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Some Fixed Point Results for Six Maps in Fuzzy Metric Space

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Abstract

In this paper, firstly we prove common fixed point theorems using concept of E.A. and common (E.A) property and then extend the main result for finite number of mappings and integral-type contractive condition in fuzzy metric spaces. Secondly, we prove new common fixed point theorems using the concept of semi weakly compatibility of maps along with the notion of weakly compatibility of maps in fuzzy metric spaces.

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1. INTRODUCTION AND PRELIMINARIES

The concept of fuzzy sets was introduced by Zadeh [9] . It has applications in applied sciences such as neural network theory, stability theory, mathematical programming, modelling theory, engineering sciences, medical sciences (medical genetics, nervous system), image processing, control theory, communication etc. In particular, fuzzy metric space was introduced by Kramosil and Michalek[3]. George and Veeramani[2] modified the definition of Kramosil and Michalek in order to introduce a Hausdorff

topology on fuzzy metric spaces. There are several fixed point results for mappings defined on fuzzy metric spaces in the sense of George and Veeramani.
Before we give our main result we need the following definitions:

Definition 1.1[9] A fuzzy set A in X is a function with domain X and values in [0, 1].

Definition 1.2[5] A binary operation $* : [0,1]\times[0,1]\rightarrow[0,1]$ is a continuous t-norm if $([0,1], *)$ is a topological abelian monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, $\forall$ a, b, c, d $\in[0,1]$.

Definition 1.3[2] The 3-tuple (X, M, *) is called a fuzzy metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set on $X^2\times[0,\infty)$ satisfying the following conditions:

(FM-1) $M(x, y, 0) > 0$,

(FM-2) $M(x, y, t) = 1$ iff $x=y$,

(FM-3) $M(x, y, t) = M(y, x, t)$,

(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,

(FM-5) $M(x, y, .) : (0,\infty) \rightarrow [0,1]$ is continuous, for all x, y, z $\in$ X and s, t > 0.

(FM-6) $\lim_{n\rightarrow\infty} M(x, y, t) = 1$, $\forall$ x, y $\in$ X and t > 0.

Example 1.4[4] Let (X ,d) be a metric space. Define $a * b = ab$ (or $a * b = \min\{a, b\}$) and for all x , y $\in$ X , t > 0, $M(x, y, t) = \frac{t}{t + d(x,y)}$.

Then (X,M, *) is a fuzzy metric space. We call this fuzzy metric M induced by the metric d the standard fuzzy metric.

Definition 1.5[2] Let (X, M, *) be fuzzy metric space . A sequence $\{x_n\}$ in X is said to be

(i) convergent to a point $x \in X$,if $\lim_{n\rightarrow\infty} M(x_n, x, t) = 1$, for all t > 0;

(ii) Cauchy sequence if $\lim_{n\rightarrow\infty} M(x_{n+p}, x_n, t) = 1$, for all t > 0 and p > 0.

Definition 1.6 [2] A fuzzy metric space (X, M, *) is said to be complete if and only if every Cauchy sequence in X is convergent.

Lemma 1.7[4] M(x, y, .) is non-decreasing for all x, y $\in$ X.

Lemma 1.8[7] Let $\{x_n\}$ be a sequence in a fuzzy metric space (X, M, *). If there exists a number k ∈(0, 1) such that M(x_{n+2}, x_{n+1}, kt) ≥M(x_{n+1}, x_n, t) ∀ t > 0 and n ∈N.

Then $\{x_n\}$ is a Cauchy sequence in X.

Definition 1.9[6] Two maps A and B from a fuzzy metric space (X, M, *) into itself are said to be weak-compatible if they commute at their coincidence points, i.e.,

Ax = Bx implies ABx = BAx.

In 2016,Sharma and Bharti [8] introduced the concept of semi weakly compatible in intuitionistic fuzzy metric space as follows:

Definition 1.10[8] Let A and B be maps from an intuitionistic fuzzy metric space

(X, M, N, *, △) into itself. Then for all t > 0, maps A and B are said to be semi weakly compatible if M(ABz, BAz, t) = 1 and N(ABz, BAz, t) = 0, where z is the fixed point of either A or B.

Semi weakly compatible mappings can be used in similar mode in fuzzy metric space.

