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Hahn-Banach extension theorem in quasi-normed linear spaces

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Abstract

We introduce here the concept of quasi sub-linear functional and give a constructive proof of Hahn-Banach extension theorem, an important fundamental theorem of functional analysis, in quasi-normed linear spaces. We establish here that the quasi-norm of extended bounded linear functional lies between an interval. Some important properties and consequences of this theorems are studied in this space.

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1. Introduction

Metric and norm structures play an important role in functional analysis. So in order to develop this subject one has to take care of the suitable generalization of these structures. Historically, the problem of generalization of the metric structure came first. All we know that an infinitely many metric can be define on a non empty set. Different authors introduced ideas of different kinds of metric space, quasi-metric space ([1], [10]) generalized metric space ([4], [7]), fuzzy metric space ([3],[6], [9]), statistical metric space[8], two metric space [5], quasi-normed linear space ([11], [12]), fuzzy normed linear space [2] etc. In [10], G. Rano introduce the concepts of Cauchy sequence, Convergent sequence, Open set, Closed set etc. in a quasi-metric space and established some basic theorems like Cantor's intersection theorem, Baire's category theorem etc. in complete quasi-metric spaces. We give the definition of Contraction mapping and established some fixed point theorem with uniqueness. In [11], some results on finite dimensional quasi-normed linear spaces are established, the idea of equivalent quasi-norm is introduced and Riesz's lemma is proved in this space. In [12], we define continuity and boundedness of linear operators in quasi-normed linear space. Quasi-norm linear space of bounded linear operators is deduced. Concept of dual space is developed. In this paper, we develop a constructive proof of Hahn-Banach extension theorem in quasi-normed linear space.

The organization of the paper is as follows:
In section 1, comprises some preliminary results. We establish the Hahn-Banach extension theorem in quasi-normed linear space in section 2. We give some consequences of this theorem in section 3.

2. Some preliminary results

In this section some preliminary results are given which are related to this paper.

Definition 2.1. [11] Let X be a linear space over the field F and θ the origin of X. Let $|.|_q : X \to [0, \infty)$ satisfying the following conditions:

(QN-1) $|x|_q = 0$ iff $x = \theta$;

(QN-2) $|ex|_q = |e||x|_q$ for $x \in X$ and $e \in F$;

(QN-3) there exists a $K \geq 1$ such that
$|x + y|_q \leq K\{|x|_q + |y|_q\}$ for $x, y \in X$.

Then $(X, |.|_q)$ is called a quasi-normed linear space(**qnls**) and the least value of the constant $K \geq 1$ is called the index of the quasi-norm $|.|_q$.

Example 2.2. [11] Let $X = R^2$ be a linear space. For $x = (x_1, x_2) \in X$ define

$$|x|_q = (\sqrt{|x_1|} + \sqrt{|x_2|})^2 .$$

Then $(X, |.|_q)$ is a quasi-normed linear space but not a normed linear space.

Definition 2.3. [11] Let $(X, |.|_q)$ be a quasi-normed linear space.

(i) A sequence $\{x_n\}_{n=1}^{\infty} \subset X$ is said

 (a) to converge to $x \in X$ denoted by $\lim_{n\to\infty} x_n = x$ if $\lim_{n\to\infty} |x_n - x|_q = 0$;

 (b) to be a Cauchy sequence if $\lim_{m,\, n\to\infty} |x_n - x_m|_q = 0$;

(ii) A subset $B \subset X$ is said to be complete if every Cauchy sequence in B converges in B;

(iii) A subset A of X is said to be bounded if there exists a real number $M > 0$ such that $|x|_q \leq M \; \forall \, x \in A$;

(iv) A subset A of X is said to be closed if for any sequence $\{x_n\}$ of points of A with $\lim_{n\to\infty} x_n = x$ implies $x \in A$;

(v) A subset A of X is said to be compact if for any sequence $\{x_n\}$ of points of A has a convergent subsequence which converges to a point in A.

Proposition 2.4. [11] Let $(X, |.|_q)$ be a quasi-normed linear space. Then

(a) the limit of a sequence $\{x_n\}$ in X if exists is unique;

(b) every subsequence of a convergent sequence converges to the same limit;

(c) every convergent sequence in X is a Cauchy sequence.

