

On the Eigenproblems of Nilpotent Lattice Matrices

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Abstract

This paper discusses the eigenproblems of nilpotent lattice matrices and describes the λ -eigenspace of a nilpotent lattice matrix. Also, this paper introduces the concept of non-singular lattice matrices. We prove that every invertible lattice matrix is non-singular and every nilpotent lattice matrix is singular.

Keywords: Distributive lattice, Nilpotent lattice matrix, λ -eigenspace, Non-singular lattice matrix, Invertible lattice matrix

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1. INTRODUCTION

The notion of lattice matrices appeared firstly in the work, 'Lattice matrices' [2] by G. Give'on in 1964. A matrix is called a lattice matrix if its entries belong to a distributive lattice. All Boolean matrices and fuzzy matrices are lattice matrices. Lattice matrices in various special cases become useful tools in various domains like the theory of switching nets, automata theory and the theory of finite graphs [2].

The concept of nilpotent lattice matrix was introduced firstly by G. Give'on [2]. A square lattice matrix A is called nilpotent if $A^m = \mathbf{0}_n$, for some positive integer m , where $\mathbf{0}_n$ is the zero matrix. In [2], Give'on proved that an $n \times n$ lattice matrix A is nilpotent if and only if $A^n = \mathbf{0}_n$. Since then, a number of researchers have carried out their research in the area of nilpotent lattice matrices (see [6, 8, 9, 10]).

The eigenvector-eigenvalue problems (for short, eigenproblems) of matrices over a

complete and completely distributive lattice with the greatest element 1 and the least element 0 (lattice matrices) were discussed firstly by Y. J. Tan [5]. In [7], Y. J. Tan raised an open problem 'How to describe the λ -eigenspace for a given lattice matrix?'. In this paper, we describe the λ -eigenspace of a given nilpotent lattice matrix.

Further, we introduce the concept of non-singular lattice matrices. In classical theory of matrices over a field, the theory of non-singular matrices coincides with the theory of invertible matrices. In the case of lattice matrices, every non-singular lattice matrix need not be invertible. Also, we prove that every nilpotent lattice matrix is singular.

2. PRELIMINARIES

We recall some basic definitions and results on lattice theory and lattice matrices which will be used in the sequel. For details see [1, 2, 3, 4, 7, 9, 10].

A partially ordered set (L, \leq) is a *lattice* if for all $a, b \in L$, the least upper bound of $\{a, b\}$ and the greatest lower bound of $\{a, b\}$ exist in L . For any $a, b \in L$, the least upper bound and the greatest lower bound is denoted by $a \vee b$ and $a \wedge b$ (or ab), respectively.

An element $a \in L$ is called *greatest element* of L if $x \leq a, \forall x \in L$. An element $b \in L$ is called *least element* of L if $b \leq x, \forall x \in L$. We use 1 and 0 to denote the greatest element and the least element of L , respectively.

A nonvoid subset K of a lattice L is a *sublattice* of L , if for any $a, b \in K$, $a \vee b, a \wedge b \in K$. A lattice L is called a *complete lattice* if for any $H \subseteq L$, both the least upper bound $\vee \{y \mid y \in H\}$ and the greatest lower bound $\wedge \{y \mid y \in H\}$ of H exist in L . A lattice (L, \leq, \vee, \wedge) is a *distributive lattice* if the operations \vee and \wedge are distributive with respect to each other.

For any $a, b \in L$, the greatest element $x \in L$ satisfying the inequality $a \wedge x \leq b$ is called the *relative pseudocomplement* of a in b and is denoted by $a \rightarrow b$. If for any $a, b \in L$, $a \rightarrow b$ exists, then L is said to be a *Brouwerian lattice*. For $a \in L$, $a \rightarrow 0$ is denoted by a^c .

A lattice L is said to be *completely distributive* if for any $x \in L$ and any $\{y_i \mid i \in I\} \subseteq L$, where I is an index set,

- (a). $x \wedge (\vee_{i \in I} y_i) = \vee_{i \in I} (x \wedge y_i)$ and
- (b). $x \vee (\wedge_{i \in I} y_i) = \wedge_{i \in I} (x \vee y_i)$ holds.

It is known [1] that a complete lattice L is Brouwerian if and only if (a) is satisfied in L . Therefore, a complete and completely distributive lattice L is Brouwerian.

