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Rough G-modules and their properties

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Abstract

Rough set theory is an effective mathematical approach to deal with vagueness and ambiguity in information systems. By combining this theory with abstract algebra, many rough algebraic structures were introduced. In this paper we shall first introduce the notion of rough G-module and then homomorphism in rough G-modules and prove some related results.

Keywords: Approximation space,rough group, rough coset, rough field,rough vector space, rough G-module, rough G-module homomorphism

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1. INTRODUCTION

Rough set theory is an extension of traditional set theory. Z.Pawlak introduced rough set theory as a framework for the construction of approximations of concepts when only incomplete information is available [13]. It has proved to be an effective mathematical tool to deal with vague, uncertain and imperfect knowledge. In Pawlak rough set theory the key concept is an equivalence relation and the equivalence classes are the building blocks for the construction of lower and upper approximations in terms of which a rough set is defined.

Combining rough set theory with abstract algebra is an emerging trend in the area of mathematical research. Some papers substituted an algebraic structure for the universal set, and investigated the roughness in algebraic structures.On the other hand, some papers directly introduced the concepts of rough algebraic structures into an approximation space. The concepts of rough group, rough subgroup, rough coset, rough quotient group and rough homomorphism are studied in [2, 12, 14]. B. Davvaz studied roughness in rings and modules [4, 5]. Some properties of rough subrings and rough ideals are studied in [10, 11]. Qun-Fen Zhang proposed the concept of rough modules in an approximation space and investigated their properties.[15]

The theory of group representation was developed by G.Frobenius in the 19th century. The works of Emmy Noether on representation theory led to the absorption of the theory of group representations into the study of modules over rings and algebra. Module theoretic approach especially G-module structure has been extensively used for the study of group representation. The concept of fuzzy G-module and its properties are studied in [6]. The aim of this paper is to introduce the concept of rough G-module in an approximation space and investigate some of its properties.
In section 2 we recall some of the basics of rough set theory and rough algebraic structures. In section 3 we deal with the concepts of rough field,rough vector space and rough G-module. Then in section 4, we define homomorphism of rough G-modules and prove some related results. We conclude in section 5 with possible future work in the area of rough G-modules.

2. PRELIMINARIES

In this section, we see some basic definitions of rough algebraic structures and results that will be needed in the sequel. For crisp algebraic concepts one may refer the books by Artin [1], Fraleigh [7] or Gallian [8].

Definition 2.1 [13] A pair (U,θ) where $U \neq \phi$ and θ is an equivalence relation on U is called an *approximation space*.

Definition 2.2 [13] For an approximation space (U,θ) and a subset X of U, the sets

$$\overline{X} = \{x \in \mathrm{U} \,|\, [x]_\theta \cap \mathrm{X} \neq \phi\}$$

$$\underline{X} = \{x \in \mathrm{U} \,|\, [x]_\theta \subseteq \mathrm{X}\}$$

$BN(X) = \overline{X} - \underline{X}$ are called respectively the *upper approximation, lower approximation* and *boundary region* of X in (U,θ).

Definition 2.3 [12] Let (U,θ)be an approximation space and let be a binary operation on U. A subset G of U is called a *rough group* if it satisfies the following properties

(1) $\forall x, y \in G, x * y \in \overline{G}$

(2) Associativity holds in $\overline{G}$

(3) $\forall x \in G, \exists e \in \overline{G}$ such that $x * e = x = e * x$; e is called the rough identity.

(4) $\forall x \in G, \exists y \in G$ such that $x * y = e = y * x$; y is called the rough inverse of x.

Definition 2.4 [12] A non empty subset H of a rough group G is called its *rough subgroup* if it is a rough group itself with respect to the same binary operation on G.

Theorem 2.5 [12] A necessary and sufficient condition for a subset H of a rough group G to be a rough subgroup is that

(1) $\forall x, y \in H, x * y \in \overline{H}$

(2) $\forall x \in H, x^{-1} \in H$

Definition 2.6 [12] A rough group G is *commutative* if $\forall x,\ y \in G, x * y = y * x.$

Definition 2.7 [12] Let G be a rough group and H be a rough subgroup of G. If we define a relationship $:$ of elements of $\overline{G}$ as $a : b$ iff $a^{-1} * b \in H \cup \{e\}$ then $:$ is a compatible relation.The *rough left coset* of H in G with respect to $a \in G$ is the compatible category $a * H = \{a * h \mid h \in H, a * h \in G\} \cup \{a\}$. Similarly *rough right coset* can also be defined .

