Advances in Fuzzy Mathematics.
ISSN 0973-533X Volume 12, Number 4 (2017), pp. 881-895

http://www.ripublication.com

New Intuitionistic Fuzzy Operator $A^{(m,n)}$ and an Application on Decision Making

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Abstract

Many operators are defined over the intuitionistic fuzzy sets. The primary objective of this paper is to propose a new intuitionistic fuzzy operator $A^{(m,n)}$. With respect to this operator some theorems related to modal, level operators, set relations and set operations over intuitionistic fuzzy sets are discussed. A numerical example in decision making is also given to clarify the developed approach and to demonstrate its application.

Keywords: Fuzzy sets, Intuitionistic fuzzy sets, Intuitionistic fuzzy numbers, Intuitionistic fuzzy operators, Decision making.

Mathematics Subject Classification: 03E72, 03E55, 03E75, 62C86.

1. INTRODUCTION

Fuzzy sets were introduced by L.A.Zadeh **[1]**. Atanassov **[3]** extended the fuzzy set into the intuitionistic fuzzy set (IFS) which is characterised by a membership function, a non-membership function and a hesitancy function. For the past few years researchers concentrate on the operators over IFS and has been applied to various fields like decision making, medical diagnosis and market prediction etc.,.

The operations and relations defined as in ordinary fuzzy sets can be defined in intuitionistic fuzzy sets (IFS). But the operators which cannot be defined in ordinary fuzzy sets can be defined in IFS. In the two books of Atanassov **[2,3]** modal operators □ , ◊ and a series of their extensions were described. One of the most powerful extension of the modal operators is the operator $X_{a,b,c,d,e,f}$. We introduce a new intuitionistic fuzzy operator $A^{(m,n)}$ to an intuitionistic fuzzy set A for every pair m, n ∈ ℕ.

As $A^{(m,n)}$ is verified as IFS, we prove some new equalities related to modal, level operators and IFS relations and operations are discussed. A decision making problem by using the operator $\Diamond A^{(m,n)} \rightarrow \Box A^{(m,n)}$ to select the future course by the students is discussed. In **[9]** Ejegwa Paul Augustine used intuitionistic fuzzy sets in career determination, medical diagnosis and pattern recognition. In **[10]** Desislava, Evdokia Sotirova, and Veselina Bureva applied inter criteria analysis approach to health related quality of life. In **[11]** Evdokia Sotirova, Veselina Bureva, Panagiotis Chountas and Maciejkrawczak applied inter criteria decision making method to the ranking of Universities in the United Kingdom.

In **[12]** Cökhan Cuvalcioglu and Esra Aykut applied some intuitionistic fuzzy modal operators to agriculture. In **[14]** Eulalia Szmidt and Janusz Kacprzyk used intuitionistic fuzzy sets in some medical applications. In **[15]** Lyubka Doukovska and Vassia Atanassova used inter criteria analysis approach in radar detection threshold analysis.

This paper is organised as follows: In section **2** some basic definitions related to IFS are presented. In section **3** a new operator $A^{(m,n)}$ is introduced and some equalities connected with the new operator are proved. In section **4** a decision making application is given to validate the new operator.

2. PRELIMINARIES

An intuitionistic fuzzy set A defined over a nonempty set X is the form of elements $< x, \mu_A(x), \nu_A(x) >$ where μ_A: X → [0,1] and ν_A: X → [0,1] are the degree of membership and degree of non-membership for every $x \in$ X respectively be such that $0 \leq \mu_A(x) + \nu_A(x) \leq 1$. The complement of $\mu_A(x) + \nu_A(x)$ from 1 namely $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is called hesistancy degree of x in A.

2.1 Set relations and operations on IFS [2]

Let IFS(X) denote the family of all intuitionistic fuzzy sets in the universe X.

For every A, B ∈ IFS(X) which are represented by A = $\{< x, \mu_A(x), \nu_A(x) > | \, x \in X\}$, B = $\{< x, \mu_B(x), \nu_B(x) > | \, x \in X\}$. Some relations and operations are defined as follows:

i. $\bar{A} = \{< x, \nu_A(x), \mu_A(x) > | x \in X\}$,
ii. $A \subset B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ $\forall x \in X$,
iii. $A = B$ iff $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$ $\forall x \in X$,
iv. $A \cap B = \{< x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x)) > | x \in X\}$,
v. $A \cup B = \{< x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x)) > | x \in X\}$,
vi. $A + B = \{< x, \mu_A(x) + \mu_B(x) - \mu_A(x)\,\mu_B(x), \nu_A(x)\,\nu_B(x) > | x \in X\}$,
vii. $A \cdot B = \{< x, \mu_A(x)\,\mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x)\,\nu_B(x) > | x \in X\}$,
viii. $A \,@\, B = \{< x, \frac{\mu_A(x) + \mu_B(x)}{2}, \frac{\nu_A(x) + \nu_B(x)}{2} > | x \in X\}$,
ix. $A \,\$\, B = \{< x, \sqrt{\mu_A(x)\,\mu_B(x)}, \sqrt{\nu_A(x)\,\nu_B(x)} > | x \in X\}$,
x. $A \# B = \{< x, \frac{2\,\mu_A(x)\,\mu_B(x)}{\mu_A(x) + \mu_B(x)}, \frac{2\,\nu_A(x)\,\nu_B(x)}{\nu_A(x) + \nu_B(x)} > | x \in X\}$.If $\mu_A(x) = \mu_B(x) = 0$ then $\frac{\mu_A(x)\,\mu_B(x)}{\mu_A(x) + \mu_B(x)} = 0$ and if $\nu_A(x) = \nu_B(x) = 0$ then $\frac{\nu_A(x)\,\nu_B(x)}{\nu_A(x) + \nu_B(x)} = 0$,
xi. $A * B = \{< x, \frac{\mu_A(x) + \mu_B(x)}{2(\mu_A(x)\,\mu_B(x) + 1)}, \frac{\nu_A(x) + \nu_B(x)}{2(\nu_A(x)\,\nu_B(x) + 1)} > | x \in X\}$,
xii. $A \to B = \{< x, \max(\nu_A(x), \mu_B(x)), \min(\mu_A(x), \nu_B(x)) > | x \in X\}$.

In **[2]** the simplest modal operators defined over IFS are

$\Box A = \{< x, \mu_A(x), 1 - \mu_A(x) > | x \in X\}$ and
$\Diamond A = \{< x, 1 - \nu_A(x), \nu_A(x) > | x \in X\}$.

Two new relations are defined by

$A \subset_{\Box} B$ iff $\mu_A(x) \leq \mu_B(x)$ $\forall x \in X$ and
$A \subset_{\Diamond} B$ iff $\nu_A(x) \geq \nu_B(x)$ $\forall x \in X$.

The level operators are defined by

$P_{\alpha,\beta}(A) = \{< x, \max(\mu_A(x), \alpha), \min(\nu_A(x), \beta) > | x \in X\}$ and
$Q_{\alpha,\beta}(A) = \{< x, \min(\mu_A(x), \alpha), \max(\nu_A(x), \beta) > | x \in X\}$ where $\alpha + \beta \leq 1$.

The empty IFS, the totally uncertain IFS and the unit IFS are defined by
$O^* = \{< x, 0, 1 > | x \in X\}$, $U^* = \{< x, 0, 0 > | x \in X\}$ and $E^* = \{< x, 1, 0 > | x \in X\}$.