Unless otherwise specified, we shall assume throughout the paper that L is a complete and completely distributive lattice with the greatest element 1 and the least element 0, respectively.

Lemma 2.1 [4, 7] *Let $\{a_i, 1 \leq i \leq n\} \subseteq L$. Then we have*

- (a). $(\vee_{1 \leq i \leq n} a_i)^c = \wedge_{1 \leq i \leq n} a_i^c$
- (b). $\vee_{1 \leq i \leq n} a_i^c \leq (\wedge_{1 \leq i \leq n} a_i)^c$.

Lemma 2.2 Let $a, b \in L$ such that $a^c = b^c = 0$. Then $(ab)^c = 0$.

Proof. If possible, assume that $(ab)^c > 0$. Then there exists $y > 0$ such that $(ab)y = 0$. That is, there exists $y > 0$ such that $(ay)(by) = 0$.

Let $y_1 = ay$ and $y_2 = by$. Then $y_1, y_2 > 0$. Otherwise, $a^c, b^c > 0$. Now $by_1 = b(ay) = (ab)y = 0$ and $ay_2 = a(by) = (ab)y = 0$.

Thus there exist $y_1, y_2 > 0$ such that $by_1 = ay_2 = 0$, which is a contradiction.

Hence $(ab)^c = 0$.

Let $M_n(L)$ be the set of all $n \times n$ matrices over L (Lattice Matrices). We shall denote by A_{ij} the element of L which stands in the $(i, j)^{\text{th}}$ entry of $A \in M_n(L)$.

For $A = (A_{ij}), B = (B_{ij}), C = (C_{ij}) \in M_n(L)$, define

$$A \vee B = C \Leftrightarrow C_{ij} = A_{ij} \vee B_{ij} \quad (i, j = 1, 2, \dots, n)$$

$$aA = C \Leftrightarrow C_{ij} = aA_{ij}, \text{ for } a \in L \quad (i, j = 1, 2, \dots, n)$$

$$AB = C \Leftrightarrow C_{ij} = \bigvee_{k=1}^n A_{ik} B_{kj} \quad (i, j = 1, 2, \dots, n) \quad (\text{Multiplication})$$

$$A^T = C \Leftrightarrow C_{ij} = A_{ji} \quad (i, j = 1, 2, \dots, n) \quad (\text{Transposition})$$

$$(I)_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \quad (\mathbf{0}_n)_{ij} = 0 \quad (i, j = 1, 2, \dots, n)$$

$$A^0 = I, A^{k+1} = A^k A \text{ for } k \geq 0, k \text{ is an integer.}$$

It is clear that $M_n(L)$ is a semigroup with the identity I and the zero $\mathbf{0}_n$ with respect to the multiplication. Also, the transposition in $M_n(L)$ has the following properties:

(a). $(AB)^T = B^T A^T$

(b). $(A^T)^T = A$.

Let $A \in M_n(L)$. Then A is called *nilpotent*, if there exists some integer $k \geq 1$ such that $A^k = \mathbf{0}_n$.

Theorem 2.3 [2] Let $A \in M_n(L)$. Then A is nilpotent if and only if $A^n = \mathbf{0}_n$.

Theorem 2.4 [10] Let $A \in M_n(L)$ be nilpotent. Then $a_{i_1} a_{i_1 i_2} \cdots a_{i_m i} = 0$, for $\{i, i_1, i_2, \dots, i_m\} \subseteq \{1, 2, \dots, n\}$.

Let $A \in M_n(L)$. Then A is said to be *invertible* if there is a $B \in M_n(L)$ such that $AB = BA = I$. Here B is called the *inverse* of A and is denoted by A^{-1} .

A set $S = \{a_1, a_2, \dots, a_m\}$ of elements of L is a *decomposition* of 1 in L if and only if $\bigvee_{i \leq i \leq m} a_i = 1$. The set S is called *orthogonal* if and only if $a_i a_j = 0$, for all $i, j = 1, 2, \dots, m, i \neq j$. Hence S is called an *orthogonal decomposition* of 1 in L if and only if it is orthogonal and a decomposition of 1 in L .

Theorem 2.5 [2] *Let $A \in M_n(L)$. Then A is invertible if and only if each row and each column of A is an orthogonal decomposition of 1 in L .*

Let $A, B \in M_n(L)$. If there exists an invertible matrix $P \in M_n(L)$ such that $B = P^{-1}AP$, then B is said to be *similar* to A .