Definition 1.11[1] A pair of self-mappings (A, B) defined on a fuzzy metric space(X, M, *) is said to satisfy the property E.A. if there exists a sequence $\{x_n\}$in Xsuch that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z \text{ for some } z \in X.$$

Definition 1.12[1]. Two pairs of self- mappings (A, S) and (B, T) defined on a fuzzy

metric space(X, M, *) are said to share common property E.A. if there exist sequences$\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z$ for some z ∈X.

2. MAIN RESULTS

In this section, we are proving new fixed point theorems with contractive condition in fuzzy metric spaces.

Theorem 2.1 Let (X,M,∗) be a fuzzy metric space. Let A, B, Sand T be self-mappings such that

(2.1) for all x, y ∈ X, t > 0 and constant k ∈ (0, 1), we have

$$M(Ax,By,kt) * [M(Tx,Ax,kt) \times M(Sy,By,kt)] \geq \{M(Tx,Ax,t) * M(Tx,Sy,t) * M(Tx,By,t)\}.$$

(2.2) the pairs (A, T) and (B, S) share the common property (E.A.),

(2.3) T(X) and S(X) are closed subsets of X.

Then the pairs (A, T) and (B, S) have a coincident point. Further, if both the pairs (A, T) and (B, S) are weakly compatible, then A, B, S and T have a unique common fixed point in X.

Proof. Since the pairs (A, T) and (B, S) share the common property (E.A), therefore, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Ty_n = m \tag{2.4}$$

for some m ∈ X.

Since S(X) is a closed subset of X, therefore $\lim_{n\to\infty} Sx_n = m \in S(X)$.

Hence, there exists a point u ∈ X such that Su=m. (2.5)

From (2.1), we have

$$M(Ay_n,Bu,kt) * [M(Ty_n,Ay_n,kt)\times M(Su,Bu,kt)]$$
$$\geq \{M(Ty_n,Ay_n,t) * M(Ty_n,Su,t) * M(Ty_n,Bu,t)\}. \tag{2.6}$$

Taking limit as n → ∞ in (2.6), we get Bu = m or Bu = m = Su.

It shows that the pair (B, S) has a coincident point. (2.7)

In a similar way, T(X) is also a closed subset of X, hence we have

$$\lim_{n\to\infty} Ty_n = m \in T(X)$$

Hence, there exists a point v ∈ X such that Tv = m. (2.8)

From condition (2.1), we get

$$M(Av, Bx_n,kt) * [M(Tv, Av, kt)\times M(Sx_n,Bx_n,kt)]$$
$$\geq \{M(Tv, Av, t) * M(Tv, Sx_n, t) * M(Tv, Bx_n, t)\}.$$

Letting n → ∞ and using (2.4), (2.7), we get Av = m or Av = m = Tv.

It shows that the pair (A, T) has a coincident point. (2.9)

Since, the pair (B, S) is weakly compatible, therefore

$$Bm = BSu = SBu = Sm. \quad (2.10)$$

By putting x_n=v, y_n=m in (2.1), we obtain

$$M(Av, Bm, kt)*[M(Tv, Av, kt)\times M(Sm, Bm, kt)]$$
$$\geq\{M(Tv, Av, t)*M(Tv, Sm, t)*M(Tv, Bm, t)\}.$$

So, by (2.9) and (2.10), we have Bm = m = Sm, which shows that m is a common fixed point of the pair (B, S).

As Av = Tv and pair (A, T) is weakly compatible, therefore

$$Am=ATv=TAv=Tm. \quad (2.11)$$

From condition (2.1), we obtain

$$M(Am, Bu, kt)*[M(Tm, Am, kt)\times M(Su, Bu, kt)]$$
$$\geq \{M(Tm, Am, t)*M(Tm, Su, t)*M(Tm, Bu, t)\}.$$

Using (2.8) and (2.11), we get Am = m = Tm, which shows that m is a common fixed point of the pair (A, T).

Hence, m is a common fixed point of A, B, S and T. Uniqueness can be easily found by (2.1).

This implies that m is a unique common fixed point of A, B, S, and T.

