[(x, T_2(x_{n-1}), t) + \varphi(x_{n-1}, T_1(x), t)] + \alpha_4\varphi(x, x_{n-1}, t) \\
&\leq \varphi(x, x_n, t) + \alpha_1\varphi(x, T_1(x), t) + \alpha_2\varphi(x_{n-1}, x_n, t) \\
&+ \alpha_3[\varphi(x, x_n, t) + \varphi(x_{n-1}, T_1(x), t)] + \alpha_4\varphi(x, x_{n-1}, t) \\
&\leq \varphi(x, x_n, t) + \alpha_1\varphi(x, T_1(x), t) + \alpha_2\varphi(x_{n-1}, x_n, t) \\
&+ \alpha_3[\varphi(x, x_n, t) + \varphi(x, x_{n-1}, t)] + \alpha_3\varphi(x, T_1(x), t) + \alpha_4\varphi(x, x_{n-1}, t)
\end{aligned}$$

$$\begin{aligned}
(1 - \alpha_1 + \alpha_3)\varphi(x, T_1(x), t) &\leq (1 + \alpha_3)\varphi(x, x_n, t) + \alpha_2\varphi(x_{n-1}, x_n, t) \\
&+ (\alpha_3 + \alpha_4)\varphi(x, x_{n-1}, t) \\
\varphi(x, T_1(x), t) &\leq \frac{1 + \alpha_3}{1 - \alpha_1 + \alpha_3}\varphi(x, x_n, t) + \frac{\alpha_2}{1 - \alpha_1 + \alpha_3}\varphi(x_{n-1}, x_n, t) \\
&+ \frac{\alpha_3 + \alpha_4}{1 - \alpha_1 + \alpha_3}\varphi(x, x_{n-1}, t) \\
&\to 0 \text{ as } n \to \infty
\end{aligned}$$

Hence $x = T_1(x)$.
Similarly we can prove that $x = T_2(x)$. Hence $x = T_1(x) = T_2(x)$. ■

Corollary 3.3. Let $(X, M, *)$ be a complete metric space and let mapping $T_1, T_2 : \mathbf{X} \to \mathbf{X}$ satisfying the following conditions:

(i) for each $x \in X$, $T_1(x), T_2(x) \in X$.

(ii) $\varphi(T(x), T(y), t) \leq q\varphi(x, y, t)$ for some $q \in [0, 1)$.

Then there exists an element $p \in X$ such that $p = T(p)$.

Proof. The proof of the corollary immediately follows by putting

$$T_1 = T_2 = T \quad \text{and} \quad \alpha_1 = \alpha_2 = \alpha_3 = 0, \alpha_4 = q$$

in the previous theorem. ■

The following theorem generalizes the Bose and Mukherjee's theorem in fuzzy metric spaces.

Theorem 3.4. Let $(X, M, *)$ be a complete fuzzy metric space and the self mappings $T_1, T_2 : \mathbf{X} \to \mathbf{X}$ are contractive and satisfying the following conditions: For each $x \in X$, and $T_1(x), T_2(x) \in X$.

$$\varphi(T_1(x), T_2(y), t) \leq \alpha_1\varphi(x, T_1(x), t) + \alpha_2\varphi(y, T_2(y), t) + \alpha_3\varphi(y, T_1(x, t)) + \alpha_4\varphi(x, T_2(y), t) + \alpha_5\varphi(x, y, t).$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ are positive real numbers and $\Sigma_{i=1}^{5}\alpha_i < 1$, $\alpha_1 = \alpha_2$ or $\alpha_3 = \alpha_4$. Then T_1, T_2 has a common fixed point in X.

Proof. Let $x_0 \in X$ and $x_1 = T_1(x_0)$ then there exists $x_2 = T_2(x_1)$ such that

$$\varphi(x_1, x_2, t) \leq \varphi(T_1(x_0), T_2(x_1), t) + \mu$$

where

$$\mu = \max\left(\frac{\alpha_1 + \alpha_3 + \alpha_5}{1 - \alpha_2 - \alpha_3}, \frac{\alpha_2 + \alpha_4 + \alpha_5}{1 - \alpha_1 - \alpha_4}\right),$$