Definition 2.5. [11] Let $(X, |.|_q)$ be a quasi-normed linear space. If X is a finite dimensional linear space then $(X, |.|_q)$ is called a finite dimensional quasi-normed linear space.

Definition 2.6. [12] Let $(X_1, |.|_{q_1})$ and $(X_2, |.|_{q_2})$ be two quasi-normed linear spaces and $T : X_1 \to X_2$ be an operator. Then T is said to be continuous at $x \in X_1$ if for any sequence $\{x_n\}$ of X_1 with $x_n \to x$ i.e. with $\lim_{n\to\infty} |x_n - x|_{q_1} = 0$ implies $T(x_n) \to T(x)$. i.e. $\lim_{n\to\infty} |T(x_n) - T(x)|_{q_2} = 0$. If T is continuous at each point of X_1, then T is said to be continuous on X_1.

Proposition 2.7. [12] Let $(X_1, |.|_{q_1})$ and $(X_2, |.|_{q_2})$ be two quasi-normed linear spaces and $T : X_1 \to X_2$ be an operator. If T is continuous at a point $x \in X_1$, then T is continuous everywhere on X_1.

Definition 2.8. [12] Let $(X_1, |.|_{q_1})$ and $(X_2, |.|_{q_2})$ be two quasi-normed linear spaces and $T : X_1 \to X_2$ be an operator. Then T is said to be bounded if $\exists\, M > 0$ such that

$$|T(x)|_{q_2} \leq M\, |x|_{q_1} \;\forall x \in X_1.$$

Theorem 2.9. [12] Let $(X_1, |.|_{q_1})$ and $(X_2, |.|_{q_2})$ be two quasi-normed linear spaces and $T : X_1 \to X_2$ be an operator. Then T is bounded iff T is continuous.

Theorem 2.10. [12] Let $(X_1, |.|_{q_1})$ and $(X_2, |.|_{q_2})$ be two quasi-normed linear spaces and $T : X_1 \to X_2$ be a linear operator. If X_1 is of finite dimensional, then T is bounded (so continuous).

Theorem 2.11. [12] Let $(X_1, |.|_{q_1})$ and $(X_2, |.|_{q_2})$ be two quasi-normed linear spaces. We denote by $B(X_1, X_2)$ the set of all bounded linear operators from $(X_1, |.|_{q_1})$ to $(X_2, |.|_{q_2})$. Then $B(X_1, X_2)$ is also a linear space.

Theorem 2.12. [12] Let $(X_1, |.|_{q_1})$ and $(X_2, |.|_{q_2})$ be two quasi-normed linear spaces. For $T \in B(X_1, X_2)$ we define

$$|T|_q = \bigvee_{x(\neq\theta)\in X_1} \frac{|T(x)|_{q_2}}{|x|_{q_1}}$$

Then $(B(X_1, X_2), |.|_q)$ is a quasi-normed linear space.

Theorem 2.13. [12] Let $(X_1, |.|_{q_1})$ be a quasi-normed linear space and $(X_2, |.|_{q_2})$ be a complete quasi-normed linear space. Then $(B(X_1, X_2), |.|_q)$ is a complete quasi-normed linear space.

Definition 2.14. [12] The space $(B(X_1, X_2), |.|_q)$ is called the Dual space of $(X_1, |.|_{q_1})$ if $X_2 = R$ and $|.|_{q_2} = |.|$. We denote the set of all bounded linear functional defined on $(X, |.|_q)$ by $B(X, |.|_q)$ which is the Dual space of $(X, |.|_q)$.

Theorem 2.15. [12] Let $(X, |.|_q)$ be a quasi-normed linear space. Then the Dual space $B(X, |.|_q)$ of $(X, |.|q)$ is a complete normed linear space.

3. Hahn-Banach extension theorem

In this section we define a quasi sub-linear functional on a linear space X and establish the Hahn-Banach extension theorem in quasi-normed linear space.

Definition 3.1. Let X be a linear space and $P : X \to R^+$ be a function. Then P is called a quasi sub-linear functional on X if the followings hold:

(i) $p(\lambda x) = |\lambda|\, p(x)$ for all $\lambda \in R, \forall x \in X$;

(ii) There exists $K \geq 1$ such that $p(x + y) \leq K\{p(x) + p(y)\} \, \forall x, y \in X$.