Definition 2.8 [12] A rough subgroup N of rough group G is called a *rough invariant (normal)subgroup* if $\forall a \in G,\ a * N = N * a$.

Definition 2.9 [15] Let N be a rough invariant subgroup of a rough group G and let $G/N = \{g * N \mid g \in G\}$. Then $(G/N, *')$ is a rough group which is called the *rough quotient group* of G with respect to N where the binary operation $*'$ is defined as $(g_1 * N) *' (g_2 * N) = (g_1 * g_2) * N$.

3. ROUGH G-MODULE

In this section, we define the concepts of rough field and rough vector space and then introduce the notion of rough G-module.

Definition 3.1 Let (U, θ) be an approximation space and let $+, *$ be two binary operations on U. A non empty subset F of U is called a *rough field* if it satisfies the following conditions

(1) $(F, +)$ is a rough commutative additive group

(2) $(F, *)$ is a rough commutative multiplicative group

(3) $(a+b)*c = a*c + b*c$ and $a*(b+c) = a*b + a*c \quad \forall a,b,c \in F$.

Definition 3.2 Consider two approximation spaces (U_1, θ_1) and (U_2, θ_2) with the binary operations $+$ and $*$ on U_1 and $+'$ on U_2. Let $F \subseteq U_1$ be a rough field and $M \subseteq U_2$ be a rough commutative group. Then M is called a *rough vector space* over the rough field F if there is a mapping $\overline{F} \times \overline{M} \to \overline{M}; \quad (a, m) \to am$ such that

(1) $a(m+' n) = am+' an$

(2) $(a+b)m = am+' bm$

(3) $(a*b)m = a(bm)$

(4) $1m = m$ where $a,b \in F$; $m,n \in M$ and 1 is the rough multiplicative identity of F .

It can be easily verified that $a0 = 0 \quad \forall a \in K$

Definition 3.3 A non empty subset N of M is called a *rough subspace* of M if $cn_1 +' n_2 \in \overline{N} \quad \forall c \in F$ and $n_1, n_2 \in N$.

Definition 3.4 Consider the approximation spaces (U_1,θ_1) , (U_2,θ_2) and (U_3,θ_3) with the binary operations $*$ on U_1 , $+$ on U_2 and $+'$ and $*'$ on U_3 . Let $G \subseteq U_1$ be a rough group. A rough vector space $M \subseteq U_2$ over a rough field $K \subseteq U_3$ is called a *rough* G *-module* if there is a mapping $\overline{G} \times \overline{M} \to \overline{M}; \quad (g,m) \to g \cdot m$ such that

(1) $1_G \cdot m = m \quad \forall m \in M$ where 1_G is the rough identity element of G

(2) $(g*h) \cdot m = g \cdot (h \cdot m) \quad \forall m \in M; \ g,h \in G$

(3) $g \cdot (k_1 m_1 + k_2 m_2) = k_1(g \cdot m_1) + k_2(g \cdot m_2) \quad \forall k_1,k_2 \in K; \ m_1,m_2 \in M; \ g \in G$

It can be easily verified that $g \cdot 0 = 0 \quad \forall g \in G$

Definition 3.5 A non empty subset N of M is said to be a *rough* G *-submodule* of M if

(1) N is a rough subspace of M

(2) $g \cdot n \in \overline{N} \ \forall g \in G$ and $n \in N$

Theorem 3.6 Let P and Q be two rough G -submodules of M .Then $P \cap Q$ is a rough G -submodule of M if $\overline{P} \cap \overline{Q} = \overline{P \cap Q}$

Proof. Let $x, y \in P \cap Q$

$\Rightarrow x, y \in P$ and $x, y \in Q$

$\Rightarrow cx + y \in \overline{P}$ and $cx + y \in \overline{Q}$

$\Rightarrow cx + y \in \overline{P} \cap \overline{Q} = \overline{P \cap Q}$

Now consider $g \in G$ and $x \in P \cap Q$

$\Rightarrow x \in P$ and $x \in Q$

$\Rightarrow g \cdot x \in \overline{P}$ and $g \cdot x \in \overline{Q}$

$\Rightarrow g \cdot x \in \overline{P} \cap \overline{Q} = \overline{P \cap Q}$

$\therefore P \cap Q$ is a rough G-submodule of M.