In **[7]** Supriyakumar De, Ranjit Biswas and Akhil Ranjan Roy introduced the operator $A^n = \{< x, (\mu_A(x))^n, 1 - (1 - \nu_A(x))^n > | x \in X\}$.

In **[8]** Beloslav Riecan and Krassimir T. Atanassov defined the extraction operator $A^{1/n} = \{< x, (\mu_A(x))^{1/n}, 1 - (1 - \nu_A(x))^{1/n} > | x \in X\}$.

2.2 Intuitionistic Fuzzy Number

An IFN A is defined as follows:

i. Intuitionistic fuzzy subset of the real line $\mathbb{R}$,
ii. Normal i.e there is an $x_0 \in \mathbb{R}$ be such that $\mu_A(x_0) = 1$ (so $\nu_A(x_0) = 0$),
iii. A convex set for the membership function $\mu_A(x)$ ie $\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$ for all $x_1, x_2 \in \mathbb{R}$, $\lambda \in [0,1]$,

iv. A convex set for the non-membership function $\nu_A(x)$ ie $\nu_A(\lambda x_1 + (1-\lambda)x_2) \geq \max(\nu_A(x_1), \nu_A(x_2))$ for all $x_1, x_2 \in \mathbb{R}$, $\lambda \in [0,1]$.

3. NEW OPERATOR ON IFS

For every $A \in IFS(X)$ and for every $m, n \in \mathbb{N}$, we define $A^{(m,n)} = \{< x, \frac{(\mu_A(x))^m}{n}, 1 - \frac{(1-\nu_A(x))^m}{n} > | \, x \in X\}$. It is clearly an IFS. From this definition we have the following: **i)** $E^{*(m,1)} = E^*$ **ii)** $U^{*(m,1)} = U^*$ **iii)** $O^{*(m,1)} = O^*$ **iv)** If $m \leq m_1$ then $A^{(m,n)} \subset A^{(m_1,n)}$ and if $n \leq n_1$ then $A^{(m,n_1)} \subset A^{(m,n)}$ where $m, n, m_1, n_1 \in \mathbb{N}$, **v)** If $A \subset B$ then $A^{(m,n)} \subset B^{(m,n)}$.

Theorem 1

i) $(A^{(m,1)})^{(n,1)} = A^{(mn,1)} = (A^{(n,1)})^{(m,1)}$

ii) $(A^{(1,m)})^{(1,n)} = A^{(1,mn)} = (A^{(1,n)})^{(1,m)}$

iii) $(A^{(m,1)})^{(1,n)} = A^{(m,n)}$

iv) $(A^{(1,m)})^{(n,1)} = A^{(n,m^n)}$

v) $\Box(A^{(m,n)}) = (\Box A)^{(m,n)}$

vi) $\Box(A \cap B)^{(m,n)} \cdot (A \cup B)^{(m,n)} = \Box A^{(m,n)} \cdot \Box B^{(m,n)}$. If we replace the set operation by any one of the following operations +, @, \$, # and $*$ the result **(vi)** is true. If we replace $\Box$ by $\Diamond$ the results **(v)** and **(vi)** are true.

Theorem 2

i) $(\Box A^{(m,n)})^{(m_1, n_1)} = \Box(A^{(mm_1, n_1 n^{m_1})})$

ii) $(\Box A^{(m,n)})^{(m_1, n_1)} \cdot (\Box A^{(n,m)})^{(n_1, m_1)} = \Box A^{(mm_1 + nn_1, \; m_1 n_1 m^{n_1} n^{m_1})}$

If we replace $\Box$ by $\Diamond$ the above theorem is true.

Proof: Obviously **(i)** is true.

We have $\Box A^{(m,n)} = \{< x, \frac{(\mu_A(x))^m}{n}, 1 - \frac{(\mu_A(x))^m}{n} > | \, x \in X\}$.Therefore

$$(\Box A^{(m,n)})^{(m_1, n_1)} = \{< x, \frac{(\mu_A(x))^{mm_1}}{n_1 n^{m_1}}, 1 - \frac{(\mu_A(x))^{mm_1}}{n_1 n^{m_1}} > | \, x \in X\} \qquad \textbf{(1)}$$

and

$$(\Box A^{(n,m)})^{(n_1, m_1)} = \{< x, \frac{(\mu_A(x))^{nn_1}}{m_1 m^{n_1}}, 1 - \frac{(\mu_A(x))^{nn_1}}{m_1 m^{n_1}} > | \, x \in X\} \qquad \textbf{(2)}$$

Now with $\cdot$ of **(1)** and **(2)**, $(\Box A^{(m,n)})^{(m_1, n_1)} \cdot (\Box A^{(n,m)})^{(m_1, n_1)}$

$$= \{< x, \frac{(\mu_A(x))^{mm_1 + nn_1}}{m_1 n_1 m^{n_1} n^{m_1}}, 1 - \frac{(\mu_A(x))^{mm_1 + nn_1}}{m_1 n_1 m^{n_1} n^{m_1}} > | \, x \in X\}$$

$$= \Box A^{(mm_1 + nn_1, \; m_1 n_1 m^{n_1} n^{m_1})}$$. This proves **(ii)**.

Theorem 3

i) $(\Box A^{(m,n)} \cdot \Diamond A^{(m,n)})$ @ $\overline{(\Box A^{(m,n)} + \Diamond A^{(m,n)})} = \Box A^{(m,n)} \cdot \Diamond A^{(m,n)}$

ii) $(\Box A^{(m,n)} \cdot \Diamond A^{(m,n)})$ @ $(\Box A^{(m,n)} + \Diamond A^{(m,n)}) = \{< x, \frac{1}{2}, \frac{1}{2} > | \, x \in X\}$

Proof: we have $\Box A^{(m,n)} = \{< x, \frac{(\mu_A(x))^m}{n}, 1 - \frac{(\mu_A(x))^m}{n} > | \, x \in X\}$, **(3)**

$\Box B^{(m,n)} = \{< x, \frac{(\mu_B(x))^m}{n}, 1 - \frac{(\mu_B(x))^m}{n} > | \, x \in X\}$, **(4)**

and $\Diamond A^{(m,n)} = \{< x, \frac{(1 - \nu_A(x))^m}{n}, 1 - \frac{(1 - \nu_A(x))^m}{n} > | \, x \in X\}$. **(5)**

Now with · of **(3)** and **(5)**, we have

$\Box A^{(m,n)} \cdot \Diamond A^{(m,n)} = \{< x, \frac{(\mu_A(x)\cdot(1 - \nu_A(x)))^m}{n^2}, 1 - \frac{(\mu_A(x)\cdot(1 - \nu_A(x)))^m}{n^2} > | \, x \in X\}$ **(6)**

Now with **+** of **(3)** and **(5)**, we have the following respectively

$\Box A^{(m,n)} + \Diamond A^{(m,n)} = \{< x, 1 - \frac{(\mu_A(x)\cdot(1 - \nu_A(x)))^m}{n^2}, \frac{(\mu_A(x)\cdot(1 - \nu_A(x)))^m}{n^2} > | \, x \in X\}$ **(7)**

$\overline{\Box A^{(m,n)} + \Diamond A^{(m,n)}} = \{< x, \frac{(\mu_A(x)\cdot(1 - \nu_A(x)))^m}{n^2}, 1 - \frac{(\mu_A(x)\cdot(1 - \nu_A(x)))^m}{n^2} > | \, x \in X\}$ **(8)**

Now with **@** of **(6)** and **(8)**, $(\Box A^{(m,n)} \cdot \Diamond A^{(m,n)})$ @ $\overline{(\Box A^{(m,n)} + \Diamond A^{(m,n)})}$

$= \{< x, \frac{(\mu_A(x).(1 - \nu_A(x)))^m}{n^2}, 1 - \frac{(\mu_A(x)\cdot(1 - \nu_A(x)))^m}{n^2} > | \, x \in X\}$

$= \Box A^{(m,n)} \cdot \Diamond A^{(m,n)}$. This proves **(i)**.