Theorem 3.2. L et X be any vector space and $P : X \to R^+$ a quasi sub-linear functional on X. Let f be a linear functional which is defined on a subspace Z of X satisfying $|f(x)| \leq p(x) \, \forall x \in Z$. Then f has a linear extension $\hat{f}$ from Z to Z_1 having higher dimension of Z satisfying $|\hat{f}(x)| \leq K\, p(x) \, \forall x \in Z_1$ and $\hat{f}(x) = f(x) \, \forall x \in Z$.

Proof. Let $x_0 \in Z - X$. Clearly $x_0 \neq \theta$ and the space Z_1 generated by $Z \cup \{x_0\}$ is also a subspace of X and has higher dimension than Z. Let $x, y \in Z$, then

$$f(x)-f(y) = f(x-y) \leq p(x-y) = p(x+x_0-x_0-y) \leq K\{p(x+x_0)+p(x_0+y)\}$$

$$\Rightarrow f(x) - Kp(x + x_0) \leq f(y) + Kp(y + x_0) \, \forall x, y \in Z$$

$$\Rightarrow \bigvee_{x\in Z} \{f(x) - Kp(x + x_0)\} \leq \bigwedge_{y\in Z} \{f(y) + Kp(y + x_0)\}$$

Let $\gamma \in R$ such that

$$\bigvee_{x\in Z} \{f(x) - Kp(x + x_0)\} \leq \gamma \leq \bigwedge_{y\in Z} \{f(y) + Kp(y + x_0)\}$$

Let $z \in Z_1$ then z is of the form $z = x + tx_0$, where $t \in R$ and $x \in Z$. Clearly this representation is unique. If we define

$$\hat{f}(z) = f(x) - t\gamma$$

Then clearly $\hat{f}$ is a linear functional defined on Z_1 such that

$$\hat{f}(x) = f(x)\ \forall\, x \in Z.$$

If $t > 0$ then

$$\hat{f}(z) = t\left\{f\left(\frac{x}{t}\right) - \gamma\right\} \leq tKp\left(\frac{x}{t} + x_0\right) = Kp(x + tx_0) = Kp(z).$$

If $t < 0$ then

$$\left\{\hat{f}\left(\frac{x}{t}\right) - \gamma\right\} \geq -Kp\left(\frac{x}{t} + x_0\right) = -\frac{1}{|t|}Kp(x + tx_0) = \frac{1}{t}Kp(z).$$

Hence

$$\hat{f}(z) = t\left\{f\left(\frac{x}{t}\right) - \gamma\right\} \leq Kp(z).$$

If $t = 0$ then

$$\hat{f}(z) = f(z) \leq p(z) \leq Kp(z)\ \forall z \in Z.$$

Now

$$-\hat{f}(z) = \hat{f}(-z) \leq Kp(-z) = |-1|Kp(z) = Kp(z).$$

Hence

$$|\hat{f}(z)| \leq Kp(z)\ \forall\, z \in Z_1.$$

■

Corollary 3.3. If X is a finite dimensional vector space and $P : X \to R^+$ a quasi sub-linear functional on X. Let f be a linear functional which is defined on a subspace Z of X satisfying $|f(x)| \leq p(x)\ \forall\, x \in Z$. Then f has a linear extension $\hat{f}$ from Z to X satisfying $|\hat{f}(x)| \leq K^n\, p(x)\ \forall\, x \in X$ and $\hat{f}(x) = f(x)\ \forall\, x \in Z$.

Theorem 3.4. Let $(X, |.|_q)$ be a quasi-normed linear space and f be a bounded linear functional which is defined on a subspace Z of X. Then f has a linear extension $\hat{f}$ from Z to Z_1 which is a higher dimensional subspace of X and bounded on Z_1 satisfying $|f|_q \leq |\hat{f}|_q \leq K|f|_q$.

