

The Deficient Quartic Spline Interpolation

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Abstract

In this paper we have to construct a method to obtained existence, uniqueness and error bounds of quartic spline interpolation.

Keywords: Quartic Spline, Error Bounds, Deficient, Interpolation.

I. INTRODUCTION.

Spline interpolation is often preferred over polynomial interpolation because the interpolation error can be made small even using lower degree polynomial. Higher order polynomial are useful because error can be controlled by mesh spacing therefore less nodes than lower order method are needed. In the direction of some higher order method, we refer Gemling, R.H.J. and Meyling, G [3] Rana & Dubey [6,7], Hall & Mayer [4], the error bounds for quartic spline interpolation obtain by Howell and Verma [5] and the error bound of spline of degree six obtained by Dubey [2], (also see Meir and Sharma [8]). The purpose of this paper is to construct a spline method for solving a interpolation problem using piecewise quartic polynomial and obtain error bound.

II. EXISTENCE AND UNIQUENESS

Let a mesh on $[0, 1]$ be given by

$P : 0 = x_0 < x_1 < \dots < x_n = 1$ with

$h_i = x_i - x_{i-1}$ for $i = 1, 2, \dots, n$.

Let $S(4, P)$ denote the set of or all algebraic polynomial of degree 4 and s_i is the restriction of s over $[x_{i-1}, x_i]$, the class $S^*(4, P)$ of deficient quartic spline is defined by

$$S^*(4, P) = \{s_i : s_i \in C^1[0, 1], s_i \in S(4, P) \text{ for } i=1, 2, \dots, n\}$$

Where in $S^*(4, P)$ denotes the class of all deficient spline $S^*(4, P)$ which satisfies the boundary condition.

$$s(x_0) = f(x_0), s(x_n) = f(x_n) \quad (2.1)$$

Problem A : For what restriction on θ there exist an unique $s(x) \in S^*(4, P)$ which satisfies the following condition.

$$s(\alpha_i) = f(\alpha_i) \quad \alpha_i = x_i + \frac{1}{3}P_i \quad (2.2)$$

$$s(\beta_i) = f(\beta_i) \quad \beta_i = x_i + \frac{1}{2}P_i \quad (2.3)$$

$$s'(\gamma_i) = f'(\gamma_i) \quad \gamma_i = x_i + \theta P_i \quad (2.4)$$

$$0 < \theta < 1$$

Let $P(t)$ be the quartic polynomial on $[0, 1]$ then we can show that

$$P(t) = P\left(\frac{1}{3}\right)Q_1(t) + P\left(\frac{1}{2}\right)Q_2(t) + P^1(\theta)Q_3(t) + P(0)Q_4(t) + P(1)Q_5(t) \quad (2.5)$$

Where

$$Q_1(t) = \left[\left\{ 36\theta^3 - \frac{81}{2}\theta^2 + 9\theta + \right\} t + \left\{ -108\theta^3 - \frac{189\theta^2}{2} - \frac{9}{2} \right\} t^2 + \left\{ 72\theta^3 - 63\theta + \frac{27}{2} \right\} t^3 \right] + \frac{\{54\theta - 9 - 54\theta^2\}t^4}{A}$$

$$Q_2(t) = \left[\left\{ (-64\theta^3)/3 + (64\theta^2)/3 - 32\theta/9 \right\} t + t^2 \left\{ 256/3 \theta^3 + 48/27 + (256\theta/3 - 208/9) \theta^2 - 208/3 \theta^2 \right\} + t^3 \left\{ -64\theta^3 + 416\theta/9 - 64/9 \right\} + t^4 \left\{ 48\theta^2 - 128\theta/3 + 16/3 \right\} \right] / A$$

