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Bipolar fuzzy near algebra over Bipolar fuzzy field

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Abstract

The notion of bipolar fuzzy field is introduced on the basis of the theory of bipolar fuzzy set. The main contribution of this work is that, the notion of bipolar fuzzy near-algebra over bipolar fuzzy field is initiated. Furthermore, pertaining to these notions some of important fundamental concepts with the illustrative examples are gained. Interestingly, this contribution opens up many enlightening theorems and propositions for future research in the field of bipolar fuzzy near-algebra.

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1. Introduction

Brown [2], Irish [4] and Srinivas [9] have discussed certain properties of a near-algebra. The concept of fuzzy set is introduced by L.A. Zadeh [11]. Narasimha Swamy [8] introduced the concept of Fuzzy-near-algebra over a fuzzy field.

Bipolar-valued fuzzy set is a fuzzy set whose membership degree value is in the interval $[-1, 1]$. K.M. Lee [5], [6] introduced the notion of bipolar valued fuzzy set. Different disciplines such as algebraic structure, medical science, graph theory, decision making, machine theory and so on, [7] adopted the idea of bipolar fuzzy set as vigorous area of research. Hyoung Gu Baik [1] introduced the notion of a bipolar fuzzy ideals in near-ring.

2. Preliminaries

Let X be the universe of discourse. A bipolar-valued fuzzy set ϕ in X is an object having the form $\phi = \{(x, \phi^+(x), \phi^-(x)) \mid x \in X\}$, where $\phi^+ : X \to [0, 1]$ and $\phi^- : X \to [-1, 0]$ are mappings. The positive membership degree $\phi^+(x)$ denotes the satisfaction degree of an element x to the property corresponding to a bipolar-valued fuzzy set $\phi = \{(x, \phi^+(x), \phi^-(x)) \mid x \in X\}$ and the negative membership degree $\phi^-(x)$ denotes the satisfaction degree of x to some implicit counter property of $\phi = \{(x, \phi^+(x), \phi^-(x)) \mid x \in X\}$.

For a bipolar fuzzy set $\phi = (X, \phi^+, \phi^-)$ and $(s, t) \in [-1, 0] \times [0, 1]$, we define

$$\phi_t^P = \{x \in X \mid \phi^+(x) \geq t\}, \phi_s^N = \{x \in X \mid \phi^-(x) \leq s\}$$

which are called the positive t-cut of $\phi = (X; \phi^+, \phi^-)$ and the negative s-cut of $\phi = (X, \phi^+, \phi^-)$ respectively. For $(s, t) \in [-1, 0] \times [0, 1]$ the set $\phi_{(t,s)} = \phi_t^P \cap \phi_s^N$ is called (t, s)-cut of $\phi = (X; \phi^+, \phi^-)$.

Definition 2.1. [3] A fuzzy subset F of a field X is called a *fuzzy field* of X if it satisfies the following four conditions for every $x, y \in X$:

(i) $F(x + y) \geq \min(F(x), F(y)) = F(x) \wedge F(y)$,

(ii) $F(-x) \geq F(x)$,

(iii) $F(xy) \geq \min(F(x), F(y)) = F(x) \wedge F(y)$,

(iv) $F(x^{-1}) \geq F(x)$ for every $x(\neq 0) \in X$. A fuzzy field F of X is denoted by (F, X).

Definition 2.2. [4] A *right near-algebra* Y over a field X is a linear space Y over X on which a multiplication is defined such that

(i) Y forms a semigroup under multiplication,

(ii) multiplication is right distributive over addition (i.e. $(a + b)c = ac + bc$ for every $a, b, c \in Y$) and

(iii) $\lambda(ab) = (\lambda a)b$ for every $a, b \in Y$ and $\lambda \in X$.

Definition 2.3. [4] A near-algebra Y is said to be a *zero symmetric near-algebra* or *near-c-algebra* if $n \cdot 0 = 0$ for every $n \in Y$, where 0 is the additive identity in Y.