For $A \in M_n(L)$, the *permanent* of A is defined as

$$|A| = \bigvee_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)},$$

where S_n denotes the symmetric group of all permutations of the indices $\{1, 2, \dots, n\}$.

Lemma 2.6 [9] *Let $A \in M_n(L)$ be invertible. Then $|A| = 1$.*

Let $V_n(L)$ be the set of all column vectors or n -vectors over L . For $\vec{x} = (x_1, x_2, \dots, x_n)^T, \vec{y} = (y_1, y_2, \dots, y_n)^T \in V_n(L)$ and $a \in L$, define

Addition: $\vec{x} \vee \vec{y} = (x_1 \vee y_1, x_2 \vee y_2, \dots, x_n \vee y_n)^T$,

scalar multiplication: $a\vec{x} = (ax_1, ax_2, \dots, ax_n)^T$ and

$\vec{0} = (0, 0, \dots, 0)^T \in V_n(L)$.

Then $V_n(L)$ is a lattice vector space over L . The elements of $V_n(L)$ are called as vectors and the elements of L are called as scalars. Also, $\vec{0}$ is called the zero vector. Denote by $\vec{e} = (1, 1, \dots, 1)^T$.

For $\vec{x} = (x_1, x_2, \dots, x_n)^T, \vec{y} = (y_1, y_2, \dots, y_n)^T \in V_n(L)$, define

$$\vec{x} \leq \vec{y} \Leftrightarrow x_i \leq y_i \quad (i = 1, 2, \dots, n)$$

$$\vec{x} \wedge \vec{y} = (x_1 \wedge y_1, x_2 \wedge y_2, \dots, x_n \wedge y_n)^T$$

$$\vec{x} \rightarrow \vec{y} = (x_1 \rightarrow y_1, x_2 \rightarrow y_2, \dots, x_n \rightarrow y_n)^T$$

$$\vec{x}^c = (x_1^c, x_2^c, \dots, x_n^c)^T$$

$$\vec{y} = A\vec{x} \Leftrightarrow y_i = \bigvee_{1 \leq j \leq n} A_{ij} \wedge x_j, \text{ for } A \in M_n(L) \quad (i = 1, 2, \dots, n).$$

Lemma 2.7 [7] *For $\vec{x}, \vec{y} \in V_n(L)$, we have $(\vec{x} \vee \vec{y})^c = \vec{x}^c \wedge \vec{y}^c$.*

A vector $\vec{x} = (x_1, x_2, \dots, x_n)^T$ is an *ortho vector* when $x_i x_j = 0$, for $i, j = 1, 2, \dots, n$ and $i \neq j$. An ortho vector $\vec{x} = (x_1, x_2, \dots, x_n)^T$ is a *stochastic vector* when $\bigvee_{1 \leq i \leq n} x_i = 1$.

Let W be a non-empty subset of $V_n(L)$. Then W is a sub lattice vector space of $V_n(L)$ if and only if W is closed under addition and scalar multiplication in $V_n(L)$. Let S be a subset of W . Then a vector $\vec{v} \in W$ is a linear combination of the vectors in S , if there exists $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \in S$ and $a_1, a_2, \dots, a_n \in L$ such that $\vec{v} = \sum_{1 \leq i \leq n} a_i \cdot \vec{v}_i$. Assume that there exists $S \subseteq W$ such that every $\vec{v} \in W$ is a linear combination of the vectors from S . Then S is called a spanning subset of W .

3. EIGENPROBLEMS OF NILPOTENT MATRICES OVER L

In this section, we consider the eigenproblems (See [7]) of nilpotent matrices over L , a complete and completely distributive lattice with 1 and 0.

Definition 3.1 [5] Let $A \in M_n(L)$. An eigenvector of A is a vector \vec{x} in $V_n(L)$ such that $A\vec{x} = \lambda\vec{x}$, for some scalar λ in L . The element λ is called the associated eigenvalue.

First, we consider a given eigenvalue λ of a nilpotent matrix A in $M_n(L)$ and discuss the properties of its eigenvectors.

For a given eigenvalue λ of a matrix A in $M_n(L)$, the set of all eigenvectors of λ is denoted by $E_A(\lambda)$. That is, $E_A(\lambda) = \{\vec{x} \in V_n(L) \mid A\vec{x} = \lambda\vec{x}\}$.