In next theorem we prove common fixed point result via E.A property in fuzzy metric spaces as follows:

Theorem 2.2 Let (X, M,∗) be a fuzzy metric space with t-norm a * b = min {a, b}

Let A, B, G, H, Sand T be self-mappings such that

(2.12) for k ∈ (0, 1) and every x, y ∈X, t >0

$$M(Ax, By, kt)*[M(THx, Ax, kt)\times M(SGy, By, kt)]$$
$$\geq\{M(THx, Ax, t)*M(THx, SGy, t)*M(THx, By, t)\},$$

(2.13) A(X)⊂SG(X), B(X)⊂TH(X),

(2.14) the pair (A, TH)or (B, SG) satisfies E.A property.

(2.15) If one of A(X), B(X), SG(X) or TH(X) is a complete subspace of X, then (A, TH) and (B, SG) have a coincident point.

Moreover, if (A, TH) and (B, SG) are weakly compatible then, A, B, TH and SG have a unique common fixed point in X.

Proof. By considering (B, SG) satisfying E.A. property, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Bx_n$ =m= $\lim_{n\to\infty} SGx_n$ where m ∈ X. (2.16)

From condition (a), there exists $\{y_n\}$ in X such that Bx_n=THy_n. (2.17)

Using (2.12), we get

$$M(Ay_n,Bx_n,kt)*[M(THy_n,Ay_n,kt)\times M(SGx_n,Bx_n,kt)]$$
$$\geq\{M(THy_n,Ayn,t)*M(THy_n,SGx_n,t)*M(THy_n,Bx_n,t)\}.$$

Taking limit as n → ∞, we obtain

$$\lim_{n\to\infty} Ay_n = m \text{ and } \lim_{n\to\infty} Ay_n = m = \lim_{n\to\infty} SGy_n. \quad (2.18)$$

The property of complete subspace SG(X) of Ximplies that m = SG(*l*) for some *l* ∈ X.

So, we get

$$\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} THy_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} SGx_n = m = TH(l) \quad (2.19)$$

The followings conditions are obtained from (2.12)

$$M(Al, Bx_n,kt)*[M(THl, Al, kt)\times M(SGx_n,Bx_n,kt)]$$
$$\geq\{M(THl, Al, t)*M(THl, SGx_n,t)*M(THl, Bx_n,t)\}. \quad (2.20)$$

From (2.20) as n → ∞, we get A(*l*) = TH(*l*) (2.21)

which show that the pair (A, TH) has a coincident point m ∈ X.

Since the pair (A, TH) is weakly compatible, therefore we have

$$A(TH)l = (TH)Al.$$

Thus, AA(*l*)=ATH(*l*)=THA(*l*)=THTH(*l*). (2.22)

By (2.13), there exists q ∈ X such that A(*l*) = SG(q). (2.23)

From (2.16), (2.21), and (2.22), we obtain A*l* = Bq which gives

$$Al = THl = SGq = Bq.$$

The weak compatibility of (B, SG) implies that BSGq = SGBq.

This implies, BSGq = SGBq = BBq = SGSGq.

By putting x = Al, y = q in (2.12), we get

$$M(AAl, Bq, kt)*[M(THAl, AAl, kt)\times M(SGq, Bq, kt)]$$
$$\geq\{M(THAl, AAl, t)*M(THAl, SGq, t)*M(THAl, Bq, t)\}.$$

Thus, Al = AAl = THA$l$is a common fixed point of Aand TH. (2.24)

In same way as discussed above, we can prove that Bq is the common fixed point of SG and B.

Since Al = Bq, so Al is the common fixed point of A, B, TH and SG.

Uniqueness: If possible, let x and x′ be two fixed points of A, B, TH and SG.

Consider x = x, y = x′ in (2.12), we obtain

$$M(Ax,B x',kt) *[M(THx, Ax, kt)\times M(SG x',B x', kt)]$$
$$\geq\{M(THx, Ax,t)*M(THx, SGx', t)*M(THx, Bx', t)\}.$$

We get x = x′ using the concept of fixed point and fuzzy metric space.

Therefore, themappings A, B, TH and SG have a unique common fixed point.

As an application of the previously proved result, Integral-type contractive condition is employed for proving the next theorem on fuzzy metric space.

Theorem 2.3 Let (X,M, ∗)be a fuzzy metric space with t-norm a * b = min{a, b}.