$$Q_3(t) = \left\{ (-z)/9 + (2z^2)/3 - (11z^3)/9 + (2z^4)/3 \right\} / A$$

$$\begin{aligned}
 Q_4(t) &= \left[1 + \left\{ -16\theta^3 + 20\theta^2 - \frac{50\theta}{9} \right\} t \right. \\
 &\quad + \left\{ \left(\frac{88\theta}{3} - \frac{85}{9} \right) h^2 + \left(\frac{88}{3}\theta^3 - \frac{85}{3}\theta^2 \right) + \frac{25}{9} \right\} t^2 \\
 &\quad \left. + \left\{ -16\theta^3 + \frac{170}{9}\theta - \frac{20}{3} \right\} t^3 + \left\{ 4 - \frac{44}{3}\theta + 12\theta^2 \right\} t^4 \right] / A \\
 Q_5(z) &= \left[\left\{ \frac{4}{3}\theta^3 - \frac{5}{6}\theta^2 + \frac{\theta}{9} \right\} t + \left\{ \frac{-20}{3}\theta^3 + \frac{19}{6}\theta^2 - \frac{1}{18} \right\} t^2 + \left\{ 8\theta^3 + \frac{5}{18} - \frac{38\theta}{18} \right\} t^3 \right. \\
 &\quad \left. + \left\{ \frac{10}{3}\theta - 6\theta^2 - \frac{1}{3} \right\} t^4 \right] / A
 \end{aligned}$$

Where

$$A = \left[8\theta^3 - \frac{4}{3}\theta^2 - \frac{4}{3}\theta - \frac{1}{9} \right]$$

Now we are set to answer the problem “A” in the following

Theorem 2.1: There exist an unique deficient quartic spline $s(x) \in S^*(4, P)$ which satisfies the condition (2.2) - (2.4)

Proof : Denoting $(x-x_i)/P_i$ by t , $0 \leq t \leq 1$ we can write (2.5) in the form of the restriction $s_i(x)$ of quartic spline $s(x)$ on $[x_i, x_{i+1}]$ as follows

$$s_i(x, h) = f(\alpha_i)Q_1(t) + f(\beta_i)Q_2(t) + P_i \wedge (\gamma_i)Q_3(t) + s_i(x_i)Q_4(t) + s_i(x_{i+1})Q_5(t) \tag{2.6}$$

From equation (2.6) we can easily verified that $s_i(x)$ is quartic on $[x_i, x_{i+1}]$ for $i = 0, 1, \dots, n-1$ satisfying (2.2) - (2.4). Now using the continuity of first derivative of $s_i(x)$ at x_i in (2.1) to see that

$$\begin{aligned}
 &P_{i+1} \left[\left\{ \frac{16}{3}\theta^3 - \frac{34}{3}\theta^2 + \frac{68}{9}\theta - \frac{14}{9} \right\} s_{i-1} \right. \\
 &+ \left\{ -12\theta^3 + \frac{37\theta^2}{2} + \frac{64\theta}{9} + \frac{11}{18} \right\} P_{i+1} + \left. \left[\left\{ -16\theta^3 + 20\theta^2 - \frac{50\theta}{9} \right\} P_i \right] s_i + \right. \\
 &s_{i+1} \left[\left\{ \frac{4}{3}\theta^3 - \frac{5}{6}\theta^2 + \frac{\theta}{9} + \left(\frac{4\theta}{3} - \frac{5}{18} \right) h^2 \right\} P_i = F_i \right] \\
 &i = 1, 2, \dots, n-1 \tag{2.7}
 \end{aligned}$$

Where,

$$\begin{aligned}
 F_i &= \left\{ 36\theta^3 + 36\theta - \frac{9}{2} - \frac{135}{1}\theta^2 \right\} P_{i+1} f(\alpha_{i-1}) - P_i \left\{ 36\theta^3 - \frac{81\theta^2}{2} + 9\theta \right\} f(\alpha_i) + \\
 &P_{i+1} \left[\left\{ \left(\frac{-128}{30}\theta^3 - \frac{224}{3}\theta^2 - \frac{320\theta}{9} + \frac{32}{9} \right) \right\} f(\beta_{i-1}) - f(\beta_i) \left[\left\{ \frac{-16\theta^3}{3} + \frac{64\theta^2}{2} - \right. \right. \right.
 \end{aligned}$$

$$\frac{32\theta}{9} \left. P_i \right] + \left[\frac{2}{9} P_{i+1} \right] f^1(\gamma_{i-1}) - f^1(\gamma_i) P_i \left[\frac{1}{9} \right]$$