Now with **@** of **(6)** and **(7)**, $(\Box A^{(m,n)} \cdot \Diamond A^{(m,n)})$ @ $(\Box A^{(m,n)} + \Diamond A^{(m,n)})$

$= \{< x, \frac{1}{2}, \frac{1}{2} > | \, x \in X\}$. This proves **(ii)**.

Theorem 4

i) $[P_{\alpha^{1/m}, \, 1-(1-\beta)^{1/m}}(\Box A)]^{(m,n)} = P_{\frac{\alpha}{n}, \, 1-\frac{(1-\beta)}{n}}[\Box A^{(m,n)}]$

ii) $[Q_{\alpha^{1/m}, \, 1-(1-\beta)^{1/m}}(\Box A)]^{(m,n)} = Q_{\frac{\alpha}{n}, \, 1-\frac{(1-\beta)}{n}}[\Box A^{(m,n)}]$

iii) $[P_{\alpha^{1/m}, \, 1-(1-\beta)^{1/m}}(\Diamond A)]^{(m,n)} = P_{\frac{\alpha}{n}, \, 1-\frac{(1-\beta)}{n}}[\Diamond A^{(m,n)}]$

iv) $[Q_{\alpha^{1/m}, \, 1-(1-\beta)^{1/m}}(\Diamond A)]^{(m,n)} = Q_{\frac{\alpha}{n}, \, 1-\frac{(1-\beta)}{n}}[\Diamond A^{(m,n)}]$

Proof: $[P_{\alpha^{1/m}, \, 1-(1-\beta)^{1/m}}(\{< x, \mu_A(x), 1 - \mu_A(x) > | \, x \in X\})]^{(m,n)}$

$= \{< x, \max(\alpha^{1/m}, \mu_A(x)), \min(1 - (1-\beta)^{1/m}, 1 - \mu_A(x)) > | \, x \in X\}^{(m,n)}$

$= \{< x, \frac{(\max(\alpha^{1/m}, \mu_A(x)))^m}{n}, 1 - \frac{(1 - \min(1 - (1-\beta)^{1/m}, 1 - \mu_A(x)))^m}{n} > | \, x \in X\}$

$= \{< x, \max(\frac{\alpha}{n}, \frac{(\mu_A(x))^m}{n}), 1 - \frac{\max((1-\beta)^{1/m}, \mu_A(x))^m}{n}) > | \, x \in X\}$

$= \{< x, \max(\frac{\alpha}{n}, \frac{(\mu_A(x))^m}{n}), 1 - \max(\frac{(1-\beta)}{n}, 1 - \frac{(\mu_A(x))^m}{n}) > | \, x \in X\}$

$= \{< x, \max(\frac{\alpha}{n}, \frac{(\mu_A(x))^m}{n}), \min(1-\frac{(1-\beta)}{n}, 1-\frac{(\mu_A(x))^m}{n}) > | x \in X\}$

$= P_{\frac{\alpha}{n}, 1-\frac{(1-\beta)}{n}}[\square A^{(m,n)}]$. This proves **(i)**.

Proof of **(ii)**, **(iii)** and **(iv)** are similar to that of **(i)**.

Theorem 5

i) $(\square A^{(m,n)} \cdot \square B^{(m,n)})\ @\ (\square A^{(m,n)} + \square B^{(m,n)}) = \square A^{(m,n)} @ \square B^{(m,n)}$

ii) $(\lozenge A^{(m,n)} \cdot \lozenge B^{(m,n)})\ @\ (\lozenge A^{(m,n)} + \lozenge B^{(m,n)}) = \lozenge A^{(m,n)} @ \lozenge B^{(m,n)}$

Proof: From **(3)** and **(4)**, We have $\square A^{(m,n)} + \square B^{(m,n)}$

$$= \{< x, \left(\frac{(\mu_A(x))^m}{n} + \frac{(\mu_B(x))^m}{n} - \frac{(\mu_A(x))^m}{n} \cdot \frac{(\mu_B(x))^m}{n}\right), (1-\frac{(\mu_A(x))^m}{n}) \cdot (1-\frac{(\mu_B(x))^m}{n}) > | x \in X\} \quad \textbf{(9)}$$

and $\square A^{(m,n)} \cdot \square B^{(m,n)}$

$$= \{< x, \left(\frac{(\mu_A(x))^m}{n} \cdot \frac{(\mu_B(x))^m}{n}\right), \left(1-\frac{(\mu_A(x))^m}{n}\right) + \left(1 \ \frac{(\mu_B(x))^m}{n}\right) - (1-\frac{(\mu_A(x))^m}{n}) \cdot (1-\frac{(\mu_B(x))^m}{n}) > | x \in X\} \quad \textbf{(10)}$$

Now with @ of **(10)** and **(9)**, $(\square A^{(m,n)} \cdot \square B^{(m,n)})\ @\ (\square A^{(m,n)} + \square B^{(m,n)})$

$$= \{< x, \frac{(\mu_A(x))^m + (\mu_B(x))^m}{2n}, \frac{\left(1-\frac{(\mu_A(x))^m}{n}\right) + \left(1-\frac{(\mu_B(x))^m}{n}\right)}{2} > | x \in X\}$$

$= \square A^{(m,n)}\ @\ \square B^{(m,n)}$. This proves **(i)**. Similarly the proof of **(ii)**.

Theorem 6

i) $[(\square A^{(m,n)} + \square B^{(m,n)}) \cup (\square A^{(m,n)} \cdot \square B^{(m,n)})]$

$@\ [(\square A^{(m,n)} + \square B^{(m,n)}) \cap (\square A^{(m,n)} \cdot \square B^{(m,n)})] = \square A^{(m,n)} @ \square B^{(m,n)}$

ii) $[(\lozenge A^{(m,n)} + \lozenge B^{(m,n)}) \cup (\lozenge A^{(m,n)} \cdot \lozenge B^{(m,n)})]$

$@\ [(\lozenge A^{(m,n)} + \lozenge B^{(m,n)}) \cap (\lozenge A^{(m,n)} \cdot \lozenge B^{(m,n)})] = \lozenge A^{(m,n)} @ \lozenge B^{(m,n)}$

Proof: With ∪ of **(9)** and **(10)**, $(\square A^{(m,n)} + \square B^{(m,n)}) \cup (\square A^{(m,n)} \cdot \square B^{(m,n)})$

$$= \{< x, \max(\frac{(\mu_A(x))^m}{n} + \frac{(\mu_B(x))^m}{n} - (\frac{(\mu_A(x))^m}{n} \cdot \frac{(\mu_B(x))^m}{n}), \frac{(\mu_A(x))^m}{n} \cdot \frac{(\mu_B(x))^m}{n}),$$
$$\min(\left(1-\frac{(\mu_A(x))^m}{n}\right) \cdot \left(1-\frac{(\mu_B(x))^m}{n}\right), \left(1-\frac{(\mu_A(x))^m}{n}\right) + \left(1-\frac{(\mu_B(x))^m}{n}\right) -$$
$$\left(1-\frac{(\mu_A(x))^m}{n}\right) \cdot \left(1-\frac{(\mu_B(x))^m}{n}\right)) > | x \in X\} \quad \textbf{(11)}$$