Theorem 3.2 [7] Let $A \in M_n(L)$ and λ be a given eigenvalue of A . Then $E_A(\lambda)$ is a sub lattice vector space of $V_n(L)$ and is called the λ -eigenspace of A . Moreover, $E_A(\lambda)$ has the least element $\vec{0}$ and the greatest element $\vec{x}_g(\lambda) = (\lambda\vec{e} \vee A^T\vec{e}) \rightarrow (\lambda A^n\vec{e})$.

Theorem 3.3 Let $A \in M_n(L)$ be nilpotent and λ be a given eigenvalue of A . Then $E_A(\lambda)$ has the least element $\vec{0}$ and the greatest element $\vec{x}_g(\lambda) = (\lambda\vec{e} \vee A^T\vec{e})^c$.

Proof. By Theorem 3.2, $E_A(\lambda)$ has the least element $\vec{0}$ and the greatest element

$$\begin{aligned} \vec{x}_g(\lambda) &= (\lambda\vec{e} \vee A^T\vec{e}) \rightarrow (\lambda A^n\vec{e}) = (\lambda\vec{e} \vee A^T\vec{e}) \rightarrow (\lambda\mathbf{0}_n\vec{e}) \text{ (By Theorem 2.3)} \\ &= (\lambda\vec{e} \vee A^T\vec{e}) \rightarrow \vec{0} = (\lambda\vec{e} \vee A^T\vec{e})^c. \end{aligned}$$

Theorem 3.4 Let $A \in M_n(L)$ be nilpotent and λ be a given eigenvalue of A . Then $E_A(\lambda)$ is spanned by

$$\{(x_1, 0, \dots, 0)^T, (0, x_2, \dots, 0)^T, \dots, (0, 0, \dots, x_n)^T\} - \{(0, 0, \dots, 0)^T\},$$

where $\vec{x}_g(\lambda) = (x_1, x_2, \dots, x_n)^T = (\lambda\vec{e} \vee A^T\vec{e})^c$ is the greatest element of $E_A(\lambda)$.

Proof. Let $\vec{x}_1 = (x_1, 0, \dots, 0)^T, \vec{x}_2 = (0, x_2, \dots, 0)^T, \dots, \vec{x}_n = (0, 0, \dots, x_n)^T$.

Firstly we prove that $\bar{x}_i = (0, \dots, x_i, \dots, 0)^T \in E_A(\lambda)$, where $x_i = (\lambda \vee \bigvee_{1 \leq j \leq n} A_{ji})^c = \lambda^c \wedge \bigwedge_{1 \leq j \leq n} A_{ji}^c$, for $i = 1, 2, \dots, n$ (By Lemma 2.1(a)).

To do this, for $i = 1, 2, \dots, n$, consider

$$\begin{aligned} A\bar{x}_i &= A(0, \dots, x_i, \dots, 0)^T \\ &= \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix} (0, \dots, x_i, \dots, 0)^T \\ &= (A_{1i}x_i, A_{2i}x_i, \dots, A_{ni}x_i)^T \\ &= (A_{1i}(\lambda^c \wedge \bigwedge_{1 \leq j \leq n} A_{ji}^c), A_{2i}(\lambda^c \wedge \bigwedge_{1 \leq j \leq n} A_{ji}^c), \dots, A_{ni}(\lambda^c \wedge \bigwedge_{1 \leq j \leq n} A_{ji}^c))^T \\ &= (0, 0, \dots, 0)^T = \vec{0}. \end{aligned}$$

Also,

$$\begin{aligned} \lambda\bar{x}_i &= \lambda(0, \dots, x_i, \dots, 0)^T = (0, \dots, \lambda x_i, \dots, 0)^T \\ &= (0, \dots, \lambda(\lambda^c \wedge \bigwedge_{1 \leq j \leq n} A_{ji}^c), \dots, 0)^T \\ &= (0, 0, \dots, 0)^T = \vec{0}. \end{aligned}$$

Therefore $A\bar{x}_i = \lambda\bar{x}_i$, for $i = 1, 2, \dots, n$. Hence $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n \in E_A(\lambda)$.