Clearly the coefficient matrix of the system of equation (2.7) is diagonally dominant and hence invertible. This complete proof of theorem (2.1).

III. ERROR BOUNDS.

Following the method of Hall and Meyer [2] in this section, we shall obtain bounds of error function, $e(x) = f(x) - s(x)$ for the spline interpolant of Theorem 2.1 which are best possible. Let $s(x)$ be the second time continuously differentiable quartic spline function satisfy the condition of theorem 2.1. Now considering $f \in C^5 [0,1]$ and writing $M_i[f, x]$ for the unique quartic which agree with $f(x_i), f(x_{i-1}), f(\alpha_i), f(\beta_i)$ and $f'(\gamma_i)$ we see that for $x \in [x_i, x_{i+1}]$ we have

$$|f(x) - s(x)| \cong |f(x) - s_i(x)| \leq |f(x) - M_i[f, x]| + |M_i[f, x] - s_i(x)| \quad (3.1)$$

First, we have to obtain bounds of right hand side of (3.1).

By Cauchy theorem Davis [4], we have obtain

$$|f(x) - M_i[f, x]| \leq \frac{h^5}{5!} |t(t-1/2)(t-1/3)(t-\theta)|(1-t) \quad (3.2)$$

$$\text{Where } t = \frac{x - x_i}{h_i} \text{ and } F = \max_{0 \leq t \leq 1} |f^{(5)}(x)|$$

To get the bounds of $|M_i[f, x] - s_i(x)|$

We have from (2.4)

$$M_i[f, x] - s_i(x) = h_i^2 [f(x_{i-1}) - s(x_{i-1})] q_4(t) + h_i^2 [f(x_i) - s(x_i)] q_5(t) \quad (3.3)$$

$$\text{Thus } |M_i[f, x] - s_i(x)| \leq |e(x_{i-1}) q_4(t)| + |e(x_i) q_5(t)| \quad (3.4)$$

Thus we have

$$|q_4(t) + q_5(t)| \leq |q_4(t)| + |q_5(t)| = K(t) \quad (3.5)$$

Let the $\max_{0 \leq t \leq 1} |e(x_i)|$ exists for $i=j$ then the equation (3.5) may be written as

$$|M_i[f, x] - s_i(x)| \leq |e(x_i)| k(t) \quad (3.6)$$

Now, we proceed to obtain $|e(x_i)|$ replacing $s(x_i)$ by $e(x_i)$ in equation (2.6), we have

$$P_{i+1} e_{i-1} [(80/3)\theta^3 - (110/3)\theta^2 - (68/9)\theta + 14/9] + [P_{i+1}(12\theta^3 - (37/2)\theta^2 + (64/9)\theta - 11/18) + p_i (16\theta^3 - 20\theta^2 + (50/9)\theta)] e_i + p_i (- (4/3)\theta^3 + (5/6)\theta^2 + \theta/9) e_{i+1} = F_i$$