With ∩ of **(9)** and **(10)**, $(\square A^{(m,n)} + \square B^{(m,n)}) \cap (\square A^{(m,n)} \cdot \square B^{(m,n)})$

$$= \{< x, \min(\frac{(\mu_A(x))^m}{n} + \frac{(\mu_B(x))^m}{n} - (\frac{(\mu_A(x))^m}{n} \cdot \frac{(\mu_B(x))^m}{n}), (\frac{(\mu_A(x))^m}{n} \cdot \frac{(\mu_B(x))^m}{n})),$$
$$\max((1-\frac{(\mu_A(x))^m}{n}) \cdot (1-\frac{(\mu_B(x))^m}{n}), \left(1-\frac{(\mu_A(x))^m}{n}\right) +$$

$$\left(1-\frac{(\mu_B(x))^m}{n}\right)-\left(1-\frac{(\mu_A(x))^m}{n}\right)\cdot\left(1-\frac{(\mu_B(x))^m}{n}\right))>|\, x\in X\}. \quad \textbf{(12)}$$

Now with **@** of **(11)** and **(12)** and by using the results max(a, b) + min(a, b) = a + b , max(a, b) · min(a, b) = a · b , we have

$[(\Box A^{(m,n)}+\Box B^{(m,n)})\cup(\Box A^{(m,n)}\cdot\Box B^{(m,n)})]\ @$

$[(\Box A^{(m,n)}+\Box B^{(m,n)})\cap(\Box A^{(m,n)}\cdot\Box B^{(m,n)})]$

$$=\{<x, \tfrac{1}{2}[\frac{(\mu_A(x))^m+(\mu_B(x))^m}{n}], \tfrac{1}{2}[(1-\frac{(\mu_A(x))^m}{n})+(1-\frac{(\mu_B(x))^m}{n})]>|\, x\in X\}$$

$=\ \Box A^{(m,n)}\ @\ \Box B^{(m,n)}$. This proves **(i)**. Similarly the proof of **(ii)**.

Theorem 7

$[(\Box A^{(m,n)}\cdot\Box B^{(m,n)})\ @\ \overline{(\Box A^{(m,n)}\cdot\Box B^{(m,n)})}]\ @$

$[(\Box A^{(m,n)}+\Box B^{(m,n)})\ @\ \overline{(\Box A^{(m,n)}+\Box B^{(m,n)})}]$

$=[(\Diamond A^{(m,n)}\cdot\Diamond B^{(m,n)})\ @\ \overline{(\Diamond A^{(m,n)}\cdot\ \Diamond B^{(m,n)})}\ ]\ @$

$[(\Diamond A^{(m,n)}+\Diamond B^{(m,n)})\ @\ \overline{(\Diamond A^{(m,n)}+\Diamond B^{(m,n)})}]\ =\{<x,\frac{1}{2},\frac{1}{2}>|\, x\in X\}$

Proof: From **(10)** and **(9)** we have

$$\overline{\Box A^{(m,n)}\cdot\Box B^{(m,n)}}$$

$$=\{<x,\left(1-\frac{(\mu_A(x))^m}{n}\right)+\left(1-\frac{(\mu_B(x))^m}{n}\right)-\left(1-\frac{(\mu_A(x))^m}{n}\right)\cdot\left(1-\frac{(\mu_B(x))^m}{n}\right),$$

$$\left(\frac{(\mu_A(x))^m}{n}\cdot\frac{(\mu_B(x))^m}{n}\right)>|\, x\in X\} \quad \textbf{(13)}$$

$$\overline{\Box A^{(m,n)}+\Box B^{(m,n)}}=\{<x,\left(1-\frac{(\mu_A(x))^m}{n}\right)\cdot\left(1-\frac{(\mu_B(x))^m}{n}\right),$$

$$\left(\frac{(\mu_A(x))^m}{n}+\frac{(\mu_B(x))^m}{n}-\left(\frac{(\mu_A(x))^m}{n}\cdot\frac{(\mu_B(x))^m}{n}\right)>|\, x\in X\} \quad \textbf{(14)}$$

With **@** of **(10)** and **(13)**, $(\Box A^{(m,n)}\cdot\ \Box B^{(m,n)})\ @\ \overline{(\Box A^{(m,n)}\cdot\ \Box B^{(m,n)})}$

$$=\{x,\frac{\left(\frac{(\mu_A(x))^m}{n}\right)\cdot\left(\frac{(\mu_B(x))^m}{n}\right)+\left(1-\frac{(\mu_A(x))^m}{n}\right)+\left(1-\frac{(\mu_B(x))^m}{n}\right)-\left(1-\frac{(\mu_A(x))^m}{n}\right)\cdot\left(1-\frac{(\mu_B(x))^m}{n}\right)}{2},$$

$$\frac{\left(1-\frac{(\mu_A(x))^m}{n}\right)+\left(1-\frac{(\mu_B(x))^m}{n}\right)-\left(1-\frac{(\mu_A(x))^m}{n}\right)\cdot\left(1-\frac{(\mu_B(x))^m}{n}\right)}{2}>|\, x\in X\} \quad \textbf{(15)}$$

Also with **@** of **(9)** and **(14)**, we have $(\Box A^{(m,n)}+\ \Box B^{(m,n)})\ @\ \overline{(\Box A^{(m,n)}+\ \Box B^{(m,n)})}$

$$=\{<x,\frac{\left(\frac{(\mu_A(x))^m}{n}\right)+\left(\frac{(\mu_B(x))^m}{n}\right)-\left(\frac{(\mu_A(x))^m}{n}\right)\cdot\left(\frac{(\mu_B(x))^m}{n}\right)+\left(1-\frac{(\mu_A(x))^m}{n}\right)\cdot\left(1-\frac{(\mu_B(x))^m}{n}\right)}{2},$$

$$\frac{\left(1-\frac{(\mu_A(x))^m}{n}\right)\cdot\left(1-\frac{(\mu_B(x))^m}{n}\right)+\frac{(\mu_A(x))^m}{n}+\frac{(\mu_B(x))^m}{n}-\left(\frac{(\mu_A(x))^m}{n}\right)\cdot\left(\frac{(\mu_B(x))^m}{n}\right)}{2}>|\, x\in X\}\textbf{(16)}$$

Now with **@** of **(15)** and **(16)**, we have$[(\Box A^{(m,n)} \cdot \Box B^{(m,n)})$ @ $\overline{(\Box A^{(m,n)} \cdot \Box B^{(m,n)})}\,]$ @ $[(\Box A^{(m,n)} + \Box B^{(m,n)})$ @ $\overline{(\Box A^{(m,n)} + \Box B^{(m,n)})}\,]$ $= \{< x, \frac{1}{2}, \frac{1}{2} > | \, x \in X\}$. Similarly we have $[(\Diamond A^{(m,n)} \cdot \Diamond B^{(m,n)})$ @ $\overline{(\Diamond A^{(m,n)} \cdot \Diamond B^{(m,n)})}]$ @

$$[(\Diamond A^{(m,n)} + \Diamond B^{(m,n)}) \text{ @ } \overline{(\Diamond A^{(m,n)} + \Diamond B^{(m,n)})}] = \{< x, \tfrac{1}{2}, \tfrac{1}{2} > | \, x \in X\}.$$

This proves the theorem.