Definition 2.4. A subset W of a near-algebra Y over a field X is said to be a *sub near-algebra* of Y if it satisfies the following four conditions:

(i) W is a linear subspace of Y,

(ii) $(W, \cdot)$ is a semigroup,

(iii) $(x + y)z = xz + yz$,

(iv) $\lambda(xy) = (\lambda x)y$ for every $x, y, z \in W, \lambda \in X$.

Definition 2.5. Let Y and Y' be two near-algebras over a field X. A mapping $f : Y \to Y'$ is called a near-algebra homomorphism if

(i) $f(x + y) = f(x) + f(y)$,

(ii) $f(\lambda x) = \lambda f(x)$ and

(iii) $f(xy) = f(x)f(y)$ for every $x, y \in Y$ and $\lambda \in X$.

A homomorphism which is one-one is called a monomorphism. A monomorphism which is onto is called an isomorphism.

Throughout this article X denotes a field and Y denotes a (right) near-algebra over a field X.

3. Bipolar fuzzy field

Definition 3.1. Let X be a field. A bipolar fuzzy set $\psi = (X; \psi^+, \psi^-)$ in X is called a bipolar fuzzy field of X if for all $x, y \in X$

(i) $\psi^+(x + y) \geq \min\{\psi^+(x), \psi^+(y)\},\ \psi^-(x + y) \leq \max\{\psi^-(x), \psi^-(y)\}$

(ii) $\psi^+(-x) \geq \psi^+(x),\ \psi^-(-x) \leq \psi^-(x)$

(iii) $\psi^+(xy) \geq \min\{\psi^+(x), \psi^+(y)\},\ \psi^-(xy) \leq \max\{\psi^-(x), \psi^-(y)\}$

(iv) $\psi^+(x^{-1}) \geq \psi^+(x),\ \psi^-(x^{-1}) \leq \psi^-(x)$ for $x(\neq 0) \in X$.

Proposition 3.2. Let $\psi = (X; \psi^+, \psi^-)$ be a bipolar fuzzy field of X. Then

(i) $\psi^+(0) \geq \psi^+(x)$ and $\psi^-(0) \leq \psi^-(x)\ \forall x \in X$.

(ii) $\psi^+(1) \geq \psi^+(x)$ and $\psi^-(1) \leq \psi^-(x)\ \forall x(\neq 0) \in X$.

(iii) $\psi^+(0) \geq \psi^+(1)$ and $\psi^-(0) \leq \psi^-(1)$.

Proposition 3.3. Let $\psi = (X; \psi^+, \psi^-)$ be a bipolar fuzzy field of X. Then $\psi^+(0) \geq \psi^+(1) \geq \psi^+(x)$ and $\psi^-(0) \leq \psi^-(1) \leq \psi^-(x) \forall\, x(\neq 0) \in X$.

Proposition 3.4. $\psi = (X; \psi^+, \psi^-)$ is a bipolar fuzzy field of X if and only if the following conditions hold

(i) for every $x, y \in X$, $\psi^+(x - y) \geq \psi^+(x) \wedge \psi^+(y))$ and $\psi^-(x - y) \leq \psi^-(x) \vee \psi^-(y)$

(ii) for every $x, y(\neq 0) \in X$, $\psi^+(xy^{-1}) \geq \psi^+(x) \wedge \psi^+(y)$ and $\psi^-(xy^{-1}) \leq \psi^-(x) \vee \psi^-(y)$.

Theorem 3.5. Let $\psi = (X; \psi^+, \psi^-)$ be a bipolar fuzzy set of a field X. Then $\psi = (X; \psi^+, \psi^-)$ is a bipolar fuzzy field of X if and only if it satisfies the following two statements;

(i) $\forall\, t \in [0, 1]$, $\psi_t^P \neq \phi \Rightarrow \psi_t^P$ is a subfield of X.

(ii) $\forall\, s \in [-1, 0]$, $\psi_s^N \neq \phi \Rightarrow \psi_s^N$ is a subfield of X.

Proof. Suppose that $\psi = (X; \psi^+, \psi^-)$ is a bipolar fuzzy field of X. We have

$$\psi_t^P = \{x \in X \mid \psi^+(x) \geq t\},\ \psi_s^N = \{x \in X \mid \psi^-(x) \leq s\}.$$

Let $(s, t) \in [-1, 0] \times [0, 1]$ be such that $\psi_t^P \neq \phi$ and $\psi_s^N \neq \phi$.