Now let $(y_1, y_2, \dots, y_n)^T$ be any eigenvector of λ . Then by Theorem 3.3,

$$\begin{aligned} (y_1, y_2, \dots, y_n)^T &\leq (x_1, x_2, \dots, x_n)^T \\ \Rightarrow y_i &\leq x_i, \text{ for } i = 1, 2, \dots, n. \end{aligned}$$

Take $a_i = y_i$, for $i = 1, 2, \dots, n$. Then $a_i \in L$ and $y_i = a_i x_i$, for $i = 1, 2, \dots, n$. Therefore

$$\begin{aligned} (y_1, y_2, \dots, y_n)^T &= (a_1 x_1, a_2 x_2, \dots, a_n x_n)^T \\ &= a_1(x_1, 0, \dots, 0)^T \vee a_2(0, x_2, \dots, 0)^T \vee \cdots \vee a_n(0, 0, \dots, x_n)^T \\ &= a_1\bar{x}_1 \vee a_2\bar{x}_2 \vee \cdots \vee a_n\bar{x}_n. \end{aligned}$$

Hence the λ -eigenspace $E_A(\lambda)$ is spanned by $\{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n\} - \{\vec{0}\}$.

Theorem 3.5 Let $A \in M_n(L)$ be nilpotent and λ be a given eigenvalue of A . Then

$\lambda^c = 0$ if and only if the only eigenvector of λ is $\vec{0}$.

Proof. First assume that $\lambda^c = 0$. Then by Lemma 2.7,

$$\bar{x}_g(\lambda) = (\lambda\vec{e} \vee A^T\vec{e})^c = (\lambda\vec{e})^c \wedge (A^T\vec{e})^c = \vec{0} \wedge (A^T\vec{e})^c = \vec{0}.$$

Hence by Theorem 3.3, the only eigenvector of λ is $\vec{0}$.

Conversely assume that the only eigenvector of λ is $\vec{0}$. Then by Theorem 3.3,

$$\vec{x}_g(\lambda) = (\lambda \vec{e} \vee A^T \vec{e})^c = \vec{0} \Rightarrow (\lambda \vee \bigvee_{1 \leq j \leq n} A_{ji})^c = 0, 1 \leq i \leq n.$$

If possible, assume that $\lambda^c \neq 0$. Then

$$\begin{aligned} (\lambda \vee \bigvee_{1 \leq j \leq n} A_{ji})^c = 0, 1 \leq i \leq n &\Rightarrow (\lambda \vee \bigvee_{1 \leq j \leq n} A_{ji}) \lambda^c \neq 0, 1 \leq i \leq n \\ &\Rightarrow (\lambda \lambda^c) \vee (\bigvee_{1 \leq j \leq n} A_{ji}) \lambda^c \neq 0, 1 \leq i \leq n \\ &\Rightarrow \bigvee_{1 \leq j \leq n} (A_{ji} \lambda^c) \neq 0, 1 \leq i \leq n \\ &\Rightarrow A_{ji} \lambda^c \neq 0, \text{ for some } i, j = 1, 2, \dots, n \end{aligned}$$

Let $i = p$ and $j = p_1$. Then $A_{p_1 p} \lambda^c \neq 0$.

Now

$$\begin{aligned} (\lambda \vee \bigvee_{1 \leq j \leq n} A_{jp_1})^c = 0 &\Rightarrow (\lambda \vee \bigvee_{1 \leq j \leq n} A_{jp_1}) A_{p_1 p} \lambda^c \neq 0 \\ &\Rightarrow (\lambda A_{p_1 p} \lambda^c) \vee (\bigvee_{1 \leq j \leq n} A_{jp_1}) A_{p_1 p} \lambda^c \neq 0 \\ &\Rightarrow \bigvee_{1 \leq j \leq n} (A_{jp_1} A_{p_1 p} \lambda^c) \neq 0 \\ &\Rightarrow A_{jp_1} A_{p_1 p} \lambda^c \neq 0, \text{ for some } j = 1, 2, \dots, n, \text{ (say } p_2) \end{aligned}$$

Hence $A_{p_2 p_1} A_{p_1 p} \lambda^c \neq 0$.

Proceeding as above, we get

$$A_{p_n p_{n-1}} \cdots A_{p_2 p_1} A_{p_1 p} \lambda^c \neq 0, \text{ where } 1 \leq p, p_1, p_2, \dots, p_n \leq n.$$

But, by Theorem 2.4,

$$A_{p_n p_{n-1}} \cdots A_{p_2 p_1} A_{p_1 p} \lambda^c = 0, \text{ where } 1 \leq p, p_1, p_2, \dots, p_n \leq n.$$

Therefore, we get a contradiction.