hence $\mu \in (0, 1)$. Then we have

$$\begin{aligned}
\varphi(x_1, x_2, t) &\leq \varphi(T_1(x_0), T_2(x_1), t) + \mu \\
&\leq \alpha_1\varphi(x_0, T_1(x_0), t) + \alpha_2\varphi(x_1, T_2(x_1), t) + \alpha_3\varphi(x_0, T_2(x_1), t) \\
&+ \alpha_4\varphi(x_1, T_1(x_0), t) + \alpha_5\varphi(x_0, x_1, t) + \mu. \\
&\leq \alpha_1\varphi(x_0, x_1, t) + \alpha_2\varphi(x_1, x_2, t) + \alpha_3\varphi(x_0, x_2, t) \\
&+ \alpha_4\varphi(x_1, x_1, t) + \alpha_5\varphi(x_0, x_1, t) + \mu. \\
&\leq \alpha_1\varphi(x_0, x_1, t) + \alpha_2\varphi(x_1, x_2, t) + \alpha_3\varphi(x_0, x_2, t) + \alpha_5\varphi(x_0, x_1, t) + \mu. \\
&\leq \alpha_1\varphi(x_0, x_1, t) + \alpha_2\varphi(x_1, x_2, t) + \alpha_3\varphi(x_0, x_1, t) + \alpha_3\varphi(x_1, x_2, t) \\
&+ \alpha_5\varphi(x_0, x_1, t) + \mu.
\end{aligned}$$

$$\text{Hence} \quad (1 - \alpha_2 - \alpha_3)\varphi(x_1, x_2, t) \leq (\alpha_1 + \alpha_3 + \alpha_5)\varphi(x_0, x_1, t) + \mu.$$

$$\text{Thus} \quad \varphi(x_1, x_2, t) \leq \left(\frac{\alpha_1 + \alpha_3 + \alpha_5}{1 - \alpha_2 - \alpha_3}\right)\varphi(x_0, x_1, t) + \mu.$$

$$\text{Therefore} \quad \varphi(x_1, x_2, t) \leq \mu\, \varphi(x_0, x_1, t) + \mu.$$

Now, there exists $x_3 \in T_1(x_2)$ such that

$$\begin{aligned}
\varphi(x_2, x_3, t) &\leq \varphi(T_1(x_2), T_2(x_1), t) + \mu^2 \\
&\leq \alpha_1\varphi(x_2, T_1(x_2), t) + \alpha_2\varphi(x_1, T_2(x_1), t) + \alpha_3\varphi(x_2, T_2(x_1), t) \\
&+ \alpha_4\varphi(x_1, T_1(x_2), t) + \alpha_5\varphi(x_1, x_2, t) + \mu^2. \\
&\leq \alpha_1\varphi(x_2, x_3, t) + \alpha_2\varphi(x_1, x_2, t) + \alpha_3\varphi(x_2, x_2, t) \\
&+ \alpha_4\varphi(x_1, x_3, t) + \alpha_5\varphi(x_1, x_2, t) + \mu^2. \\
&\leq \alpha_1\varphi(x_2, x_3, t) + \alpha_2\varphi(x_1, x_2, t) + \alpha_4\varphi(x_1, x_2, t) \\
&+ \alpha_4\varphi(x_2, x_3, t) + \alpha_5\varphi(x_1, x_2, t) + \mu^2.
\end{aligned}$$

$$\text{Hence}\quad (1-\alpha_1-\alpha_4)\varphi(x_2,x_3,t) \leq (\alpha_2+\alpha_4+\alpha_5)\varphi(x_1,x_2,t)+\mu^2.$$

$$\begin{aligned}\text{Thus}\quad \varphi(x_2,x_3,t) &\leq \left(\frac{\alpha_2+\alpha_4+\alpha_5}{1-\alpha_1-\alpha_4}\right)\varphi(x_1,x_2,t)+\mu^2.\\ &\leq \mu\{\mu\varphi(x_0,x_1,t)+\mu\}+\mu^2.\\ &\leq \mu^2\varphi(x_0,x_1,t)+2\mu^2.\end{aligned}$$

Continuing this process, we get a sequence $\{x_n\}$ such that $x_{n+1} = T_2(x_n)$ or $x_{n+1} = T_1(x_n)$ and $\varphi(x_{n+1}, x_n, t) \leq \mu^n\varphi(x_0, x_1, t)+n\mu^n$. Since $\mu < 1$, it follows from Cauchy's root test that $\sum n\mu^n$ is convergent and hence $\{x_n\}$ is a Cauchy sequence in X. Then there exists $x \in X$ such that $x_n \to x$ as $n \to \infty$.