Definition 3.7 Let N be a rough G-submodule of M. Then N is a normal rough subgroup of M. The rough quotient group $M' = M/N$ is also a commutative rough group. Let $k \in K$ and $x' \in M'$. Set $kx' = k(x+N) = kx+N$ and $g \cdot x' = g \cdot x + N$ where $x \in M$ is a representative from x'. This definition is independent of the representative from x'. Let y be any other representative from x'. Then $y = x + z$ where $z \in N$ and $ky + N = k[y+N] = k[x+z+N] = k[x+N] = kx+N$. It is easy to verify that this definition satisfies all the conditions in the definition of a rough G-module. So M' is also a rough G-module and it is called the *rough quotient G-module* of M with respect to the rough G-module N.

4. ROUGH G-MODULE HOMOMORPHISM

In this section,we define the homomorphism of two rough G-modules and study some of its properties. Let (U_1, θ_1) , (U_2, θ_2) be two approximation spaces.

Definition 4.1 Let $M_1 \subseteq U_1$ and $M_2 \subseteq U_2$ be two rough G-modules. A mapping $\phi : \overline{M_1} \to \overline{M_2}$ is called a *rough G-module homomorphism* if

(1) $\phi(k_1 m_1 + k_2 m_2) = k_1 \phi(m_1) + k_2 \phi(m_2)$ and

(2) $\phi(g \cdot m) = g \cdot \phi(m)$ $\forall k_1, k_2 \in K; m, m_1, m_2 \in M$ and $g \in G$

Remark. In this case we simply say that $\phi : M_1 \to M_2$ is a rough G-module homomorphism which means that the mapping ϕ is in fact from $\overline{M_1}$ to $\overline{M_2}$.

Definition 4.2 A rough G-module homomorphism $\phi : M_1 \to M_2$ is a *rough G-module isomorphism* if $\phi : \overline{M_1} \to \overline{M_2}$ is both one-one and onto.

Definition 4.3 Let $M_1 \subseteq U_1$ and $M_2 \subseteq U_2$ be two rough G-modules and $\phi : M_1 \to M_2$ a rough G-module homomorphism. Then $\{x \in \overline{M_1} \mid \phi(x) = 0\}$ where 0 is the rough identity element of M_2 is called the *rough G-module homomorphism kernel* of ϕ denoted by $Ker\phi$

Theorem 4.4 Let $M_1 \subseteq U_1$ and $M_2 \subseteq U_2$ be two rough G-modules and $\phi : M_1 \to M_2$ a rough G-module homomorphism. If $Ker\phi$ is a subset of M_1 then it is a rough G-submodule of M_1

Proof. Let $x_1, x_2 \in Ker\phi$ and $c \in K$
$\Rightarrow x_1, x_2 \in M_1$
$\Rightarrow cx_1 + x_2 \in \overline{M_1}$
Also $\phi(cx_1 + x_2) = c\phi(x_1) + \phi(x_2) = c0 + 0 = 0$
$\therefore cx_1 + x_2 \in Ker\phi$
Now let $x \in Ker\phi$ and $g \in G$
$\Rightarrow g \cdot x \in \overline{M_1}$
Also $\phi(g \cdot x) = g \cdot \phi(x) = g \cdot 0 = 0$
$\therefore g \cdot x \in ker\phi$
Thus $Ker\phi$ is a rough G-submodule of M_1

Theorem 4.5 Let $M_1 \subseteq U_1$ and $M_2 \subseteq U_2$ be two rough G-modules and $\phi : M_1 \to M_2$ a rough G-module homomorphism. Let N be a rough G-submodule of M_1. Then $\phi(N)$ is a rough G-submodule of M_2 if $\phi(\overline{N}) = \overline{\phi(N)}$