Hence $\lambda^c = 0$.

Next, we consider a given eigenvector \vec{x} of a nilpotent matrix A in $M_n(L)$ and discuss the properties of its eigenvalues. For $a, b \in L$, the interval $[a, b] = \{x \in L \mid a \leq x \leq b\}$.

Theorem 3.6 [5] *Let $A \in M_n(L)$ and \vec{x} be an eigenvector of A . Then the set of all eigenvalues of \vec{x} form a sublattice of L consisting of the interval $[\lambda^l, \lambda^u]$, where $\lambda^l = \vec{e}^T A \vec{x}$ and $\lambda^u = \vec{e}^T \vec{x} \rightarrow \lambda^l$.*

A modification for the value of λ^l in Theorem 3.6 can be given as $\lambda^l = \vec{x}^T A \vec{x}$, since

$$\begin{aligned} \lambda^l &= \vec{x}^T A \vec{x} = \vec{x}^T \lambda \vec{x}, \text{ for some } \lambda \text{ in } L \\ &= \lambda \vec{x}^T \vec{x} = \lambda \vec{e}^T \vec{x} = \vec{e}^T \lambda \vec{x} = \vec{e}^T A \vec{x}. \end{aligned}$$

Theorem 3.7 Let $A \in M_n(L)$ be nilpotent and \bar{x} be an eigenvector of A . If \bar{x} is an ortho vector of A with $(\bar{e}^T \bar{x})^c = 0$, then \bar{x} has the unique eigenvalue 0.

Proof. We have by Theorem 2.4,

$$\lambda^l = \bar{x}^T A \bar{x} = \bigvee_{1 \leq i, j \leq n} A_{ij} x_i x_j = \bigvee_{1 \leq i, j \leq n, i \neq j} A_{ij} x_i x_j = 0 \text{ and}$$

$$\lambda^n = \bar{e}^T \bar{x} \rightarrow \lambda^l = \bar{e}^T \bar{x} \rightarrow 0 = (\bar{e}^T \bar{x})^c = 0.$$

Hence by Theorem 3.6, \bar{x} has the unique eigenvalue 0.

Corollary 3.8 Let $A \in M_n(L)$ be nilpotent and \bar{x} be an eigenvector of A . If \bar{x} is a stochastic vector of A , then \bar{x} has the unique eigenvalue 0.

Theorem 3.9 [7] If an eigenvector \bar{x} of $A \in M_n(L)$ has the unique eigenvalue λ , then $0 \leq \lambda \leq \bar{e}^T A^n \bar{e}$.

Theorem 3.10 Let $A \in M_n(L)$ be nilpotent. If an eigenvector \bar{x} of A has the unique eigenvalue λ , then $\lambda = 0$.

Proof. Since A is nilpotent, by Theorem 2.3, we have $A^n = \mathbf{0}_n$. Therefore by Theorem 3.9, $0 \leq \lambda \leq \bar{e}^T \mathbf{0}_n \bar{e} = 0$. Hence $\lambda = 0$.

Theorem 3.11 Let $A \in M_n(L)$ be nilpotent and \bar{x} be an eigenvector of A with the associated eigenvalue $\lambda \neq 0$, then \bar{x} is an eigenvector of A with the associated eigenvalue 0.

Proof. Let \bar{x} be an eigenvector of A with the associated eigenvalue $\lambda \neq 0$, then $A\bar{x} = \lambda\bar{x}$. By Theorem 2.3, we have

$$\bar{0} = A^n \bar{x} = A^{n-1} A\bar{x} = A^{n-1} \lambda\bar{x} = \dots = A\lambda\bar{x} = \lambda A\bar{x} = \lambda\lambda\bar{x} = \lambda\bar{x}.$$

Therefore, $A\bar{x} = \lambda\bar{x} = \bar{0} = 0\bar{x}$.

Hence the proof is complete.

4. NON-SINGULAR LATTICE MATRICES

In this section, we introduce the concept of non-singular matrices over a complete and completely distributive lattice with 1 and 0.

Definition 4.1 Let $A \in M_n(L)$. If the sub lattice vector space

$E_A(\bar{0}) = \{\bar{x} \in V_n(L) \mid A\bar{x} = \bar{0}\}$ of $V_n(L)$ contains a non-zero vector, then A is said to be singular. Otherwise, A is non-singular.