(i) Let $x, y \in \psi_t^P$, then $x, y \in X$, $\psi^+(x) \geq t$, $\psi^+(y) \geq t$.
Since X is a field and $x, y \in X \Rightarrow x - y \in X$and $xy^{-1} \in X$ for $y \neq 0$.

$$\begin{aligned} \text{Also}\psi^+(x - y) &\geq \psi^+(x) \wedge \psi^+(y) \\ &\geq t \wedge t = t \\ \text{and for } y \neq 0\psi^+(xy^{-1}) &\geq \psi^+(x) \wedge \psi^+(y) \\ &\geq t \wedge t = t. \end{aligned}$$

Thus $x - y \in \psi_t^P$ and $xy^{-1} \in \psi_t^P$ for every $y(\neq 0) \in X$. Hence ψ_t^P is a subfield of X.

(ii) Let $u, v \in \psi_s^N$. Then $u, v \in X$ and $\psi^-(u) \leq s,\ \psi^-(v) \leq s$.
Since X is a field, then $u, v \in X \Rightarrow u - v \in X$ and $uv^{-1} \in X$ for $v(\neq 0) in X$.

$$\begin{aligned} \text{Also}\psi^-(u - v) &\leq \psi^-(u) \vee \psi^-(v) \\ &\leq s \vee s = s. \\ \text{and for } v \neq 0\psi^-(uv^{-1}) &\leq \psi^-(u) \vee \psi^-(v) \\ &\leq s \vee s = s \end{aligned}$$

Thus $u - v \in \psi_s^N$ and $uv^{-1} \in \psi_s^N$ for $y(\neq 0) \in X$. Hence ψ_s^N is a subfield of X.
Conversely, suppose that (i) and (ii) of the hypothesis holds.
We have to prove that $\psi = (X; \psi^+, \psi^-)$ is a bipolar fuzzy field of X.
If possible suppose that there exists $u, v \in X$ such that $\psi^+(u - v) < \psi^+(u) \wedge \psi^+(v)$ and $\psi^-(u - v) > \psi^-(u) \vee \psi^-(v)$.
Put $u_0 = \frac{1}{2}(\psi^+(u - v) + (\psi^+(u) \wedge \psi^+(v))$
and $v_0 = \frac{1}{2}(\psi^-(u - v) + (\psi^-(u) \vee \psi^-(v))$
Then we have

$$\psi^+(u - v) < u_0 < \psi^+(u) \wedge \psi^+(v)$$

and

$$\psi^-(u - v) > v_0 > \psi^-(u) \vee \psi^-(v).$$

$$\Rightarrow \psi^+(u - v) < u_0, \psi^+(u) \wedge \psi^+(v) > u_0$$

and

$$\psi^-(u - v) > v_0, \psi^-(u) \vee \psi^-(v) < v_0$$

i.e. $u, v \in X \Rightarrow u - v \in X$ and $\psi^+(u - v) < u_0, \psi^+(u) > u_0, \psi^+(v) > u_0$ and $u - v \in X, \psi^-(u - v) > v_0, \psi^-(u) < v_0, \psi^-(v) < v_0$.
$\Rightarrow u - v \notin \psi_{u_0}^p, u, v \in \psi_{u_0}^p$ and $u - v \notin \psi_{v_0}^N, u, v \in \psi_{v_0}^N$
$\Rightarrow u - v \notin \psi_{u_0}^p \bigcap \psi_{v_0}^N$ and $u, v \in \psi_{u_0}^p \bigcap \psi_{v_0}^N$,
which is a contradiction to the fact that $\psi_{u_0}^N \bigcap \psi_{v_0}^N$is a subfield of X.