Let A, B, G, H, Sand T be self-mappings such that

(2.25) for k ∊ (0, 1)and every x,y∈X, t>0,

$$\int_0^{M(Ax,By,kt)*[M(THx,Ax,kt)\times M(SGy,By,kt)]}\psi(t)\,dt \geq\int_0^{U(x,y,t)}\psi(t)dt$$

where ψ : $\mathbb{R}^+ \rightarrow \mathbb{R}$ is Lebesgue integrable mapping which is summable, non-negative and

$$U(x, y, t)=\{M(THx, Ax, t)*M(THx, SGy, t)*M(THx, By, t)\},M(x, y, t)\}$$

(2.26) A(X)⊂SG(X), B(X)⊂TH(X),

(2.27) the pair (A, TH) or (B, SG) satisfies E.A property.

(2.28) If one of A(X), B(X), SG(X) or TH(X) is a complete subspace of X,

Then, (A, TH) and (B, SG) have a coincident point. Moreover, if (A, TH) and (B, SG) are weakly compatible then A, B, TH and SG have a unique common fixed point in X.

Proof Since (B, SG) satisfies E.A. property, then there exists a sequence $\{x_n\}$ in X, such that

$$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} SGx_n = \text{m where m} \in \text{X}. \tag{2.29}$$

The condition (2.26) gives a sequence $y_n \in$ X such that Bx_n=THy_n.

Using (2.25), we get

$$\int_0^{\mathrm{M}(\mathrm{A}y_n,\mathrm{B}x_n,\mathrm{kt})*[\mathrm{M}(\mathrm{TH}y_n,\mathrm{A}y_n,\mathrm{kt})\times\mathrm{M}(\mathrm{SG}x_n,\mathrm{B}x_n,\mathrm{kt})]} \psi(t)\,dt \geq \int_0^{U(y_n,x_n,t)} \psi(t)dt,$$

U(y_n, x_n, t)={M(THy_n,Ay_n,t)∗M(THy_n,SGx_n,t)∗M(THy_n,Bx_n,t)},M(y_n,x_n,t)}

This implies $\lim_{n\to\infty} Ay_n$ = m and $\lim_{n\to\infty} Ay_n$=m=$\lim_{n\to\infty} SGy_n$. (2.30)

The concept of complete subspace SG(X) of X gives m = SG(l) for some $l \in$ X.

This gives

$$\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} THy_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} SGx_n = \text{m} = \text{TH}(l).$$

Taking x = l, y = x_n in (2.25), we have

$$\int_0^{\mathrm{M}(\mathrm{A}l,\mathrm{B}x_n,\mathrm{kt})*[\mathrm{M}(\mathrm{TH}l,\mathrm{A}y_n,\mathrm{kt})\times\mathrm{M}(\mathrm{SG}x_n,\mathrm{B}x_n,\mathrm{kt})]} \psi(t)\,dt \geq \int_0^{U(l,x_n,t)} \psi(t)dt,$$

U(l, x_n, t)={M(THl,Ay_n,t)∗M(THl,SGx_n,t)∗M(THl,Bx_n,t)},M(l,x_n,t)}

By considering n → ∞, we get A(l) = TH(l). (2.31)

This implies (A, TH) have a coincident point m ∈ X.

The weak compatibility of (A, TH) implies that A(TH)l = (TH)Al.

Thus, AA(l) = ATH(l) = THA(l) = THTH(l). (2.32)

As A(X) ⊂ SG(X), there exists q ∈ Xsuch that A(l) = SG(q). (2.33)

From (2.25), we obtain

$$\int_0^{M(Al,Bq,kt)*[M(THl,Ay_n,kt)\times M(SGq,Bq,kt)]} \psi(t)\,dt \geq \int_0^{U(l,q,t)} \psi(t)dt,$$

$$U(l,q,t)=\{M(THl,Ay_n,t)*M(THl,SGq,t)*M(THl,Bq,t)\},M(l,q,t)\}.$$

Hence, we obtain Al = Bq. Thus we have Al = THl = SGq = Bq. (2.34)

The weak compatibility of (B, SG) implies that BSGq = SGBq.

This implies, BSGq = SGBq = BBq = SGSGq.