Proof. Let $p(x) = |f|_q \; |x|_q \; \forall \, x \in X$.
Then clearly p is a quasi sub-linear functional on X. So by Theorem 2.2, f has a linear extension $\hat{f}$ from Z to Z_1 satisfying

$$|\hat{f}(x)| \leq Kp(x) \; \forall \, x \in Z_1.$$

$$\Rightarrow \; |\hat{f}(x)| \leq K|f|_q \; |x|_q \; \forall \, x \in Z_1.$$

Thus $\hat{f}$ is a bounded linear functional on Z_1 and

$$|\hat{f}|_q = \bigvee_{x(\neq\theta)\in Z_1} \frac{|\hat{f}(x)|}{|x|_q}.$$

Since Z is a subspace of Z_1 and $\hat{f}(x) = f(x) \; \forall \, x \in Z$,

$$\bigvee_{x(\neq\theta)\in Z_1} \frac{|\hat{f}(x)|}{|x|_q} \geq \bigvee_{x(\neq\theta)\in Z} \frac{|f(x)|}{|x|_q};$$

$$\Rightarrow \; |\hat{f}|_q \geq |f|_q.$$

Hence $|f|_q \leq |\hat{f}|_q \leq K|f|_q$. ■

Corollary 3.5. If $(X, \; |.|_q)$ is a finite dimensional quasi-normed linear space and f be a bounded linear functional which is defined on a subspace Z of X. Then f has a linear extension $\hat{f}$ from Z to X which is bounded on X satisfying $|f|_q \leq |\hat{f}|_q \leq K^n \; |f|_q$.

4. Some consequences of Hahn-Banach extension theorem

In this section, we give some application of Hahn-Banach extension theorem on quasi-normed linear spaces.

Theorem 4.1. Let $(X, \; |.|_q)$ be a n-dimensional quasi-normed linear space and $x_0(\neq \theta) \in X$. Then there exists a bounded linear functional $\hat{f}$ on X such that $1 \leq |\hat{f}|_q \leq K^n$ and $\hat{f}(x_0) = |x_0|_q$.

Proof. We consider the subspace Z of X consisting of all elements $x = cx_0$ where c is a scalar. On Z we define a linear functional f by $f(x) = f(cx_0) = c|x_0|_q$. Then f is bounded since $|f(x)| = |c||x_0|_q = |cx_0|_q = |x|_q$ and $|f|_q = 1$. By theorem 2.4, f has a linear extension $\hat{f}$ from Z to X with $|f|_q \leq |\hat{f}|_q \leq K^n \; |f|_q$ i.e. $1 \leq |\hat{f}|_q \leq K^n$ and $\hat{f}(x_0) = f(x_0) = |x_0|_q$. ■

Corollary 4.2. For every $x \in X$ we have

$$|x|_q = \bigvee_{f\neq O, \; f\in B(X,Q)} \frac{|f(x)|}{|f|_q}.$$

Hence if x is such that $f(x) = \theta$ for all $f \in B(X, Q)$, then $x = \theta$.

Proof. From the above theorem we have, writing x for x_0,

$$\bigvee_{f \neq O, f \in B(X,Q)} \frac{|f(x)|}{|f|_q} \geq \frac{|f(x)|}{|f|_q} = |x|_q$$

and from $|f(x)| \leq |f|_q |x|_q$ we have

$$\bigvee_{f \neq O, f \in B(X,Q)} \frac{|f(x)|}{|f|_q} \leq |x|_q.$$

Hence proved. ■

Theorem 4.3. Let Let $(X, |.|_q)$ be a n-dimensional quasi-normed linear space. Let $y_0 \in X - Z$. Let $d_q = \bigwedge_{x \in Z} |y_0 - x|_q > 0$, then there exists a bounded linear functional f on X such that

1) $f(x) = 0 \,\forall\, x \in Z$,
2) $f(y_0) = 1$,
3) $1 \leq |f|_q \leq \dfrac{K^n}{d_q}$.

Proof. The subspace $\{Z + y_0\}$ is uniquely representable in the form $y = x + ty_0$ where $x \in Z$ and t is real. Let us define a functional ϕ on $\{Z + y_0\}$ by $\phi(y) = t$ for $y = x + ty_0 \in \{Z + y_0\}$. Then ϕ is a linear functional on $\{Z + y_0\}$. Also $\phi(x) = 0 \,\forall\, x \in Z$ and $\phi(y_0) = 1$.