Theorem 4.2 Let $A \in M_n(L)$. Then A is non-singular if and only if $(A^T \bar{e})^c = \bar{0}$.

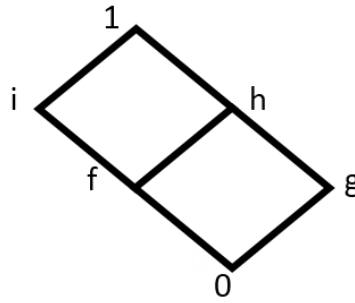
Proof. We have A is non-singular $\Leftrightarrow E_A(\bar{0}) = \{\bar{0}\}$. By Theorem 3.2, $\bar{x}_g(\bar{0}) = (0\bar{e} \vee A^T \bar{e}) \rightarrow (0A^n \bar{e}) = \bar{0} \Leftrightarrow A^T \bar{e} \rightarrow \bar{0} = \bar{0} \Leftrightarrow (A^T \bar{e})^c = \bar{0}$.

Theorem 4.3 *Let $A \in M_n(L)$. If A is invertible, then A is non-singular.*

Proof. Assume that A is invertible. Then by Theorem 2.5, $A^T \vec{e} = \vec{e}$. Therefore, $(A^T \vec{e})^c = \vec{e}^c = \vec{0}$. Hence by Theorem 4.2, A is non-singular.

However, the converse need not be true. The following example shows that if A is non-singular, then A need not be invertible.

Example 4.4 *Consider the lattice $L = \{0, f, g, h, i, 1\}$, whose diagrammatical representation is as follows:*



It is easy to see that L is a distributive lattice.

Let $A = \begin{bmatrix} h & f \\ f & h \end{bmatrix} \in M_2(L)$. Then $(A^T \vec{e})^c = (h, h)^c = \vec{0}$. Hence by Theorem 4.2,

A is non-singular. But by Theorem 2.5, we can see that A is not invertible.

Theorem 4.5 *Every nilpotent lattice matrix is singular.*

Proof. Let $A \in M_n(L)$ be nilpotent. Since $0^c = 1$, by Theorem 3.5, $E_A(0)$ contains a non-zero vector. Hence A is singular.

Theorem 4.6 *Let $A, B \in M_n(L)$ such that A is non-singular and B is invertible.*

Then we have

- (a). A is non-singular if and only if AB is non-singular.
- (b). A is non-singular if and only if BA is non-singular.

Proof. (a) First assume that A is non-singular.

Then by Theorem 4.2, $(A^T \vec{e})^c = \vec{0}$.

Let $C = AB$. Then

$$((AB)^T \vec{e})^c = (C^T \vec{e})^c = ((\bigvee_{1 \leq k_1 \leq n} C_{k_1 1})^c, (\bigvee_{1 \leq k_2 \leq n} C_{k_2 2})^c, \dots, (\bigvee_{1 \leq k_n \leq n} C_{k_n n})^c)^T.$$

Consider

$$\begin{aligned}
& (\bigvee_{1 \leq k_1 \leq n} C_{k_1})^c \vee (\bigvee_{1 \leq k_2 \leq n} C_{k_2})^c \vee \cdots \vee (\bigvee_{1 \leq k_n \leq n} C_{k_n})^c \\
&= (\bigwedge_{1 \leq k_1 \leq n} C_{k_1}^c) \vee (\bigwedge_{1 \leq k_2 \leq n} C_{k_2}^c) \vee \cdots \vee (\bigwedge_{1 \leq k_n \leq n} C_{k_n}^c) \text{ (By Lemma 2.1(a))} \\
&= \bigwedge_{1 \leq k_1, k_2, \dots, k_n \leq n} (C_{k_1}^c \vee C_{k_2}^c \vee \cdots \vee C_{k_n}^c) \\
&\leq \bigwedge_{1 \leq k_1, k_2, \dots, k_n \leq n} (C_{k_1} \wedge C_{k_2} \wedge \cdots \wedge C_{k_n})^c \text{ (By Lemma 2.1(b))} \\
&\leq (\bigvee_{1 \leq k_1, k_2, \dots, k_n \leq n} C_{k_1} \wedge C_{k_2} \wedge \cdots \wedge C_{k_n})^c \text{ (By Lemma 2.1(a))}
\end{aligned}$$