Where

$$\begin{aligned}
 F_1 = & \left\{ 36\theta^3 + 36\theta - \frac{9}{2} - \frac{135}{1}\theta^2 \right\} P_{i+1}f(\alpha_{i-1}) - P_i \left\{ 36\theta^3 - \frac{81\theta^2}{2} + 9\theta \right\} f(\alpha_i) + \\
 & P_{i+1} \left[\left\{ \left(\frac{-128}{30}\theta^3 - \frac{224}{3}\theta^2 - \frac{320\theta}{9} + \frac{32}{9} \right) \right\} f(\beta_{i-1}) - f(\beta_i) \left[\left\{ \frac{-16\theta^3}{3} + \frac{64\theta^2}{2} - \right. \right. \right. \\
 & \left. \left. \left. \frac{32\theta}{9} \right\} P_i \right] + \left[\frac{2}{9} P_{i+1} \right] f^1(\gamma_{i-1}) - f^1(\gamma_i) P_i \left[\frac{1}{9} \right] \right. \\
 & \left. - h_{i+1} f_{i-1} [(80/3)\theta^3 - (110/3)\theta^2 - (68/9)\theta + (14/9)] - \{ h_{i+1}(12\theta^3 - (37/2)\theta^2 + (64/9)\theta - (11/18) + h_i(16\theta^3 - 20\theta^2 + (50/9)\theta \right. \right. \\
 & \left. \left. \} f_i - h_i f_{i+1} (-4/3)\theta^3 + (5/6)\theta^2 + \theta/9 \right] \right. \tag{3.7}
 \end{aligned}$$

In view of that $M(f)$ is linear functional which is zero for polynomials of degree 4 or less, we can apply the Peano Theorem Davis [1] to obtain

$$M(f) = \int_{x_{i-1}}^{x_{i+1}} \frac{f^{(5)}(y)}{4!} M[(x-y)_+^4] dy \tag{3.8}$$

Thus, from (3.8), we have

$$|M(f)| \leq \frac{1}{4!} F \int_{x_{i-1}}^{x_{i+1}} |M(x-y)_+^4| dy \tag{3.9}$$

Further it can be observed that from (3.9) that in $x_{i-1} \leq x \leq x_{i+1}$

$$\begin{aligned}
 \mathbf{M(x-y)}^4 = & \left\{ 36\theta^3 + 36\theta - \frac{9}{2} - \frac{135}{1}\theta^2 \right\} P_{i+1}(\alpha_{i-1} - y)^4 - P_i \left\{ 36\theta^3 - \frac{81\theta^2}{2} + 9\theta \right\} (\alpha_i - \\
 & y)^4 + P_{i+1} \left[\left\{ \left(\frac{-128}{30}\theta^3 - \frac{224}{3}\theta^2 - \frac{320\theta}{9} + \frac{32}{9} \right) \right\} f(\beta_{i-1} - y)^4 - (\beta_i - \\
 & y)^4 \left[\left\{ \frac{-16\theta^3}{3} + \frac{64\theta^2}{2} - \frac{32\theta}{9} \right\} P_i \right] + \left[\frac{2}{9} P_{i+1} \right] (\gamma_{i-1} - y)^3 - (\gamma_i - y)^3 P_i \left[\frac{1}{9} \right] \right. \\
 & \left. \{ h_{i+1}(12\theta^3 - (37/2)\theta^2 + (64/9)\theta - (11/18) + h_i(16\theta^3 - 20\theta^2 + (50/9)\theta \} (x_i - y)^4 - h_i (x_{i+1} - y)^4 \right. \\
 & \left. [-4/3)\theta^3 + (5/6)\theta^2 + \theta/9] \right. \tag{3.10}
 \end{aligned}$$

In order to estimate the interval of r.h.s. of (3.10), we rewrite the above expression in the following symmetric form about x_i , to get