Therefore our assumption is wrong. Hence $\psi^+(x - y) \geq \psi^+(x) \wedge \psi^+(y)$ and $\psi^-(x - y) \leq \psi^-(x) \vee \psi^-(y) \forall x, y \in X$. Now, if possible suppose that there exists $u, v(\neq 0) \in X$ such that $\psi^+(uv^{-1}) < \psi^+(u) \wedge \psi^+(v)$ and

$$\psi^-(uv^{-1}) > \psi^-(u) \vee \psi^-(v).$$

Put

$$u_1 = \frac{1}{2}(\psi^+(uv^{-1}) + \min(\psi^+(u), \psi^+(v))$$

and

$$v_1 = \frac{1}{2}(\psi^-(uv^{-1}) + \max(\psi^-(u), \psi^-(v)).$$

Then we have

$$\psi^+(uv^{-1}) < u_1 < min(\psi^+(u), \psi^+(v))$$

and

$$\psi^-(uv^{-1}) > v_1 > \max(\psi^-(u), \psi^-(v))$$

$$\Rightarrow \psi^+(uv^{-1}) < u_1, \psi^+(u) \wedge \psi^+(v) > u_1 \Rightarrow \psi^+(u) > u_1, \psi^+(u) > v_1$$

and

$$\psi^-(uv^{-1}) > v_1, \psi^-(u) \vee \psi^-(v) < v_1 \Rightarrow \psi^-(u) < v_1, \psi^-(u) < v_1$$

$$\Rightarrow uv^{-1} \notin \psi^{p}_{u_1}, u, v \in \psi^{p}_{u_1}$$

and

$$uv^{-1} \notin \psi^{N}_{v_1}, u, v \in \psi^{N}_{v_1}$$

$$\Rightarrow uv^{-1} \notin \psi^{p}_{u_1} \bigcap \psi^{N}_{v_1}$$

and

$$u, v \in \psi^{p}_{u_1} \bigcap \psi^{N}_{v_1},$$

which is a contradiction to the fact that $\psi^{p}_{u_1} \bigcap \psi^{N}_{v_1}$ is a subfield of X. Therefore our assumption is wrong. Hence $\psi^+(xy^{-1}) \geq \psi^+(x) \wedge \psi^+(y)$ and $\psi^-(xy^{-1}) \leq \psi^-(x) \vee \psi^-(y)$ $\forall x, y(\neq 0) \in X$.
Thus $\psi = (X; \psi^+, \psi^-)$ is a bipolar fuzzy field of X. ■

Corollary 3.6. If $\psi = (X; \psi^+, \psi^-)$ is a bipolar fuzzy field of X, then the sets $\psi^{p}_{\psi^+(0)}$ and $\psi^{N}_{\psi^-(0)}$ are subfields of X.

Definition 3.7. Let X, Y be two non empty sets and $f : X \to Y$ be a mapping. Let $\psi = (X; \psi^+, \psi^-)$ be a bipolar fuzzy set of X. Then $f(\psi) = (Y; f(\psi^+), f(\psi^-))$ the image of ψ under f is a bipolar fuzzy set of Y defined by

$$(f(\psi^+)(y) = \begin{cases} \sup\limits_{f(x)=y(or)x\in f^{-1}(y)} \psi^+(x) & \text{if } f^{-1}(y) \neq \phi, \\ 0 & \text{if } f^{-1}(y) = \phi \end{cases}$$

for each $y \in Y$ and

$$(f(\psi^-))(y) = \begin{cases} \inf\limits_{f(x)=y(or)x\in f^{-1}(y)} \psi^-(x) & \text{if } f^{-1}(y) \neq \phi. \\ 0 & \text{if } f^{-1}(y) = \phi \end{cases}$$

Definition 3.8. Let X, Y be two non-empty set and $f : X \to Y$ be a mapping. Let $\phi = (Y; \phi^+, \phi^-)$ be a bipolar fuzzy set of Y. Then $f^{-1}(\phi) = (X; f^{-1}(\phi^+), f^{-1}(\phi^-))$, then the pre-image of ϕ under f is a bipolar fuzzy set of X defined by

$$(f^{-1}(\phi^+))(x) = \phi^+(f(x))$$

$$(f^{-1}(\phi^-))(x) = \phi^-(f(x)) \ \forall x \in X.$$

Theorem 3.9. Let X, Y be two fields and $f : X \to Y$ be an onto homo morphism. Suppose that $\psi_1 = (X; \psi_1^+, \psi_1^-)$ and $\psi_2 = (Y; \psi_2^+, \psi_2^-)$ be two bipolar fuzzy fields of X and Y respectively. Then

(a) $f(\psi_1) = (Y; f(\psi_1^+), f(\psi_1^-))$ is a bipolar fuzzy field of Y.