$$\begin{aligned}\varphi(T_1(x),x,t) &\leq \varphi(x,x_n,t)+\varphi(x_n,T_1(x),t)\\ &\leq \varphi(x,x_n,t)+\varphi(T_2(x_{n-1}),T_1(x),t)\\ &\leq \varphi(x,x_n,t)+\alpha_1\varphi(x,T_1(x),t)+\alpha_2\varphi(x_{n-1},T_2(x_{n-1}),t)\\ &\quad+\alpha_3\varphi(p,T_2(x_{n-1}),t)\\ &\quad+\alpha_4\varphi(x_{n-1},T_1(x),t)+\alpha_5\varphi(x,x_{n-1},t)\\ &\leq \varphi(x,x_n,t)+\alpha_1\varphi(x,T_1(x),t)+\alpha_2\varphi(x_{n-1},x_n,t)+\alpha_3\varphi(x,x_n,t)\\ &+\alpha_4\varphi(x_{n-1},T_1(x),t)+\alpha_5\varphi(x,x_{n-1},t)\\ &\leq \varphi(x,x_n,t)+\alpha_1\varphi(x,T_1(x),t)+\alpha_2\varphi(x_{n-1},x_n,t)+\alpha_3\varphi(x,x_n,t)\\ &+\alpha_4\varphi(x,x_{n-1},t)+\alpha_4\varphi(x,T_1(x),t)+\alpha_5\varphi(x,x_{n-1},t).\end{aligned}$$

$$\begin{aligned}\text{Hence}\quad (1-\alpha_1+\alpha_4)\varphi(x,T_1(x),t) &\leq (1+\alpha_3)\varphi(x,x_n,t)+\alpha_2\varphi(x_{n-1},x_n,t)\\ &\quad+(\alpha_4+\alpha_5)\varphi(x,x_{n-1},t).\end{aligned}$$

$$\begin{aligned}\text{Thus}\quad \varphi(x,T_1(x),t) &\leq \left(\frac{1+\alpha_3}{1-\alpha_1+\alpha_4}\right)\varphi(x,x_n,t)+\left(\frac{\alpha_2}{1-\alpha_1+\alpha_4}\right)\varphi(x_{n-1},x_n,t)\\ &\quad+\left(\frac{\alpha_4+\alpha_5}{1-\alpha_1+\alpha_4}\right)\varphi(x,x_{n-1},t)\\ &\to 0 \text{ as } n\to\infty\end{aligned}$$

Therefore $\varphi(T_1(x), x, t) = 0$. And so $x = T_1(x)$.
Similarly we can prove that $x = T_2(x)$. Hence $x = T_1(x) = T_2(x)$. ■

3.1. Convex structure

Definition 3.5. Let X and Y be topological spaces. A mapping $T : X \to Y$ is called compact if $T(X)$ is contained in a compact subset of Y. If T is compact, then $T(X)$ is compact.

Definition 3.6. Let $(X, M, *)$ be a fuzzy metric space. A continuous mapping $W : X' \times X \times [0, 1]' \to X$ is said to be a convex structure on X if for all $x, y \in X$ and $\lambda \in [0, 1]$,

$$\varphi(u, W(x, y, \lambda), t) \leq \lambda\varphi(u, x, t) + (1 - \lambda)\varphi(u, y, t)$$

holds for all $u \in X$. The fuzzy metric space $(X, M, *)$ together with a convex structure is called a **fuzzy convex metric space**.

Definition 3.7. A subset K of a fuzzy convex metric space $(X, M, *)$ is said to be a **convex set** if $W(x, y, \lambda) \in K$ for all $x, y \in K$ and $\lambda' \in [0, 1]$.

Definition 3.8. A subset K of a fuzzy convex metric space $(X, M, *)$ is said to be a **p-star shaped** if there exists a $p \in K$ such that $W(x, p, \lambda) \in K$ for all $x \in K$ and $\lambda \in [0, 1]$.

Note 3.9. Clearly, each convex set is p-star shaped but not conversely.

Definition 3.10. A fuzzy convex metric space $(X, M, *)$ is said to satisfy the **Property (I),** if for all $x, y, z \in X$ and $\lambda \in [0, 1]$,

$$\varphi(W(x, z, \lambda), W(y, z, \lambda), t) \leq \varphi(x, y, t).$$

Definition 3.11. Let K be a subset of a fuzzy metric space $(X, M, *)$ and $\mathfrak{F} = \{f_\alpha : \alpha \in K\}$ a family of functions from $[0, 1]$ into K, having the property $f_\alpha(1) = \alpha$ for each $\alpha \in K$.

Such a family $\mathfrak{F}$ is said to be **Contractive** if there exists a function $\phi : (0, 1) \to (0, 1)$ such that for all $\alpha, \beta \in K$ and for $x \in (0, 1)$, we have

$$\varphi(f_\alpha(x), f_\beta(x)), t) \leq \phi(x)\varphi(\alpha, \beta, t).$$

Definition 3.12. Let K be a subset of a fuzzy metric space $(X, M, *)$ and $\mathfrak{F} = \{f_\alpha : \alpha \in K\}$ a family of functions from $[0, 1]$ into K, having the property $f_\alpha(1) = \alpha$ for each $\alpha \in K$.