Theorem 8

i) If $A \subset_{\Diamond} B$ then $[\Diamond(A \cup B)^{(m,n)} + \Diamond(A \cap B)^{(m,n)}]$ @

$$[\Diamond(A \cup B)^{(m,n)} \cdot \Diamond(A \cap B)^{(m,n)}] = \Diamond A^{(m,n)} \text{ @ } \Diamond B^{(m,n)}$$

ii) If $A \subset_{\Box} B$ then $[\Box(A \cup B)^{(m,n)} + \Box(A \cap B)^{(m,n)}]$ @

$$[\Box(A \cup B)^{(m,n)} \cdot \Box(A \cap B)^{(m,n)}] = \Box A^{(m,n)} \text{ @ } \Box B^{(m,n)}$$

Proof: Let $A \subset_{\Diamond} B$ then $\nu_A(x) \geq \nu_B(x) \; \forall \, x \in X$. Therefore

$$\Diamond(A \cup B)^{(m,n)} = \{< x, \max\left(\frac{(1-\nu_A(x))^m}{n}, \frac{(1-\nu_B(x))^m}{n}\right), \min\left(1 - \frac{(1-\nu_A(x))^m}{n}, 1 - \frac{(1-\nu_B(x))^m}{n}\right) > | \, x \in X\} \quad \textbf{(17)}$$

$$= \{< x, \frac{(1-\nu_B(x))^m}{n}, \; 1 - \frac{(1-\nu_B(x))^m}{n} > | \, x \in X\} \text{ and}$$

$$\Diamond(A \cap B)^{(m,n)} = \{< x, \min\left(\frac{(1-\nu_A(x))^m}{n}, \frac{(1-\nu_B(x))^m}{n}\right), \max\left(1 - \frac{(1-\nu_A(x))^m}{n}, 1 - \frac{(1-\nu_B(x))^m}{n}\right) > | \, x \in X\} \quad \textbf{(18)}$$

$$= \{< x, \frac{(1-\nu_A(x))^m}{n}, 1 - \frac{(1-\nu_A(x))^m}{n} > | \, x \in X\}.$$

From **(17)**, **(18)** and by applying the results max(a, b) + min(a, b) = a + b and max(a, b) · min(a, b) = a · b we have $\Diamond(A \cup B)^{(m,n)} \cdot \Diamond(A \cap B)^{(m,n)}$

$$= \{< x, \frac{(1-\nu_A(x))^m}{n} \cdot \frac{(1-\nu_B(x))^m}{n}, 1 - \frac{(1-\nu_A(x))^m}{n} + 1 - \frac{(1-\nu_B(x))^m}{n} \left(1 - \frac{(1-\nu_A(x))^m}{n}\right) \cdot \left(1 - \frac{(1-\nu_B(x))^m}{n}\right) > | \, x \in X\} \quad \textbf{(19)}$$

and $\Diamond(A \cup B)^{(m,n)} + \Diamond(A \cap B)^{(m,n)}$

$$= \{< x, \frac{(1-\nu_A(x))^m}{n} + \frac{(1-\nu_B(x))^m}{n} - \frac{(1-\nu_A(x))^m}{n} \cdot \frac{(1-\nu_B(x))^m}{n}, \left(1 - \frac{(1-\nu_A(x))^m}{n}\right) \cdot \left(1 - \frac{(1-\nu_B(x))^m}{n}\right) > | \, x \in X\} \quad \textbf{(20)}$$

Now with **@** of **(20)** and **(19)**,

$[\Diamond(A \cup B)^{(m,n)} + \Diamond(A \cap B)^{(m,n)}]$ @ $[\Diamond(A \cup B)^{(m,n)} \cdot \Diamond(A \cap B)^{(m,n)}]$

$$= \{< x, \frac{1}{2}\left[\frac{(1-\nu_A(x))^m}{n} + \frac{(1-\nu_B(x))^m}{n}\right], \frac{1}{2}\left[1 - \frac{(1-\nu_A(x))^m}{n} + 1 - \frac{(1-\nu_B(x))^m}{n}\right] > | \, x \in X\}$$

$= \{< x, \frac{1-(\nu_A(x))^m}{n}, 1 - \frac{1-(\nu_A(x))^m}{n} > | \, x \in X\}$ @

$\{< x, \frac{1-(\nu_B(x))^m}{n}, 1 - \frac{1-(\nu_B(x))^m}{n} > | \, x \in X\} = \Diamond A^{(m,n)}$ @ $\Diamond B^{(m,n)}$. This proves **(i)**.

Similarly the proof of **(ii)**.

Theorem 9

i) $[(A \cup B)^{(m,n)}$ # $(A \cap B)^{(m,n)}]$ \$ $[(A \cup B)^{(m,n)}$ @ $(A \cap B)^{(m,n)}] = \Box A^{(m,n)}$ \$ $\Box B^{(m,n)}$

ii) $[\Box A^{(m,n)}$ # $\Box B^{(m,n)}]$ \$ $[\Box A^{(m,n)}$ @ $\Box B^{(m,n)}] = \Box A^{(m,n)}$ \$ $\Box B^{(m,n)}$

Proof: We have $(A \cup B)^{(m,n)} = \{< x, \max(\frac{(\mu_A(x))^m}{n}, \frac{(\mu_B(x))^m}{n}),$

$\min(1 - \frac{(1-\mu_A(x))^m}{n}, 1 - \frac{(1-\mu_B(x))^m}{n}) > | \, x \in X\}$ and

$(A \cap B)^{(m,n)} = \{< x, \min(\frac{(\mu_A(x))^m}{n}, \frac{(\mu_B(x))^m}{n}),$

$\max(1 - \frac{(1-\mu_A(x))^m}{n}, 1 - \frac{(1-\mu_B(x))^m}{n}) > | \, x \in X\}.$

Since max(a, b) + min(a, b) = a + b and max(a, b) · min(a, b) = a · b , we have

$$(A \cup B)^{(m,n)} \# (A \cap B)^{(m,n)} = \{< x, \frac{2(\frac{(\mu_A(x))^m}{n} \cdot \frac{(\mu_B(x))^m}{n})}{(\frac{(\mu_A(x))^m}{n} + \frac{(\mu_B(x))^m}{n})}, \frac{2(1 - \frac{(1-\mu_A(x))^m}{n}) \cdot (1 - \frac{(1-\mu_B(x))^m}{n})}{(1 - \frac{(1-\mu_A(x))^m}{n}) + (1 - \frac{(1-\mu_B(x))^m}{n})} > | \, x \in X\} \quad \textbf{(21)}$$

$$(A \cup B)^{(m,n)} @ (A \cap B)^{(m,n)} = \{< x, \frac{(\frac{(\mu_A(x))^m}{n} + \frac{(\mu_B(x))^m}{n})}{2}, \frac{(1 - \frac{(1-\mu_A(x))^m}{n} + 1 - \frac{(1-\mu_B(x))^m}{n})}{2} > | \, x \in X\} \quad \textbf{(22)}$$

Now with **\$** of **(21)** and **(22)**,

$[(A \cup B)^{(m,n)}$ # $(A \cap B)^{(m,n)}]$ \$ $[(A \cup B)^{(m,n)}$ @ $(A \cap B)^{(m,n)}]$

$= \{< x, \sqrt{\frac{(\mu_A(x))^m}{n} \cdot \frac{(\mu_B(x))^m}{n}}, \sqrt{(1 - \frac{(1-\mu_A(x))^m}{n}) \cdot (1 - \frac{(1-\mu_B(x))^m}{n})} > | \, x \in X\}$

$= \Box A^{(m,n)}$ \$ $\Box B^{(m,n)}$. This proves **(i)**.