Again taking x = Al, y = q in contractive condition of this theorem, we get Al = AAl = THAl is a common fixed point of A and TH. In the same way, We can prove that Bq is the common fixed point of SG and B. Since Al = Bq. So, Al is the common fixed point of A, B, TH and SG.

In next theorem, we generalize Theorem (2.2) for finite number of mappings:

Theorem 2.4 Let (X,M,∗) be a fuzzy metric space. Let T_1,T_2,T_3,…T_z,S_1, S_2,S_3,…S_z, A

and B be mappings from X into itself such that

(2.35) A(X) ⊂ $S_1S_2S_3...S_z$(X), B(X) ⊂$T_1T_2T_3,\ldots T_z$(X),

(2.36) the pair (A, $T_1T_2T_3,\ldots T_z$) or (B, $S_1S_2S_3.. .S_z$) satisfies E.A property,

(2.37) there exists k ∈ (0, 1) such that every x,y∈X, t>0,

M(Ax, By, kt) ∗[M($T_1T_2T_3,\ldots T_z$x,Ax,kt)×M($S_1S_2S_3.. .S_z$y,By,kt)].

≥ [M($T_1T_2T_3,\ldots T_z$x, Ax ,t)∗M($T_1T_2T_3, ... T_z$x , $S_1S_2S_3...S_z$y,t)]∗M($T_1T_2T_3,\ldots T_z$x, By, t).

If one of A(X), B(X),$T_1T_2T_3\ldots T_z$ (X), $S_1S_2S_3.. .S_z$ (X) is complete subspace of X then

(A,$T_1T_2T_3\ldots T_z$)and (B,$S_1S_2S_3.. .S_z$) have a coincident point.

Further, if (A,$T_1T_2T_3,\ldots T_z$) and (B,$S_1S_2S_3.. .S_z$) are weakly compatible, then A(X), B(X),$T_1T_2T_3\ldots T_z$ (X) and $S_1S_2S_3.. .S_z$ (X) have unique fixed point in X.

Proof. Since (B, $S_1S_2S_3...S_z$) satisfies E.A. property, then there exists a sequence $\{x_n\}$ ∊ X, such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} S_1S_2S_3\ldots S_z x_n$ = m, where m ∈ X.

Also, B(X) $\subset T_1T_2T_3\ldots T_z$(X),then there exists a sequence $\{y_n\}$ in X such that Bx_n=$T_1T_2T_3\ldots T_z y_n$.

Using the method of proof of Theorem (2.2), we can see that this result holds.

In next, we prove common fixed point theorem using commuting ,weakly compatible, semi weakly compatible in fuzzy metric space as follows:

Theorem 2.5 Let (X, M, *) be a fuzzy metric space with continuous t-norm * defined by t*t ≥ t for all t ∈ [0, 1]. Let A, B, S, T, P and Q be mappings from X into itself such that:

(2.38) P(X) ⊆ ST(X), Q(X) ⊆ AB(X)

(2.39) There exists a constant k ∈ (0, 1) such that

$$M^2(Px, Qy, kt) * [M(ABx, Px, kt).M(STy, Qy, kt)]$$
$$\geq [\alpha\, M(ABx, Px, t) + \beta\, M(ABx, STy, t)]M(ABx, Qy, 2kt)$$

for all x, y ∈ X and t > 0 ,where 0 <α, β< 1 such that α + β = 1.

(2.40) If one of P(X), ST(X), AB(X), Q(X) is a complete subspace of X then,

(a) P and AB have a coincidence point and

(b) Q and ST have a coincidence point.

Moreover, if

(2.41) pairs (P, AB) and (Q, ST) are weakly compatible.

(2.42) pairs (A, B) and (S, T) are commuting maps,

(2.43) pairs (P, A), (P, B), (S, Q) and (T, Q) are semi weakly compatible maps.

Then A, B, S, T, P and Q have a unique common fixed point in X.

Proof. Let $x_0$be an arbitrary point of X. By (2.38), there existx_1,x_2∈ X such that

Px_0 = STx_1= y_0 and Qx_1 = ABx_2= y_1.

Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

Px_{2n} = STx_{2n+1} = y_{2n} and Qx_{2n+1} = ABx_{2n+2}= y_{2n+1} for n = 0, 1, 2, … .