Now $|\phi(y)| = |t| = \dfrac{|t||y|_q}{|y|_q}$

$= \dfrac{|ty|_q}{|y|_q} = \dfrac{|ty|_q}{|x + ty_0|_q}$

$= \dfrac{|y|_q}{|\frac{x}{t} + y_0|_q} = \dfrac{|y|_q}{|y_0 - (-\frac{x}{t})|_q} \leq \dfrac{|y|_q}{d_q}$.

So ϕ is a bounded linear functional on $\{Z + y_0\}$ and

$|\phi|_q = \bigvee_{y \in \{Z + y_0\},\ |y|_q \leq 1} \{|\phi(y)|\} \leq \dfrac{1}{d_q}$

Since $d_q = \bigwedge_{x \in Z} |y_0 - x|_q$, there exists a sequence $\{x_n\}$ in Z such that

$\Rightarrow \lim_{n \to \infty} |x_n - y_0|_q = d_q$......................................(i).

Now $|\dfrac{\phi(x_n - y_0)}{|x_n - y_0|_q}| \leq |\phi|_q$

$\Rightarrow\ |\phi(x_n - y_0)| \leq\ |\phi|_q\ |x_n - y_0|_q$.
But $|\phi(x_n - y_0)| = |\phi(x_n) - \phi(y_0)| = 1$
$\Rightarrow\ |\phi|_q\ |x_n - y_0|_q \geq\ 1$
$\Rightarrow\ \lim_{n\to\infty} |x_n - y_0|_q|\phi|_q = d_q|\phi|_q \geq\ 1$
$\Rightarrow\ \lim_{n\to\infty} |\phi|_q d_q \geq\ 1$
$\Rightarrow\ |\phi|_q \geq\ \frac{1}{d_q}$..(ii)
From (i) and (ii) we have, $|\phi|_q = \frac{1}{d_q}$.
By Corollary 2.5, ϕ has a linear extension f from $\{Z + y_0\}$ to X which is a bounded linear functional on X such that the conditions (1), (2) and (3) hold. ■

5. Conclusions

In this work, we have to establish Hahn-Banach extension theorem in quasi-normed linear space to some extent.We could not get success spreading this theorem in full generality in the whole space. So there is a scope of further research and developing the concept to a large extent.Since this is a fundamental theorem of functional analysis, if we success, it will be a great achievement for the researcher. We hope there is a big possibility to develop this subject in many directions.

References

[1] C.T. Aage and J.N. Salunke, Some Results of Fixed point Theorem in Dislocated Quasi-metric Spaces, Bulletin of the Marathwada Mathematical Society, 9(2) (2008) 1–5.

[2] T. Bag, S.K. Samanta, Finite dimensional fuzzy normed linear spaces, The Journal of Fuzzy Mathematics (3) (2003) 687–705.

[3] T. Bandyopadhyay, S.K. Samanta, P. Das, Fuzzy metric spaces redefined and a fixed point theorem, Bull. Cal. Math. Soc. 81 (1989) 247–252.

[4] P. Das and L. K. Dey, A fixed point theorem in Generalized metric space, Soochow Journal of Mathematics, 33(2007), 33–39.

[5] B. Fisher, Fixed point in two metric spaces, Glasnik Mat, 16(36), (1981), 333–337.

[6] O. Kaleva, S. Seikkala, On Fuzzy metric spaces, Fuzzy Sets and Systems 12(1984) 215–229.

[7] L. Kikina, K. Kikina, Fixed point on two generalized metric spaces, Int. Journal of Math. Analysis, 5 (2011) no. 30, 1459–1467.

[8] I. Kramosil, J. Michalek, Fuzzy metric and statistical metric spaces, Kybernetica (1975) 326–334.

[9] P.V. Subrahmanyam, A common fixed point theorems in Fuzzy metric spaces, Information Seiences 83(3&4)(1995) 109–112.

[10] G. Rano, Quasi-metric space and fixed point theorems, International Journal of Mathematics and Scientific Computing 3(2) (2013) 27–31.

[11] G. Rano, T. Bag, Finite dimensional quasi-normed linear space, The Journal of Fuzzy Mathematics, 22(2) (2014) 669–676.

[12] G. Rano, T. Bag, Bounded linear operators in quasi-normed linear space, The Journal of The Egyptian Mathematical society, 23 (2015) 303–308.