Now with #, @ of **(3)** and **(4)** respectively, we have $\Box A^{(m,n)}$ # $\Box B^{(m,n)}$

$$= \{< x, \frac{2(\frac{(\mu_A(x))^m}{n} \cdot \frac{(\mu_B(x))^m}{n})}{(\frac{(\mu_A(x))^m}{n} + \frac{(\mu_B(x))^m}{n})}, \frac{2(1 - \frac{(\mu_A(x))^m}{n}) \cdot (1 - \frac{(\mu_B(x))^m}{n})}{(1 - \frac{(\mu_A(x))^m}{n}) + (1 - \frac{(\mu_B(x))^m}{n})} > | \, x \in X\} \quad \textbf{(23)}$$

$$\Box A^{(m,n)} @ \Box B^{(m,n)} = \{< x, \frac{1}{2}(\frac{(\mu_A(x))^m + (\mu_B(x))^m}{n}), \frac{1}{2}[(1 - \frac{\mu_A(x))^m}{n}) + (1 - \frac{\mu_B(x))^m}{n})] > | \, x \in X\} \quad \textbf{(24)}$$

Now with **\$** of **(23)** and **(24)**, $[\Box A^{(m,n)}$ # $\Box B^{(m,n)}]$ \$ $[\Box A^{(m,n)}$ @ $\Box B^{(m,n)}]$

$= \{< x, \sqrt{\frac{(\mu_A(x))^m}{n} \cdot \frac{(\mu_B(x))^m}{n}}, \sqrt{(1 - \frac{(\mu_A(x))^m}{n}) \cdot (1 - \frac{(\mu_B(x))^m}{n})} > | \, x \in X\}$

$= \{< x, \frac{(\mu_A(x))^m}{n}, 1 - \frac{(\mu_A(x))^m}{n} > | \, x \in X\} \$ \{< x, \frac{(\mu_B(x))^m}{n}, 1 - \frac{(\mu_B(x))^m}{n} > | \, x \in X\}$

$= \Box A^{(m,n)} \$ \Box B^{(m,n)}$. This proves **(ii)**.

Theorem 10

$$\Box(A \cup B)^{(m,n)} \rightarrow \Box(A \cap B)^{(m,n)} = \begin{cases} \Box A^{(m,n)} \rightarrow \Box B^{(m,n)} \text{ if } B \subset_{\Box} A \\ \Box B^{(m,n)} \rightarrow \Box A^{(m,n)} \text{ if } A \subset_{\Box} B \end{cases}$$

Proof: case (i) Let $B \subset_{\Box} A$ then $\mu_B(x) \leq \mu_A(x)$ $\forall x \in X$. Therefore $\frac{(\mu_B(x))^m}{n} \leq \frac{(\mu_A(x))^m}{n}$, $1 - \frac{(\mu_A(x))^m}{n} \leq 1 - \frac{(\mu_B(x))^m}{n}$ and

$\Box(A \cup B)^{(m,n)} = \{< x, \max(\frac{(\mu_A(x))^m}{n}, \frac{(\mu_B(x))^m}{n}), 1 - \max(\frac{(\mu_A(x))^m}{n}, \frac{(\mu_B(x))^m}{n}) > | \, x \in X\}$

$= \{< x, \max(\frac{(\mu_A(x))^m}{n}, \frac{(\mu_B(x))^m}{n}), \min(1 - \frac{(\mu_A(x))^m}{n}, 1 - \frac{(\mu_B(x))^m}{n}) > | \, x \in X\}$ **(25)**

$= \{< x, \frac{(\mu_A(x))^m}{n}, 1 - \frac{(\mu_A(x))^m}{n} > | \, x \in X\} = \Box A^{(m,n)}$. Similarly

$\Box(A \cap B)^{(m,n)} = \{< x, \min(\frac{(\mu_A(x))^m}{n}, \frac{(\mu_B(x))^m}{n}), 1 - \min(\frac{(\mu_A(x))^m}{n}, \frac{(\mu_B(x))^m}{n}) > | \, x \in X\}$

$= \{< x, \min(\frac{(\mu_A(x))^m}{n}, \frac{(\mu_B(x))^m}{n}), \max(1 - \frac{(\mu_A(x))^m}{n}, 1 - \frac{(\mu_B(x))^m}{n}) > | \, x \in X\}$ **(26)**

$= \{< x, \frac{(\mu_B(x))^m}{n}, 1 - \frac{(\mu_B(x))^m}{n} > | \, x \in X\} = \Box B^{(m,n)}$.

Therefore $\Box(A \cup B)^{(m,n)} \rightarrow \Box(A \cap B)^{(m,n)} = \Box A^{(m,n)} \rightarrow \Box B^{(m,n)}$.

Proof of **case (ii)** is similar to that of **case (i)**.

Theorem 11

If $B \subset_{\Box} A$ then $\left(\Box(A \cup B)^{(m,n)} \rightarrow \Box(A \cap B)^{(m,n)}\right) \cdot \overline{(\Box(A \cap B)^{(m,n)} \rightarrow \Box(A \cup B)^{(m,n)})}$ @ $(\Box(A \cup B)^{(m,n)} \rightarrow \Box(A \cap B)^{(m,n)}) + \overline{(\Box(A \cap B)^{(m,n)} \rightarrow \Box(A \cup B)^{(m,n)})}$

$= \overline{\Box A^{(m,n)}}$ @ $\Box B^{(m,n)}$

Proof: Let $B \subset_{\Box} A$ then $\mu_B(x) \leq \mu_A(x)$ $\forall x \in X$. From **(25)** and **(26)** we have

$\Box(A \cup B)^{(m,n)} = \{< x, \frac{(\mu_A(x))^m}{n}, 1 - \frac{(\mu_A(x))^m}{n} > | \, x \in X\}$

and $\Box(A \cap B)^{(m,n)} = \{< x, \frac{(\mu_B(x))^m}{n}, 1 - \frac{(\mu_B(x))^m}{n} > | \, x \in X\}$.