$$\begin{aligned}
 -h_{i+1} [(x_{i+1} - y)^4 \gamma_{i+1} \leq y \leq x_{i+1} - h_i \{ -4/3 + (5/6)\theta^2 + \theta/9 \} (x_i - y)^4 + \{ -4h_i h_{i+1} \{ \{ -4/3 \\
 + (5/6)\theta^2 + \theta/9 \} - (4/9)h_i \} (x_i - y)^3 + \{ -6 h_i h_{i+1}^2 \{ -4/3 + (5/6)\theta^2 + \theta/9 \} - (4/9)h_i \} (x_i - y)^2 + \{ - \\
 4h_i \{ -4/3 + (5/6)\theta^2 + \theta/9 \} h_{i+1}^3 - (4/9)h_{i+1}^3 h_i \} (x_i - y) - h_i h_{i+1}^4 - \theta^3/9 h_i h_{i+1}^4] \beta_{i+1} \leq y \leq \\
 \gamma_{i+1} \\
 h_i [(x_i - y)^4 \{ -(64/2)\theta^2 - (\theta^2 41/2) - (11/3)\theta \} + \{ -4 h_{i+1} (-4/3 + (5/6)\theta^2 + \theta/9) - 4/9 - 2h_{i+1} \\
 (-64/3 \theta^2 - 32/9 \theta) \} (x_i - y)^3 + \{ 3h_{i+1}^2 (-64/3 \theta^3 - (32/9)\theta + (64/3) \theta^2 - 32/9 \theta) + 3 h_{i+1}^2 - \\
 6h_{i+1}^2 (-4/3 + (5/6)\theta^2 + \theta/9) - 1/3 h_{i+1}^2 \} (x_i - y)^2 + (x_i - y) \{ h_{i+1}^3/2 (-64/3 \theta^3 + 64/3
 \end{aligned}$$

$$\theta^2 - 32/9 \theta) - 4 h_{i+1}^2(-4/3 + (5/6)\theta^2 + \theta/9) h_{i+1}^2 - (1/3) h_{i+1}^2 - (h_{i+1}^4/16) (-64/3 \theta^3 + 64/3 \theta^2 - 32/9 \theta) - 4 h_{i+1}^4(-4/3 + (5/6)\theta^2 + \theta/9) h_{i+1}^2 - \theta^3 h_{i+1}^4\}$$

$$\alpha_{i+1} \leq y \leq \beta_{i+1}$$

$$h_{i+1}(x_i - y)^4 \{44/3 \theta^3 - 61 \theta^2 + 88/9 \theta\} + (x_i - y)^2 \{66/3 \theta^3 h_i + 614/3 \theta^2 h_{i+1} + 388/9 \theta h_{i+1} - 4 h_{i+1} (-4/3 + 5/6 \theta^2 + \theta/9) - 4/9 h_i\} + (x_i - y)^2 \{6 h_{i+1}^2 (36 \theta^3 - 81/2 \theta^2 + 9\theta) + 3 h_i ((-64/3 \theta^3 + 64/3 \theta^2 - 32/9 \theta) - 6 h_{i+1}^2 (-4/3 + (5/6)\theta^2 + \theta/9) h_{i+1}^2 - 1/3 h_{i+1})\} + (x_i - y) \{h_{i+1}^2/2(-64/3 \theta^3 + 64/3 \theta^2 - 32/9 \theta) + 4 h_{i+1}^3 (36 \theta^3 - 81/2 \theta^2 + 9\theta) - 4 (-4/3 + (5/6)\theta^2 + \theta/9) h_{i+1}^3 - 1/3 h_{i+1}^4\} + h_{i+1}^4 (36 \theta^3 - 81/2 \theta^2 + 9\theta) + h_{i+1}^4 (-64/3 \theta^3 + 64/3 \theta^2 - 32/9 \theta) - (-4/3 + 5/6 \theta^2 + \theta/9) h_{i+1}^4 - \theta^3/9 h_{i+1}^3\} \quad x_i \leq y \leq \alpha_{i+1}$$