(b) $f^{-1}(\psi_2) = (X; f^{-1}(\psi_2^+), f^{-1}(\psi_2^-))$ is a bipolar fuzzy field of X.

4. Bipolar fuzzy near-algebra

Definition 4.1. A bipolar fuzzy set $\phi = (Y; \phi^+, \phi^-)$ in Y is called a bipolar fuzzy near-algebra of Y over a bipolar fuzzy field $\psi = (X; \psi^+, \psi^-)$, if it satisfies following four conditions;

(i) $\phi^+(x + y) \geq min\{\phi^+(x), \phi^+(y)\},\ \phi^-(x + y) \leq max\{\phi^-(x), \phi^-(y)\}$

(ii) $\phi^+(\lambda x) \geq min\{\psi^+(\lambda), \phi^+(x)\},\ \phi^+(\lambda x) \leq max\{\psi^-(\lambda), \phi^-(x)\}$

(iii) $\phi^+(xy) \geq min\{\phi^+(x), \phi^+(y)\},\ \phi^-(xy) \leq max\{\phi^-(x), \phi^-(y)\}$

(iv) $\psi^+(1) \geq \phi^+(x),\ \psi^-(1) \leq \phi^-(x)\ \forall x, y \in Y, \lambda \in X$ and 1 is the unity in X.

Proposition 4.2. If $\phi = (Y; \phi^+, \phi^-)$ is a bipolar fuzzy near-algebra of Y over a bipolar fuzzy field $\psi = (X; \psi^+, \psi^-)$ in X, then $\psi^+(0) \geq \phi^+(x)$ and $\psi^-(0) \leq \phi^-(x)$ for every $x \in Y$.

Proposition 4.3. If $\phi = (Y; \phi^+, \phi-)$ is a bipolar fuzzy near-algebra over bipolar fuzzy field $\psi = (X; \psi^+, \psi^-)$, then $\phi^+ \geq \phi^+(x)$ for every $x \in Y$ and $\phi^-(0) \leq \phi^-(x)$ for all $x \in Y$.

Theorem 4.4. $\phi = (Y; \phi^+, \phi^-)$ is a bipolar fuzzy near-algebra over a bipolar fuzzy field $\psi = (X; \psi^+, \psi^-)$ if and only if the following three conditions hold:

(i) $\phi^+(\lambda x + \mu y) \geq \min(\min(\psi^+(\lambda), \phi^+(x)), \min(\psi^+(\mu), \phi^+(y)))$ and
$\phi^-(\lambda x + \mu y) \leq \max(\max(\psi^-(\lambda), \phi^-(x)), \max(\psi^-(\mu), \phi^-(y)))$

(ii) $\phi^+(xy) \geq \min(\phi^+(x), \phi^+(y))$ and
$\phi^-(xy) \leq \max(\phi^-(x), \phi^-(y))$

(iii) $\psi^+(1) \geq \phi^+(x)$ and $\psi^-(1) \leq \phi^-(x)\ \forall x, y \in Y; \lambda, \mu \in X$.

Proof. Let $\lambda, \mu \in X;\ x, y \in Y$. Then

$$\begin{aligned} \phi^+(\lambda x + \mu y) &\geq \min(\phi^+(\lambda x), \phi^+(\mu y)) \\ &\geq \min(\min(\psi^+(\lambda), \phi^+(x)), \min(\psi^+(\mu), \phi^+(y))) \text{ and} \\ \phi^-(\lambda x + \mu y) &\leq \max(\phi^-(\lambda x), \phi^-(\mu y)) \\ &\leq \max(\max(\psi^-(\lambda), \phi^-(x)), \max(\psi^-(\mu), \phi^-(y))). \end{aligned}$$