Therefore $\Box(A \cup B)^{(m,n)} \rightarrow \Box(A \cap B)^{(m,n)}$

$= \{< x, \max(1 - \frac{(\mu_A(x))^m}{n}, \frac{(\mu_B(x))^m}{n}), \min(\frac{(\mu_A(x))^m}{n}, 1 - \frac{(\mu_B(x))^m}{n}) > | \, x \in X\}$ **(27)**

and $\Box(A \cap B)^{(m,n)} \rightarrow \Box(A \cup B)^{(m,n)}$

$= \{< x, \max(1 - \frac{(\mu_B(x))^m}{n}, \frac{(\mu_A(x))^m}{n}), \min(\frac{(\mu_B(x))^m}{n}, 1 - \frac{(\mu_A(x))^m}{n}) > | \, x \in X\}$ **(28)**

From **(28)**, we have $\overline{\Box(A \cap B)^{(m,n)} \rightarrow \Box(A \cup B)^{(m,n)}}$

$= \{< x, \min(1 - \frac{(\mu_A(x))^m}{n}, \frac{(\mu_B(x))^m}{n}), \max(\frac{(\mu_A(x))^m}{n}, 1 - \frac{(\mu_B(x))^m}{n}) > | x \in X\}$ **(29)**

Now with · of **(27)** and **(29)** and by applying max(a, b) + min(a, b) = a + b and max(a, b) · min(a, b) = a · b ,

we have $\left(\Box(A \cup B)^{(m,n)} \rightarrow \Box(A \cap B)^{(m,n)}\right) \cdot \overline{(\Box(A \cap B)^{(m,n)} \rightarrow \Box(A \cup B)^{(m,n)})}$

$= \{< x, (1 - \frac{(\mu_A(x))^m}{n}) \cdot (\frac{(\mu_B(x))^m}{n}),$

$\frac{(\mu_A(x))^m}{n} + (1 - \frac{(\mu_B(x))^m}{n}) - (\frac{(\mu_A(x))^m}{n}) \cdot (1 - \frac{(\mu_B(x))^m}{n}) > | x \in X\}$ **(30)**

Now with + of **(27)** and **(29)**, we have

$[\Box(A \cup B)^{(m,n)} \rightarrow \Box(A \cap B)^{(m,n)}] + \overline{[\Box(A \cup B)^{(m,n)} \rightarrow \Box(A \cap B)^{(m,n)}]}$

$= \{< x, (1 - \frac{(\mu_A(x))^m}{n}) + \frac{(\mu_B(x))^m}{n} - \left(1 - \frac{(\mu_A(x))^m}{n}\right) \cdot (\frac{(\mu_B(x))^m}{n}),$

$(\frac{(\mu_A(x))^m}{n}) \cdot (1 - \frac{(\mu_B(x))^m}{n}) > | x \in X\}$ **(31)**

Now with **@** of **(30)** and **(31)**, L.H.S of the above theorem

$= \{< x, \frac{\left(1 - \frac{(\mu_A(x))^m}{n}\right) + (\frac{(\mu_B(x))^m}{n})}{2}, \frac{\left(1 - \frac{(\mu_B(x))^m}{n}\right) + (\frac{(\mu_A(x))^m}{n})}{2} > | x \in X\}$

$= \{< x, 1 - \frac{(\mu_A(x))^m}{n}, \frac{(\mu_A(x))^m}{n} > | x \in X\}$ @

$\{< x, \frac{(\mu_B(x))^m}{n}, 1 - \frac{(\mu_B(x))^m}{n} > | x \in X\} = \overline{\Box A^{(m,n)}}$ @ $\Box B^{(m,n)}$

= R.H.S of the theorem.

Theorem 12. If A $\subset_\Diamond$ B then

i) $\left(\Diamond(A \cup B)^{(m,n)} \rightarrow \Diamond(A \cap B)^{(m,n)}\right) \cdot \overline{(\Diamond(A \cap B)^{(m,n)} \rightarrow \Diamond(A \cup B)^{(m,n)})}$

$= \Diamond A^{(m,n)} \cdot \overline{\Diamond B^{(m,n)}}$

ii) $(\Diamond(A \cap B)^{(m,n)} \rightarrow \Diamond(A \cup B)^{(m,n)}) \cdot \overline{(\Diamond(A \cup B)^{(m,n)} \rightarrow \Diamond(A \cap B)^{(m,n)})}$

$= \overline{\Diamond A^{(m,n)}} \cdot \Diamond B^{(m,n)}$

Proof: Let A $\subset_\Diamond$ B then $\nu_A(x) \geq \nu_B(x)\ \forall\ x \in X$. From **(17)** and **(18)**, we have

$\Diamond(A \cup B)^{(m,n)} = \{< x, \frac{(1 - \nu_B(x))^m}{n}, 1 - \frac{(1 - \nu_B(x))^m}{n} > | x \in X\},$

$\Diamond(A \cap B)^{(m,n)} = \{< x, \frac{(1 - \nu_A(x))^m}{n}, 1 - \frac{(1 - \nu_A(x))^m}{n} > | x \in X\}.$

Therefore $\Diamond(A \cup B)^{(m,n)} \rightarrow \Diamond(A \cap B)^{(m,n)}$

$= \{< x, \max(1 - \frac{(1 - \nu_B(x))^m}{n}, \frac{(1 - \nu_A(x))^m}{n}), \min(\frac{(1 - \nu_B(x))^m}{n}, 1 - \frac{1 - \nu_A(x))^m}{n}) > | x \in X\}$

$\Diamond(A \cap B)^{(m,n)} \rightarrow \Diamond(A \cup B)^{(m,n)}$

$= \{< x, \max(1 - \frac{(1 - \nu_A(x))^m}{n}, \frac{(1 - \nu_B(x))^m}{n}), \min(\frac{(1 - \nu_A(x))^m}{n}, 1 - \frac{1 - \nu_B(x))^m}{n}) > | x \in X\}$

and $\overline{\Diamond(A \cap B)^{(m,n)} \rightarrow \Diamond(A \cup B)^{(m,n)}}$

$= \{< x, \min(1 - \frac{(1-\nu_B(x))^m}{n}, \frac{(1-\nu_A(x))^m}{n}), \max(\frac{(1-\nu_B(x))^m}{n}, 1 - \frac{1-\nu_A(x))^m}{n}) > | \, x \in X\}$

From the above three equations we have

$\left(\Diamond(A \cup B)^{(m,n)} \to \Diamond(A \cap B)^{(m,n)}\right) \cdot \overline{(\Diamond(A \cap B)^{(m,n)} \to \Diamond(A \cup B)^{(m,n)})}$

$= \{< x, \left(\frac{(1-\nu_A(x))^m}{n}\right) \cdot \left(1 - \frac{(1-\nu_B(x))^m}{n}\right),$

$(\frac{1-\nu_B(x))^m}{n}) + (1 - \frac{1-\nu_A(x))^m}{n}) - (\frac{1-\nu_B(x))^m}{n}) \cdot (1 - \frac{1-\nu_A(x))^m}{n}) > | \, x \in X\}$

$= \Diamond A^{(m,n)} \cdot \overline{\Diamond B^{(m,n)}}$. This proves **(i)**. Proof of **(ii)** is similar to that of **(i)**

4 APPLICATION ON DECISION MAKING

Let S = { s_1, s_2, s_3, s_4, s_5 } be the set of students, C = {Medical, Pharmacy, Biotechnology, Agriculture, B.Sc Degree, Computer field, Charted Accountant } be the set of courses and Su = {Tamil, English, Maths, Biology, Physics, Chemistry} be the subjects related to the selection of courses. For the purpose of selecting their future course, we assume the above students have written the examination in all the above subjects. The following table shows the courses and related subjects.