$$-h_{i+1}(x_i - y)^4 \{44/3 \theta^3 - 61 \theta^2 + 88/9 \theta\} + (x_i - y)^2 \{66/3 \theta^3 h_i + 614/3 \theta^2 h_{i+1} + 388/9 \theta h_{i+1} - 4 h_{i+1} (-4/3 + 5/6 \theta^2 + \theta/9) - 4/9 h_i\} + (x_i - y)^2 \{6 h_{i+1}^2 (36 \theta^3 - 81/2 \theta^2 + 9\theta) + 3 h_i ((-64/3 \theta^3 + 64/3 \theta^2 - 32/9 \theta) - 6 h_{i+1}^2 (-4/3 + (5/6)\theta^2 + \theta/9) h_{i+1}^2 - 1/3 h_{i+1})\} + (x_i - y) \{h_{i+1}^2/2(-64/3 \theta^3 + 64/3 \theta^2 - 32/9 \theta) + 4 h_{i+1}^3 (36 \theta^3 - 81/2 \theta^2 + 9\theta) - 4 (-4/3 + (5/6)\theta^2 + \theta/9) h_{i+1}^3 - 1/3 h_{i+1}^4\} + h_{i+1}^4 (36 \theta^3 - 81/2 \theta^2 + 9\theta) + h_{i+1}^4 (-64/3 \theta^3 + 64/3 \theta^2 - 32/9 \theta) - (-4/3 + 5/6 \theta^2 + \theta/9) h_{i+1}^4 - \theta^3/9 h_{i+1}^3\} \quad \gamma_i \leq y \leq x_i$$

$$-h_{i+1}[(x_i - y)^4 \{-(64/2)\theta^2 - (\theta^2 41/2) - (11/3)\theta\} + \{-4 h_{i+1} (-4/3 + (5/6)\theta^2 + \theta/9) - 4/9 - 2 h_{i+1} (-64/3 \theta^2 - 32/9 \theta) (x_i - y)^3 + \{3 h_{i+1}^2 (-64/3 \theta^3 - (32/9)\theta + (64/3) \theta^2 - 32/9 \theta) + 3 h_{i+1}^2 - 6 h_{i+1}^2 (-4/3 + (5/6)\theta^2 + \theta/9) - 1/3 h_{i+1}^2\} (x_i - y)^2 + (x_i - y) \{h_{i+1}^3/2 (-64/3 \theta^3 + 64/3 \theta^2 - 32/9 \theta) - 4 h_{i+1}^2 (-4/3 + (5/6)\theta^2 + \theta/9) h_{i+1}^2 - (1/3) h_{i+1}^2 - (h_{i+1}^4/16) (-64/3 \theta^3 + 64/3 \theta^2 - 32/9 \theta) - 4 h_{i+1}^4 (-4/3 + (5/6)\theta^2 + \theta/9) h_{i+1}^2 - \theta^3 h_{i+1}^4\}$$

$$\beta_i \leq y \leq \gamma_i$$

$$h_{i+1} \{-4/3 + (5/6)\theta^2 + \theta/9\} (x_i - y)^4 + \{-4 h_i h_{i+1} \{-4/3 + (5/6)\theta^2 + \theta/9\} - (4/9) h_i\} (x_i - y)^3 + \{-6 h_i h_{i+1}^2 \{-4/3 + (5/6)\theta^2 + \theta/9\} - (4/9) h_i\} (x_i - y)^2 + \{-4 h_i \{-4/3 + (5/6)\theta^2 + \theta/9\} h_{i+1}^3 - (4/9) h_{i+1}^3 h_i\} (x_i - y) - h_i h_{i+1}^4 - \theta^3/9 h_i h_{i+1}^4\}$$

$$\alpha_i \leq y \leq \beta_i$$

$$h_{i+1}[(x_{i+1} - y)^4$$

$$x_{i-1} \leq y \leq \alpha_i$$

Thus its clear from above expression that $M[(x - y)_+^4]$ is non negative for $x_{i-1} \leq y \leq x_{i+1}$

$$\int_{n_{i-1}}^{n_{i+1}} |M(x-y)_+^4| dy = K_1(\theta^3) + K_2(\theta^2) + K_3(\theta)h_i h_{i+1} (h_{i+1}^4 + h_i^4) \tag{3.11}$$