And the remaining two conditions hold directly, from the definition of ϕ. Conversely, let $x, y \in Y; \lambda \in X$. Then

$$\begin{aligned} \phi^+(x + y) &= \phi^+(1x + 1y) \\ &\geq \min\{\min(\psi^+(1), \phi^+(x)), \min(\psi^+(1), \phi^+(y))\} \\ &\geq \min\{\min(\phi^+(x), \phi^+(x)), \min(\phi^+(x), \phi^+(y))\} \\ &= \min(\phi^+(x), \phi^+(y))) \end{aligned}$$

and

$$\begin{aligned}\phi^-(x+y) &= \phi^-(1x+1y)\\ &\le \max\{\max(\psi^-(1),\phi^-(x)),\max(\psi^-(1),\phi^-(y))\}\\ &\le \max\{\max(\phi^-(x),\phi^-(x)),\max(\phi^-(x),\phi^-(y))\}\\ &= \max(\phi^-(x),\phi^-(y)).\end{aligned}$$

Also

$$\begin{aligned}\phi^+(\lambda x) &= \phi^+(\lambda x+0x)\\ &\ge \min\{\min(\psi^+(\lambda),\phi^+(x)),\min(\psi^+(0),\phi^+(x))\}\\ &\ge \min\{\min(\psi^+(\lambda),\phi^+(x)),\min(\phi^+(x),\phi^+(x))\}\\ &= \min\{\psi^+(\lambda),\phi^+(x))\}\end{aligned}$$

and

$$\begin{aligned}\phi^-(\lambda x) &= \phi^-(\lambda x+0x)\\ &\le \max\{\max(\psi^-(\lambda),\phi^-(x)),\max(\psi^-(0),\phi^-(x))\}\\ &\le \max\{\max(\psi^-(\lambda),\phi^-(x)),\max(\phi^-(x),\phi^-(x))\}\\ &= \max\{\psi^-(\lambda),\phi^-(x)\}.\end{aligned}$$

From sufficient part, we have

$$\begin{aligned}\phi^+(xy) &\ge \min(\phi^+(x),\phi^+(y)) \text{ and } \phi^-(xy)\le\max(\phi^-(x),\phi^-(y)).\\ \psi^+(1) &\ge \phi^+(x)\ and\ \psi^-(1)\le\phi^-(x).\end{aligned}$$

Thus $\phi = (Y;\phi^+,\phi^-)$ is a bipolar fuzzy near-algebra over a bipolar fuzzy field $\psi = (X;\psi^+,\psi^-)$. ■

Proposition 4.5. If $\phi = (Y;\phi^+,\phi^-)$ is a bipolar fuzzy near-algebra over a bipolar fuzzy field $\psi = (X;\psi^+,\psi^-)$ then $\phi^+(x-y)\ge\min\{\phi^+(x),\phi^+(y)\}$, and $\phi^-(x-y)\le\max\{\phi^-(x),\phi^-(y)\}\ \forall\, x,y\in Y$.

Example 4.6. Let $X = Z_2 = \{0,1\}_{\oplus_2,\otimes_2}$, for any $x,y\in X$, we have $x-y\in X$ and particularly for $y\ne 0$, $xy^{-1}\in X$. This shows that X is a field. Let $\psi = (X;\psi^+,\psi^-)$ be a bipolar fuzzy set of X.
Now consider two mappings $\psi^+ : X\to[0,1]$ and $\psi^- : X\to[-1,0]$, which are respectively defined by

$$\psi^+(x) = \begin{cases} 0.9 & \text{if } x=0\\ 0.8 & \text{if } x=1.\end{cases}$$

and

$$\psi^-(x) = \begin{cases} -0.1 & \text{if } x=1\\ -0.2 & \text{if } x=0.\end{cases}$$

It is clear that $\psi = (X;\psi^+,\psi^-)$ is a bipolar fuzzy field of X. Let $Y = \{0,a,b,c\}$ be a set with two binary operations " + " and "." as follows:

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

.	0	a	b	c
0	0	0	0	0
a	0	b	0	b
b	0	0	0	0
c	0	b	0	b

Define a scalar multiplication on Y as $0.x = 0$, $1.x = x$ for all $x \in Y$, $0, 1 \in X$. A direct verification shows that Y is a near-algebra over a field X. Let $\phi = (Y; \phi^+, \phi^-)$ be a bipolar fuzzy sub set of Y defined by $\phi^+ : Y \to [0, 1]$, $\phi^- : Y \to [-1, 0]$ where

$$\phi^+(x) = \begin{cases} 0.6 & \text{if } x = 0 \\ 0.4 & \text{otherwise.} \end{cases}$$

and

$$\phi^-(x) = \begin{cases} -0.8 & \text{if } x = 0 \\ -0.6 & \text{otherwise.} \end{cases}$$

A direct verification shows that $\phi = (Y; \phi^+, \phi^-)$ is a bipolar fuzzy near-algebra of Y over a bipolar fuzzy field $\psi = (X; \psi^+, \psi^-)$ of X.

Theorem 4.7. $\phi = (Y; \phi^+, \phi^-)$ is a bipolar fuzzy near-algebra over a bipolar fuzzy field $\psi = (X; \psi^+, \psi^-)$ if and only if it satisfies the following two assertions:

(i) $\forall\, t \in [0, 1], \phi_t^P \neq \phi \Rightarrow \phi_t^P$ is a sub near-algebra over a field ψ_t^P.

(ii) $\forall\, s \in [-1, 0], \phi_s^N \neq \phi \Rightarrow \phi_s^N$ is a sub near-algebra over a field ψ_s^N.

Theorem 4.8. Let Y, Z be two near-algebras over a field X and $f : Y \to Z$ be an onto near-algebra homomorphism. If $\phi = (Y; \phi^+, \phi^-)$ is a bipolar fuzzy near-algebra over a bipolar fuzzy field $\psi = (X; \psi^+, \psi^-)$, then $f(\phi) = (Z; f(\phi^+), f(\phi^-))$ is a bipolar fuzzy near-algebra over a bipolar fuzzy field $\psi = (X; \psi^+, \psi^-)$.

Proof.

(i) Let $u, v \in Z$. Then $u = f(x)$, $v = f(y)$ for some $x, y \in Y$ and $f : Y \to Z$ Consider

$$\begin{aligned} f(\phi^+)(u+v) &= sup\{\phi^+(z) \mid z \in Y, f(z) = u + v \text{ or } z \in f^{-1}(u+v)\} \\ &= sup\{\phi^+(x+y) \mid x, y \in Y, f(x) = u, f(y) = v\} \\ &\geq sup\{\phi^+(x) \wedge \phi^+(y) \mid x, y \in Y, f(x) = u, f(y) = v\} \\ &= (sup\{\phi^+(x) \mid x \in Y, f(x) = u\}) \wedge (sup\{\phi^+(y) \mid y \in Y, \\ &\quad f(y) = v\}) \\ &= f(\phi^+)(u) \wedge f(\phi^+)(v) \end{aligned}$$

and

$$
\begin{aligned}
f(\phi^-)(u+v) &= \inf\{\phi^-(z) \mid z \in Y, f(z) = u+v \text{ or } z \in f^{-1}(u+v)\} \\
&= \inf\{\phi^-(x+y) \mid x, y \in Y, f(x) = u, f(y) = v\} \\
&\leq \inf\{\phi^-(x) \vee \phi^-(y) \mid x, y \in Y, f(x) = u, f(y) = v\}
\end{aligned}
$$

$$
\begin{aligned}
&= (\inf\{\phi^-(x) \mid x \in Y, f(x) = u\}) \vee (\inf\{\phi^-(y) \mid y \in Y, f(y) = v\}) \\
&= f(\phi^-)(u) \vee f(\phi^-)(v).
\end{aligned}
$$

(ii) Let $u \in Z, \lambda \in X$. Then $u = f(x)$ for some $x \in Y$.