Such a family $\mathfrak{F}$ is said to be **Jointly continuous** if $x \to x_0$ in $[0, 1]$ and $\alpha \to \alpha_0$ in K imply $f_\alpha(x) \to f_{\alpha_0}(x_0)$.

If K is a star shaped subset with star center q of a fuzzy convex metric space $(X, M, *)$ with property (I), then the family $\mathfrak{F} = \{f_\alpha : \alpha \in K\}$ defined by $f_\alpha(x) = W(\alpha, q, \lambda)$ satisfies

$$\varphi(f_\alpha(x), f_\alpha(y), t) = \varphi(W(\alpha, q, \lambda), W(\beta, q, \lambda), t) \leq \lambda\varphi(\alpha, \beta, t).$$

So taking $\phi(x) = x, 0 < x < 1$, the family $\mathfrak{F}$ is a contractive jointly continuous family and therefore the class of subsets of X with the property of contractiveness and joint continuity contains the class of star shaped sets which in turn contains the class of convex sets.

Theorem 3.13. Let $(X, M, *)$ be a fuzzy metric space and K a non-empty closed subset with a contractive jointly continuous family $\{f_\alpha : \alpha \in K\}$. If $T : K \to K$ is a nonexpansive compact mapping, then T has a fixed point.

Proof. Let $k_n = \dfrac{n}{n+1}$, $n = 1, 2, 3, \ldots$. Define $T_n : K \to K$ as

$$T_n x = f_{Tx}(k_n), \quad x \in K.$$

Since $T(K) \subset K$ and $k_n < 1$, $T_n : K \to K$ is well defined. Consider

$$\begin{aligned} \varphi(T_n(x), T_n(y), t) &= \varphi(f_{Tx}(k_n), f_{Ty}(k_n), t) \\ &\leq \phi(k_n)\varphi(Tx, Ty, t) \\ &\leq \phi(k_n)\varphi(x, y, t), \quad x, y \in K \end{aligned}$$

and so each T_n is a contraction mapping on K. Since cl $T_n(K)$ is T_n-invariant and also compact for each n and hence complete, by Banach contraction principle, each T_n has a unique fixed point $x_n \in K \Rightarrow T_n x_n = x_n$.

Since $T(K)$ lies in a compact subset of K, (Tx_n) has a subsequence (Tx_{n_i}) such that $(Tx_{n_i}) \to x_0 \in K$.

Now $x_{n_i} = T_{n_i}(x_{n_i}) = f_{Tx_{n_i}}(k_{n_i}) \to f_{x_0}(1) = x_0$.

Since T is a continuous, $(Tx_{n_i}) \to Tx_0$ and hence $x_0 = Tx_0$.
$\Rightarrow x_0$ is a fixed point of T. ■

References

[1] Bose R.K. and Mukerjee R.N., "Common fixed points of some multivalued mappings", Tamkang J. Math., Vol. 8(2), pp. 245–249, 1977.

[2] Badard, R., Fixed point theorems for fuzzy numbers, Fuzzy Sets Systems 13 (1984), 291–302.

[3] Butnariu, D., Fixed points for Fuzzy maps, Fuzzy Sets Systems 7 (1982), 191–207.

[4] Grabiec, M., Fixed points in fuzzy metric spaces, Fuzzy Sets Systems 27 (1988), 385–389.

[5] George, A. and Veeramai, P., on some result in fuzzy metric space, Fuzzy Sets and System, 64 (1994), 395–399.

[6] Jungck, G., Compatible mappings and common fixed points, Inter'nat. J. Math. and Math. Sci. 9 (1986), 771–779.

[7] Kramosil. I and Michalek. J, Fuzzy metric and statistical metric spaces, Kybernetica 11 (1975), 326–334.

[8] Mishra. S. N., Sharma N. and Singh S. L., Common fixed points of maps on fuzzy metric spaces, Int. J. Math. Math. Sci. 17(1994), 253–258.

[9] Pant R.P., Common fixed points of non commuting mappings, J. Math. Anal. Appl. 188(1994), 436–440.

[10] Rhoades B.E., A fixed point theorem for some non-self maps, Math. Japonica. 23 (4) (1978), 457–459.

[11] Weiss, M.D., Fixed points, separation and induced topologies for fuzzy sets, J. Math. Anal. Appl. 50 (1975), 142–150.

[12] Zadeh, L. A., Fuzzy sets, Information Control 8(1965), 338–353.