By taking x = $x_{2n}$and y = x_{2n+1}in (2.39), we have

M^2(Px_{2n}, Qx_{2n+1}, kt)*[M(ABx_{2n}, Px_{2n}, kt).M(STx_{2n+1}, Qx_{2n+1}, kt)]

≥ [αM(ABx_{2n}, Px_{2n}, t) + βM(ABx_{2n}, STx_{2n+1}, t)]M(ABx_{2n}, Qx_{2n+1}, 2kt).

i.e., M^2(y_{2n}, y_{2n+1}, kt) * [M(y_{2n-1}, y_{2n}, kt).M(y_{2n}, y_{2n+1}, kt)]

≥ [α M(y_{2n}, y_{2n-1}, t) + β M(y_{2n-1}, y_{2n}, t)]M(y_{2n-1}, y_{2n+1}, 2kt)

i.e., M^2(y_{2n}, y_{2n+1}, kt) * [M(y_{2n-1}, y_{2n}, kt).M(y_{2n}, y_{2n+1}, kt)]

≥ (α + β)M(y_{2n}, y_{2n-1}, t)M(y_{2n-1}, y_{2n+1}, 2kt)

M(y_{2n}, y_{2n+1}, kt)[M(y_{2n-1}, y_{2n}, kt)*M(y_{2n}, y_{2n+1}, kt)]

≥ (α + β)M(y_{2n}, y_{2n-1}, t)M(y_{2n-1}, y_{2n+1}, 2kt)

Thus it follows that

M(y_{2n}, y_{2n+1}, kt)M(y_{2n-1}, y_{2n+1}, 2kt) ≥ M(y_{2n-1}, y_{2n}, t)M(y_{2n-1}, y_{2n+1}, 2kt)

Hence we have M(y_{2n}, y_{2n+1}, kt) ≥ M(y_{2n-1}, y_{2n}, t) .

Similarly, M(y_{2n+1}, y_{2n+2}, kt) ≥ M(y_{2n}, y_{2n+1}, t).

In general, for all n either even or odd, we have

M(y_n, y_{n+1}, kt) ≥ M(y_{n-1}, y_n, t).

Hence by Lemma 1.8, {y_n} is a Cauchy sequence in X.

Now suppose AB(X) is a complete.

Note that the subsequence {y_{2n+1}} is contained in AB(X) and has a limit in AB(X) call it z.

Let w = AB^{-1}(z), then ABw = z.

We shall use the fact that subsequence {y_{2n}} also converges to z.

By putting x = w, y = x_{2n+1} in (2.39) and taking limit as n→∞, we have

M2(Pw, z, kt) * [M(z, Pw, kt).M(z, z, kt)] ≥ [α M(z, Pw, t) + β M(z, z, t)]M(z, z, 2kt)

Thus it follows that

M(z, Pw, kt) ≥ α M(z, Pw, t) + β ≥ α M(z, Pw, kt) + β implies M(z, Pw, kt) ≥ [β /(1-α)] =1

Hence z = Pw. Since ABw = z thus we have Pw = z = ABw , that is ,w is a coincidence point

of P and AB. Since P(X) ⊂ ST(X), Pw = z implies that z ∈ ST(X).

Let v = ST^{-1}z. Then STv = z.

By putting x = x_{2n} and y = v in (2.39) and taking limit as n → ∞, we have

M^2(z, Qv, kt)*[M(z, z, kt).M(z, Qv, kt)] ≥ [αM(z, z, t) + βM(z, z, t)]M(z, Qv, 2kt)

Thus we have M(z, Qv, kt) ≥ 1 .Thus, z = Qv. Since STv = z, we have

Qv = z = STv that is v is coincidence point of Q and ST. This proves (b).

The remaining two cases pertain essentially to the previous cases.

Indeed if P(X) or Q(X) is complete then by (2.38) z ∈ P(X) ⊂ ST(X)

or z ∈ Q(X) ⊂ AB(X).

Thus (a) and (b) are completely established.

Since the pair (P, AB) is weakly compatible therefore P and AB commute at their coincidence

point that is P(ABw) = (AB)Pw, that is Pz = ABz.

Similarly, the pair (Q, ST) is weakly compatible therefore Q and ST commute at their coincidence point that is Q(STv) = (ST)Qv, that is ,Qz = STz.