Combining (3.10) and (3.11), we have

$$|M(f)| \leq \frac{K(\theta)F}{5!} (h_{i-1}^4 + h_i^4) h_i h_{i+1} \tag{3.12}$$

Where $K(\theta) = K_1(\theta^3) + k_2(\theta^2) + k_3(\theta)$

Thus from (3.8) and (3.11)

$$|e(x_i)| \leq |e_i| \leq \frac{F}{(5!)} \left(\frac{h_{i-1}^4 + h_i^4}{K_1^*(\theta)h_{i-1} + k_2^*(\theta)h_i} \right) \tag{3.13}$$

Where $K_1^*(\theta) = (44/3)\theta^3 - (115/6)\theta^2 + (51/9)\theta$

$K_2^*(\theta) = (116/3)\theta^3 - (331/6)\theta^2 - (4/3)\theta + (17/18)$

Now, using (3.2), (3.7) along with (3.13) in (3.1), we have

$$\begin{aligned} |e(x)| &\leq \frac{h^5}{5} \left| t^2(1-t)^2(t - \frac{1}{2}) \right| F + |e(x_i)|k(t) \\ &= \frac{h^5}{5!} \left| t(t-1/3)(t-\theta)(1-t) \left(t - \frac{1}{2} \right) \right| F + \frac{Fh^5}{5!} k(t) / K_1^*(\theta) + K_2^*(\theta) \end{aligned} \tag{3.14}$$

$$\leq \frac{h^5 F}{5!} |c(t)| \tag{3.15}$$

$$\text{Where } c(t) = \left[t(1-t)(t-1/3)(t-\theta)(t-\frac{1}{2}) + k(t) / K_1^*(\theta) + K_2^*(\theta) \right] \tag{3.16}$$

Thus we prove the following :

THEOREM 3.1 : Suppose $s(x)$ is the quartic spline of Theorem 2.1 interpolating a function $f(x)$ and $f \in C^5[0,1]$, then

$$|e(x)| \leq K \frac{h^5}{5!} \max_{0 \leq t \leq 1} |f^{(5)}(x)| \quad (3.17)$$

Where $K = \max_{0 \leq t \leq 1} |c(t)|$ and

$$c(t) = \left[t(1-t)(t-1/3)(t-\theta)(t-\frac{1}{2}) + k(t)/K_1^*(\theta) + K_2^*(\theta) \right]$$

$$|e(x_i)| \leq \frac{1}{5!} h^5 / ((k_1^*(\theta) + k_2^*(\theta))) \max_{0 \leq t \leq 1} |f^{(5)}(x)| \quad (3.18)$$

Further more, it can be seen easily that K in (3.17) be improved for an equally spaced partition. Inequality (3.18) is also best possible.

Where K_1 is positive constant Equation (3.14) proves inequality (3.17) whereas inequality (3.18) is direct consequence of (3.13).

Now we turn to see that the inequality (3.17) is best possible in the limit.

Considering $f(x) = \frac{x^5}{5!}$ and using Cauchy formula Davis [4] we have

$$M_i \left[\frac{x^5}{5!}, x \right] - \frac{x^5}{5!} = h^5 (1-t)(t-1/3)(t-\theta)(t-\frac{1}{2})t/5! \quad (3.20)$$

Moreover, for the function under consideration (3.7) the following relation holds for equally spaced knots

$$e_{i-1} [(80/3)\theta^3 - (110/3)\theta^2 - (68/9)\theta + 14/9] + [28\theta^3 - (3/2)\theta^2 + (114/9)\theta - 11/18] e_i + (- (4/3)\theta^3 + (5/6)\theta^2 + \theta/9) e_{i+1} = F_i \quad (3.21)$$

Considering for a moment

$$e_j = h^5 / (160\theta^3 + (224/6)\theta^2 + (47/9)\theta - 11/18) 5! = e_{j-1} = e_{j+1} \quad (3.22)$$