Proof. Let $y_1, y_2 \in \phi(N)$. Then there exists $x_1, x_2 \in N$ such that $\phi(x_1) = y_1$ and $\phi(x_2) = y_2$
Consider $cy_1 + y_2 = c\phi(x_1) + \phi(x_2) = \phi(cx_1 + x_2) \in \phi(\overline{N}) = \overline{\phi(N)}$
$\therefore cy_1 + y_2 \in \overline{\phi(N)}$
Now consider $g \in G$
$g \cdot y_1 = g \cdot \phi(x_1) = \phi(g \cdot x_1) \in \phi(\overline{N}) = \overline{\phi(N)}$
$\therefore g \cdot y_1 \in \overline{\phi(N)}$
Thus $\phi(N)$ is a rough G-submodule of M_2

Theorem 4.6 (Fundamental theorem of rough G-module homomorphism) Let $M_1 \subseteq U_1$ and $M_2 \subseteq U_2$ be two rough G-modules and $\phi : M_1 \to M_2$ a rough G-module homomorphism such that $Ker\phi \subseteq M_1$ and $\phi(\overline{M_1}) = \overline{\phi(M_1)}$. Then $\phi(M_1)$ is a rough G-submodule of M_2 and ϕ induces a rough G-module isomorphism ϕ' from the rough G-module $M_1/Ker\phi$ onto $\phi(M_1)$ defined by $\phi'(x + Ker\phi) = \phi(x)$

Proof. By theorem 4.3 we have $Ker\phi$ is a rough G-submodule of M_1 and hence $M_1/Ker\phi$ is the rough quotient G-module of M_1 with respect to $Ker\phi$. Also by theorem 4.4 $\phi(M_1)$ is a rough G-submodule of M_2. Now we will prove that ϕ' is

well defined ie, independent of the choice of the representative x. Let $y \in x + Ker\phi$.
We will show that $\phi(x) = \phi(y)$

$y \in x + Ker\phi \Rightarrow y = x + z$ where $z \in Ker\phi$

Then $0 = \phi(z) = \phi(y - x) = \phi(y) + \phi(-x) = \phi(y) - \phi(x)$

$\Rightarrow \phi(x) + 0 = \phi(y)$

$\Rightarrow \phi(x) = \phi(y)$

Thus ϕ' is well defined.

Now we will show that ϕ' is one-one.

Let $\phi'(x + Ker\phi) = \phi'(y + Ker\phi)$

$\Rightarrow \phi(x) = \phi(y)$

We have $0 = \phi(x) - \phi(x) = \phi(y) - \phi(x) = \phi(y) + \phi(-x) = \phi(y - x)$

$\Rightarrow y - x \in Ker\phi$

$\Rightarrow y \in x + Ker\phi$

$\Rightarrow y + Ker\phi = x + Ker\phi$

Thus ϕ' is $1-1$

Obviously ϕ' is onto.

Now $\phi'[(x + Ker\phi) + (y + Ker\phi)]$

$= \phi'[x + y + Ker\phi]$

$= \phi(x + y)$

$= \phi(x) + \phi(y)$

$= \phi'(x + Ker\phi) + \phi'(y + Ker\phi)$

And $\phi'[k(x + Ker\phi)] = \phi'[kx + Ker\phi] = \phi(kx) = k\phi(x) = k\phi'(x + Ker\phi)$

Also $\phi'[g \cdot (x + Ker\phi)] = \phi'[g \cdot x + Ker\phi] = \phi(g \cdot x) = g \cdot \phi(x) = g \cdot \phi'(x + Ker\phi)$

Thus ϕ' is a rough G-module isomorphism from the rough G-module $M_1/Ker\phi$ onto $\phi(M_1)$.

5. CONCLUSION

In this paper, we have introduced the concept of rough G-module.We have also defined homomorphism in rough G-modules and investigated some of its properties. The theory of rough sets can be extended to other areas in the traditional module theory in a similar manner.

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