By putting x = z, y = x_{2n+1} in (2.39) and taking limit as n → ∞, we have

M^2(Pz, z, kt) * [M(ABz, Pz, kt).M(z, z, kt)]

≥ [αM(ABz, Pz, t) + β M(ABz, z, t)]M(ABz, z, 2kt).

Then we have M(z, Pz, kt) ≥ 1, therefore z = Pz.

Hence Pz = ABz = z.

Similarly ,by putting x = x_{2n}, y = z in (2.39) and taking limit as n → ∞, we have

M(z, Qz, kt) ≥ 1 ,therefore , z = Qz.

Hence Qz = STz = z.

Hence Pz = Qz = ABz = STz = z, i.e., z is the common fixed point of P, Q, AB and ST.

Now for the uniqueness of z let z' be another common fixed point of P, Q, AB and ST then by

putting x = z, y = z' in (2.39), we have

$$M^2(Pz, Qz', kt) * [M(ABz, Pz, kt).M(STz', Qz', kt)] \geq [\alpha M(ABz, Pz, t) + \beta M(ABz, STz', t)]M(ABz, Qz', 2kt).$$

i.e., $M^2(z, z', kt) * [M(z, z, kt).M(z', z', kt)] \geq [\alpha M(z, z, t) + \beta M(z, z', t)]M(z, z', 2kt)$

Then ,we have M(z, z', kt) ≥ 1, therefore z = z' i.e, z is the unique common fixed point of P, Q, AB and ST.

From (2.42) and (2.43)), we have Az = A(ABz) = A(BAz) = (AB)Az,Az = APz = PAz

and Bz = B(ABz) = (BA)Bz = (AB)Bz,Bz = BPz = PBz, implies that Az and Bz are common fixed points of (AB, P) ,therefore, z = Az = Bz = Pz = ABz.

Similarly, Sz and Tz are common fixed points of (ST, Q) ,therefore, z = Sz = Tz = Qz = STz. Hence z is the common fixed point of A, B, S, T, P and Q. Further, since z is the unique common fixed point of P, Q, AB and ST, consequently it is the unique common fixed point of A, B, S, T, P and Q. This completes the proof.

If we put B = T = IX (The identity map on X) in Theorem 2.5, we have the following:

Corollary 2.6. Let (X, M, *) be a fuzzy metric space with continuous t-norm * defined by

t* t ≥ t for all t ∈ [0, 1]. Let A, S, P and Q be mappings from X into itself such that

(2.44) P(X) ⊂S(X), Q(X) ⊂A(X),

(2.45) There exists a constant k∈ (0, 1) such that

$$M^2(Px, Qy, kt) * [M(Ax, Px, kt).M(Sy, Qy, kt)] \geq [\alpha M(Ax, Px, t) + \beta M(Ax, Sy, t)] M(Ax, Qy, 2kt)$$,for all x, y ∈ X

and t > 0 where 0 <α, β< 1 such that α + β = 1.

(2.46) If one of P(X), S(X), A(X), Q(X) is a complete subspace of X then,

(a) P and A have a coincidence point and

(b) Q and S have a coincidence point.

Moreover, if

(2.47) pairs (P, A) and (Q, S) are weakly compatible.

Then A, S, P and Q have a unique common fixed point in X.

If we take A = S and P = Q in Corollary 2.6, we have the following:

Corollary 2.7. Let (X, M, *) be a fuzzy metric space with continuous t-norm * defined by

t *t ≥ t for all t ∈ [0, 1]. Let A and P be mappings from X into itself such that

(2.48) P(X) ⊂A(X),

(2.49) There exists a constant k ∈ (0, 1) such that

$$M^2(Px, Py, kt)*[M(Ax, Px, kt).M(Ay, Py, kt)]$$

$$\geq [\alpha M(Ax, Px, t) + \beta\, M(Ax, Ay, t)]\, M(Ax, Py, 2kt)$$

for all x, y ∈ X and t > 0 where 0 <α, β< 1 such that α + β = 1.

(2.50) If one of A(X), P(X) is a complete subspace of X then P and A have a coincidence point. Moreover, if

(2.51) pair (P, A) is weakly compatible.

Then A and P have a unique common fixed point in X.

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