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Equivalent Conditions for Irreducibility of Prime Spectrum of a Ring

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Abstract

In this paper, it is proved that the equivalent conditions for prime spectrum of a ring is irreducible.

Keywords: prime ideal, prime spectrum, dense set, irreducible.

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1. INTRODUCTION:

Let R be a commutative ring with identity and X be the set of all prime ideals of R. For any $A \subseteq R$, let $X(A) = \{P \in X \mid A \not\subseteq P\}$. It is clear that $\{X(A) \mid A \subseteq R\}$ forms a topology on X, for which $\{X(a) \mid a \in R\}$ is a base ([1], [2] and[5]). The set of all prime ideals of a commutative ring R with identity together with this topology is called the prime spectrum of a ring R and is denoted by $Spec\,R$ ([4], [6]). A. V. S. N. Murty etc., all ([7] and [8]) studied the properties of Prime spectrum of a ring and some homeomorphic theorems on Prime spectrum of a ring. Also A. V. S. N. Murty etc., all [9] characterized certain ring theoretic properties of R by the corresponding topological properties of $Spec\,R$. For example, a semi prime ring R is regular if and only if $Spec\,R$ is a Hausdorff space. In this paper, it is studied about the equivalent conditions for prime spectrum of a ring is irreducible. Throughout this paper we consider only commutative rings with identity and hence we preferred to call these as rings for simplicity.

2. PRELIMINARIES:

Definition 2.1: An ideal P of a ring R is called prime if

(1) $P \neq R$

(2) For any $a, b \in R$, $ab \in P$ implies either $a \in P$ or $b \in P$

Definition 2.2: Let R be a ring. An element $a \in R$ is said to be nilpotent if there exist a positive integer n such that $a^n = 0$.

Definition 2.3: An ideal I of a ring R is said to be nil ideal of R if each element of I is nilpotent.

Theorem 2.4: The set N of all nilpotent elements of a ring R is an ideal and R/N has no nilpotent element other than zero element in R/N. Also N coincides with the intersection of all prime ideals of R.

Here N is called the nil radical of R and is denoted by $N(R)$.

Definition 2.5: A ring R is called semi prime if $N(R) = \{0\}$.

Definition 2.6: If I, J are any two ideals of a ring R then their ideal quotient is defined as $(I : J) = \{x \in R \mid xJ \subseteq I\}$ which is an ideal. For any ideal I and an element $a \in R$, we write $(I : a)$ for $(I : <a>)$ and $<a>^*$ for $(0 : a)$; i.e., $<a>^* = \{x \in R \mid xa = 0\}$. $<a>^*$ is called the annihilator of a.

Definition 2.6: A subset A of a topological space X is called dense in X if the closure $\overline{\mathrm{A}}$ of A is X; i.e., $\overline{\mathrm{A}} = X$.

Definition 2.7: A topological space X is said to be irreducible if any non-empty open set is dense in X.

3. MAIN THEOREM:

Theorem 3.1: For any ring R, the following are equivalent:

1) $Spec\, R$ is irreducible
2) $N(R)$ is a prime ideal
3) For any ideal $I \not\subseteq N(R)$, $< N(R):I > = N(R)$

Proof: $(1) \Rightarrow (2)$: Clearly $N(R)$ is a proper ideal of R. Let $a, b \in R.$ If $ab \in R,$ then $X(A) \cap X(B) = \varnothing$ and hence $X(A) = \varnothing$ or $X(B) = \varnothing.$ Therefore, $a \in N(R)$ or $b \in N(R).$ Thus $N(R)$ is a prime ideal.

$(2) \Rightarrow (3)$: Let I be an ideal such that $I \subseteq N(R).$ Then, clearly $N(R) \subseteq \langle N(R):I \rangle$.

Note that $I \cap \langle N(R):I \rangle \subseteq N(R).$ Since $N(R)$ is prime and $I \not\subseteq N(R)$, we have $\langle N(R):I \rangle \subseteq N(R).$ Thus $\langle N(R):I \rangle = N(R).$

$(3) \Rightarrow (1)$: Let $X(I)$ be a non-empty set in $Spec\, R$. Then, $I \not\subseteq N(R)$ and hence $\langle N(R):I \rangle \subseteq N(R).$ Now, $\overline{X(I)} = H(K(X(I))) = H\langle N(R):I \rangle = H(N(R)) = X.$

Thus $Spec\, R$ is irreducible.

CONCLUSION

In this paper, it is studied that the equivalent conditions for prime spectrum of a ring is irreducible. These conditions may be studied for **fuzzy prime spectrum of a ring.**

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