$$
\begin{aligned}
\text{Now } f(\phi^+)(\lambda u) &= \sup\{\phi^+(z) \mid z \in Y, f(z) = \lambda u\} \\
&= \sup\{\phi^+(\lambda x) \mid x \in Y, \lambda \in X, f(x) = u\} \\
&\geq \sup\{\psi^+(\lambda) \wedge \phi^+(x) \mid x \in Y, \lambda \in X, f(x) = u\} \\
&= \min(\psi^+(\lambda), \sup\{\phi^+(x) \mid x \in Y, f(x) = u\} \\
&= \min(\psi^+(\lambda), f(\phi^+)(u)).
\end{aligned}
$$

$$
\begin{aligned}
\text{and } f(\phi^-)(\lambda u) &= \inf\{\phi^-(z) \mid z \in Y, f(z) = \lambda u\} \\
&= \inf\{\phi^-(\lambda x) \mid x \in Y, \lambda \in X, f(x) = u\} \\
&\leq \inf\{\psi^-(\lambda) \vee \phi^-(x) \mid x \in Y, \lambda \in X, f(x) = u\} \\
&= \max(\psi^-(\lambda), \inf\{\phi^-(x) \mid x \in Y, f(x) = u\}) \\
&= \max(\psi^-(\lambda), f(\phi^-)(u)).
\end{aligned}
$$

(iii) Let $u, v \in Z$. then $u = f(x), v = f(y)$ for some $x, y \in Y$.,

$$
\begin{aligned}
\text{Now } f(\phi^+)(uv) &= \sup\{\phi^+(z) \mid z \in Y, f(z)) = uv\} \\
&= \sup\{\phi^+(xy) \mid x, y \in Y, f(x) = u, f(y) = v\} \\
&\geq \sup\{\phi^+(x) \wedge \phi^+ f(y) \mid x, y \in Y, f(x) = u, f(y) = v\} \\
&= (\sup\{\phi^+(x) \mid x \in Y, f(x) = u\}) \wedge (\sup\{\phi^+(y) \mid y \in Y, \\
&\quad f(y) = v\}) \\
&= \min\{f(\phi^+)(u), f(\phi^+)(v)\}
\end{aligned}
$$

$$
\begin{aligned}
\text{and } f(\phi^-)(uv) &= \inf\{\phi^-(z) \mid z \in Y, f(z)) = uv\} \\
&= \inf\{\phi^-(xy) \mid x, y \in Y, f(x) = u, f(y) = v\} \\
&\leq \inf\{\phi^-(x) \vee \phi^- f(y) \mid x, y \in Y, f(x) = u, f(y) = v\} \\
&= (\inf\{\phi^-(x) \mid x \in Y, f(x) = u\}) \vee (\inf\{\phi^-(y) \mid y \in Y, \\
&\quad f(y) = v\}) \\
&= \max\{f(\phi^-)(u), f(\phi^-)(v)\}.
\end{aligned}
$$

(iv) We have $\psi^+(1) \geq \phi^+(x)$ for all $x \in Y$.

$$\begin{aligned} \text{Thus for all } u \in Z, \psi^+(1) &\geq \sup\{\phi^+(x) \mid x \in Y, f(x) = u\} \\ &= f(\phi^+)(u). \\ \text{Also we have } \psi^-(1) &\leq \phi^-(x) \; \forall\, x \in Y. \\ \text{Thus for all } u \in Z, \psi^-(1) &\leq \inf\{\phi^-(x) \mid x \in Y, f(x) = u\} \\ &= f(\phi^-(u). \end{aligned}$$

Hence $f(\phi) = (Z; f(\phi^+), f(\phi^-))$ is a bipolar fuzzy near-algebra over a bipolar fuzzy field $\psi = (X; \psi^+, \psi^-)$. ■

Theorem 4.9. Let Y, Z be two near-algebras over a field X and $f : Y \to Z$ be an onto near-algebra homomorphism. If $\phi = (Z; \phi^+, \phi^-)$ is bipolar fuzzy near-algebra over a bipolar fuzzy field $\psi = (X; \psi^+, \psi^-)$. Then $f^{-1}(\phi) = (Y; f^{-1}(\phi^+), f^{-1}(\phi^-))$ is a bipolar fuzzy near-algebra over a bipolar fuzzy field $\psi = (X; \psi^+, \psi^-)$.

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