We have from (3.4)

$$s(x) - M_i[f, x] = -\frac{h^5}{(5!)} (q_4(t) + q_5(t)) k(t) / (k_1^*(\theta) + k_2^*(\theta))$$

$$= \frac{h^5}{5!} k(t) / k^*_1(\theta) + k^*_2(\theta) \tag{3.23}$$

Now combining (3.20) and (3.23) we get, for $x_i \leq x \leq x_{i+1}$

$$= \frac{h^5}{5!} \left[t(t-1/3)(t-\theta)(1-t) \left(t - \frac{1}{2} \right) \right] + \frac{h^5}{5!} k(t) / K_1^*(\theta) + K_2^*(\theta) \tag{3.24}$$

From (3.24) it is clearly observed that (3.17) is best possible, provided we could prove that

$$e_{i-1} = e_i = e_{i+1} = \frac{h^5}{(160\theta^3 + 224\theta^2(47/9)\theta - 11/18)5!} \tag{3.25}$$

In fact (3.25) is attained only in the limit, the difficulty will appear in the case of boundary conditions i.e. $e(x_0) = e(x_n) = 0$. However, it can be shown that

as we were many subintervals away from the boundaries $e(x_i) \rightarrow \frac{h^5}{5!} / K_1^*(\theta) + K_2^*(\theta)$. For that, we shall apply (3.20) inductively to move away from the end conditions $e(x_0) = e(x_n) = 0$.

The first step in this direction is to establish that $e(x_i) \geq 0$ for some $i, i = 1, 2, \dots, n$, which can be shown by contradictory result. Let $e(x_i) < 0$ for some $i = 1, 2, \dots, n-1$. Now making use of (3.18), we get

$$\frac{1}{5!} h^5 / ((k^*_1(\theta) + k^*_2(\theta)) > 0 \leq x \leq 1 \max |e(x_i)| \geq$$

$$e_{i-1} [(80/3)\theta^3 - (110/3)\theta^2 - (68/9)\theta + 14/9] + [28\theta^3 - (3/2)\theta^2 + (114/9)\theta - 11/18] e_i + (-4/3)\theta^3 + (5/6)\theta^2 + (0/9)e_{i+1}]/2 >$$

$$h^5 / (160\theta^3 + (224/6)\theta^2 + (47/9)\theta - 11/18)5!$$

This is the contradiction, hence

$$e(x_j) \geq 0$$

Now from (3.21)

$$- [280^3 - (3/2)\theta^2 + (114/9)\theta - 11/18] e_i = \frac{h^5}{5!} (160\theta^3 + 224\theta^2(47/9)\theta - 11/18)5!$$

$$-e_{i-1}[(80/3)\theta^3 - (110/3)\theta^2 - (68/9)\theta + 14/9] - (- (4/3)\theta^3 + (5/6)\theta^2 + \theta/9)e_{i+1}$$

e_i

$$< \frac{h^5}{5!} (160\theta^3 + (224/6)\theta^2 + (47/9)\theta - 11/18)5! [280^3 - (3/2)\theta^2 + (114/9)\theta - 11/18]$$

$$e_i << \frac{h^5}{5!} (160\theta^3 + (224/6)\theta^2 + (47/9)\theta - 11/18)5! [280^3 - (3/2)\theta^2 + (114/9)\theta - 11/18]$$

$$[1 + (84/3)\theta^3 - (225/6)\theta^2 - (69/9)\theta + 14/9 \dots\dots\dots]$$

Type equation here. $e(x_i) \rightarrow \frac{h^5}{5!} / K_1^*(\theta) + K_2^*(\theta)$

Now it can be seen easily that right hand side of (3.28) $e(x_i) \rightarrow \frac{h^5}{5!} / K_1^*(\theta) + K_2^*(\theta)$ and

hence in the limiting case $|e(x_i)| \leq \frac{h^5}{5!} / K_1^*(\theta) + K_2^*(\theta)$ which

verify proof the inequality (3.19). This is the complete proof of theorem 3.2.1

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