Now

$$\begin{aligned}
& C_{k_1} \wedge C_{k_2} \wedge \cdots \wedge C_{k_n} \\
&= (\bigvee_{1 \leq i_1 \leq n} A_{k_1 i_1} B_{i_1}) \wedge (\bigvee_{1 \leq i_2 \leq n} A_{k_2 i_2} B_{i_2}) \wedge \cdots \wedge (\bigvee_{1 \leq i_n \leq n} A_{k_n i_n} B_{i_n}) \\
&= \bigvee_{1 \leq i_1, i_2, \dots, i_n \leq n} (A_{k_1 i_1} A_{k_2 i_2} \cdots A_{k_n i_n}) \wedge (B_{i_1} B_{i_2} \cdots B_{i_n}) \\
&= \bigvee_{\sigma \in S_n} (A_{k_1 \sigma(1)} A_{k_2 \sigma(2)} \cdots A_{k_n \sigma(n)}) \wedge (B_{\sigma(1)} B_{\sigma(2)} \cdots B_{\sigma(n)}) \text{ (By Theorem 2.5)}
\end{aligned}$$

Therefore,

$$\begin{aligned}
& (\bigvee_{1 \leq k_1 \leq n} C_{k_1})^c \vee (\bigvee_{1 \leq k_2 \leq n} C_{k_2})^c \vee \cdots \vee (\bigvee_{1 \leq k_n \leq n} C_{k_n})^c \\
&\leq (\bigvee_{1 \leq k_1, k_2, \dots, k_n \leq n} C_{k_1} \wedge C_{k_2} \wedge \cdots \wedge C_{k_n})^c \\
&\leq (\bigvee_{1 \leq k_1, k_2, \dots, k_n \leq n} (\bigvee_{\sigma \in S_n} (A_{k_1 \sigma(1)} A_{k_2 \sigma(2)} \cdots A_{k_n \sigma(n)}) \wedge (B_{\sigma(1)} B_{\sigma(2)} \cdots B_{\sigma(n)})))^c \\
&\leq ((\bigvee_{\sigma \in S_n} B_{\sigma(1)} B_{\sigma(2)} \cdots B_{\sigma(n)}) \wedge (\bigvee_{1 \leq k_1, k_2, \dots, k_n \leq n} A_{k_1} A_{k_2} \cdots A_{k_n}))^c \\
&\leq (\bigvee_{1 \leq k_1, k_2, \dots, k_n \leq n} A_{k_1} A_{k_2} \cdots A_{k_n})^c \text{ (By Lemma 2.6)} \\
&\leq (\bigwedge_{1 \leq i \leq n} (\bigvee_{1 \leq j \leq n} A_{ji}))^c \\
&\leq 0 \text{ (By Theorem 4.2 and Lemma 2.2).}
\end{aligned}$$

Thus $(\bigvee_{1 \leq k_i \leq n} C_{k_i})^c = 0$, for $i = 1, 2, \dots, n$. Therefore, $((AB)^T \vec{e})^c = 0$. Hence AB is non-singular.

Conversely assume that AB is non-singular. Then by first part $(AB)B^{-1} = A(BB^{-1}) = A$ is non-singular.

(b) First assume that A is non-singular. Then by Theorem 4.2, $(A^T \vec{e})^c = \vec{0}$.

We have by Theorem 2.5, $((BA)^T \vec{e})^c = ((A^T B^T) \vec{e})^c = (A^T (B^T \vec{e}))^c = (A^T \vec{e})^c = \vec{0}$ Hence BA is non-singular.

Conversely assume that BA is non-singular. Then by first part $B^{-1}(BA) = (B^{-1}B)A = A$ is non-singular.

Theorem 4.7 *Let $A \in M_n(L)$. Then A is non-singular if and only if every lattice matrix similar to A is non-singular.*

Proof. Let $B \in M_n(L)$ such that B is similar to A . Then there exists an invertible matrix $P \in M_n(L)$ such that $B = P^{-1}AP$. By Theorem 4.6, A is non-singular if and only if B is non-singular. Hence the proof is complete.

5. CONCLUSION

In this paper, we partially solved the open problem 'How to describe the λ -eigenspace of a given lattice matrix?', for a given nilpotent matrix over a complete and completely distributive lattice with 1 and 0. Also, we introduced the concept of non-singular lattice matrices. We hope this research would be able to throw light on at least a few new concepts in lattice matrices including Boolean and fuzzy matrices and its applications which will further enrich the subject in science and technology.

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