Table 1: Subject talents related to the courses

R	MEDICAL	PHARMACY	BIO-TECHNOLOGY	AGRICULTURE	B.Sc	COMPUTER FIELD	C.A
TAM	(0.7,0.3,0.0)	(0.7,0.0,0.3)	(0.6,0.3,0.1)	(0.7,0.1,0.2)	(0.8,0.2,0.0)	(0.7,0.2,0.1)	(0.6,0.2,0.2)
ENG	(0.6,0.4,0.0)	(0.8,0.2,0.0)	(0.7,0.2,0.0)	(0.6,0.3,0.1)	(0.6,0.3,0.1)	(0.6,0.3,0.1)	(0.7,0.1,0.2)
MAT	(0.7,0.3,0.0)	(0.7,0.2,0.2)	(0.6,0.4,0.0)	(0.7,0.1,0.2)	(0.5,0.3,0.2)	(0.6,0.3,0.1)	(0.8,0.1,0.1)
BIO	(0.6,0.3,0.1)	(0.7,0.1,0.2)	(0.7,0.1,0.2)	(0.8,0.2,0.0)	(0.7,0.2,0.1)	(0.7,0.2,0.1)	(0.7,0.2,0.1)
PHY	(0.7,0.1,0.2)	(0.6,0.3,0.1)	(0.8,0.1,0.1)	(0.7,0.1,0.2)	(0.7,0.1,0.2)	(0.6,0.2,0.2)	(0.6,0.1,0.3)
CHE	(0.7,0.2,0.1)	(0.5,0.3,0.2)	(0.6,0.3,0.1)	(0.8,0.2,0.0)	(0.7,0.2,0.1)	(0.8,0.1,0.1)	(0.7,0.1,0.2)

TAM-Tamil, ENG-English, MAT-Maths, BIO-Biology, PHY-Physics, CHE-Chemistry

Table 2: The score of the students in the examinations

Q	*TAMIL*	*ENGLISH*	*MATHS*	*BIOLOGY*	*PHYSICS*	*CHEMISTRY*
s_1	(0.8,0.1,0.1)	(0.6,0.1,0.3)	(0.4,0.6,0.0)	(0.6,0.1,0.3)	(0.7,0.2,0.0)	(0.7,0.2,0.1)
s_2	(0.8,0.2,0.0)	(0.7,0.2,0.1)	(0.6,0.4,0.0)	(0.7,0.2,0.1)	(0.8,0.0,0.2)	(0.7,0.1,0.2)
s_3	(0.8,0.1,0.1)	(0.8,0.0,0.2)	(0.7,0.1,0.2)	(0.7,0.1,0.2)	(0.8,0.1,0.1)	(0.8,0.2,0.0)
s_4	(0.9,0.1,0.1)	(0.8,0.1,0.1)	(0.7,0.1,0.2)	(0.7,0.0,0.3)	(0.8,0.2,0.0)	(0.8,0.1,0.1)
s_5	(0.8,0.2,0.0)	(0.7,0.1,0.2)	(0.8,0.1,0.1)	(0.7,0.1,0.1)	(0.7,0.1,0.2)	(0.8,0.1,0.1)

Table 3: Consider the composition T= R ∘ Q. It gives the data in which the students s_i interms of the subject performance fit of the courses c_j ∀ i = 1 to 5, j = 1 to 7. The composition T = R ∘ Q is defined by $\mu_T(s_i, c_j)$ = $\max\{\min[\mu_Q(s_i, su), \mu_R(su, c_j)]\}$ and $\nu_T(s_i, c_j) = \min\{\max[\mu_Q(s_i, su), \mu_R(su, c_j)]\}$. By using this composition we get the following table.

Data corresponding to the students in terms of the course perform

T	MEDICAL	PHARMACY	BIO-TECH NOLOGY	AGRI CULTURE	B.Sc	COMPUTER FIELD	C.A
s_1	(0.7,0.2,0.0)	(0.7,0.1,0.2)	(0.7,0.1,0.2)	(0.7,0.1,0.2)	(0.8,0.2,0.0)	(0.7,0.1,0.2)	(0.7,0.1,0.2)
s_2	(0.7,0.2,0.1)	(0.7,0.2,0.1)	(0.8,0.1,0.1)	(0.7,0.1,0.2)	(0.8,0.1,0.1)	(0.7,0.1,0.2)	(0.7,0.1,0.2)
s_3	(0.7,0.1,0.2)	(0.8,0.1,0.1)	(0.8,0.1,0.1)	(0.8,0.1,0.1)	(0.8,0.1,0.1)	(0.8,0.2,0.0)	(0.7,0.1,0.2)
s_4	(0.7,0.2,0.1)	(0.8,0.1,0.1)	(0.8,0.1,0.1)	(0.8,0.1,0.1)	(0.8,0.2,0.0)	(0.8,0.1,0.1)	(0.7,0.1,0.2)
s_5	(0.7,0.1,0.2)	(0.7,0.2,0.1)	(0.7,0.1,0.2)	(0.8,0.1,0.1)	(0.8,0.1,0.1)	(0.8,0.1,0.1)	(0.8,0.1,0.1)

Table 4. Decision making by using the operator $\Diamond A^{(m,n)} \rightarrow \Box A^{(m,n)}$

By definition **(xii)** of section **2.1** we have $\Diamond A^{(5,7)} \rightarrow \Box A^{(5,7)}$

$= \{< x, \max(1 - \frac{1-(\nu_A(x))^5}{7}, \frac{\mu_A(x))^5}{7}), \min(1 - \frac{(\mu_A(x))^5}{7}, \frac{1-(\nu_A(x))^5}{7}) > | x \in X\}$.

Applying this operator to each entry of **table 3** we get the following table corresponding to $\Diamond A^{(5,7)} \rightarrow \Box A^{(5,7)}$

T	MEDICAL	PHARMACY	BIO-TECHNO LOGY	AGRICULTURE	B.Sc	COMPUTER FIELD	C.A
s_1	.953188571 .046811429	.915644286 .084355714	.915644286 .084355714	.915644286 .084355714	.953188571 .046811429	.915644286 .084355714	.915644286 .084355714
s_2	.953188571 .046811429	.953188571 .046811429	.915644286 .084355714	.915644286 .084355714	.915644286 .084355714	.915644286 .084355714	.915644286 .084355714
s_3	.915644286 .084355714	.915644286 .084355714	.915644286 .084355714	.915644286 .084355714	.915644286 .084355714	.953188571 .046811429	.915644286 .084355714
s_4	.953188571 .046811429	.915644286 .084355714	.915644286 .084355714	.915644286 .084355714	.953188571 .046811429	.915644286 .084355714	.915644286 .084355714
s_5	.915644286 .084355714	.953188571 .046811429	.915644286 .084355714	.915644286 .084355714	.915644286 .084355714	.915644286 .084355714	.915644286 .084355714

From the above table we infer $\mathbf{s_1}$ is suitable to study Medical or B.Sc, $\mathbf{s_2}$ is suitable to study Medical or Pharmacy, $\mathbf{s_3}$ is suitable to study Computer field, $\mathbf{s_4}$ is suitable to study Medical or B.Sc and $\mathbf{s_5}$ is suitable to study pharmacy.

CONCLUSION

In this paper more realistic intuitionistic fuzzy set operator $A^{(m,n)}$ has been introduced and some equalities are proved. An example is given to verify the developed approach and to demonstate its practicality and effectiveness. Based on the calculations of the example, a suitable course is selected to study. From the above application we conclude that the introduced operator $A^{(m,n)}$ is reliable and valuable tool for making decision rather than distance method and similarity measure method.

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