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The Number of Fuzzy Subgroups for finite Abelian p-Group of rank three

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Abstract

In this paper, we give formula for the number of fuzzy subgroups of the group $Z_{p^m} \times Z_{p^n} \times Z_{p^r}$ where p is a prime and $m, n, r \in Z^+ \cup \{0\}$. This is achieved by using the maximal chains of subgroups and the recurrence relation technique already used by authors in their research papers on the number of fuzzy subgroups of Abelian p-group of rank two, number of fuzzy subgroups of Abelian group of rank two and Dihedral group.

Mathematics Subject Classification: Primary 20N25; 03E72; Secondary 20K01; 20K27

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1. INTRODUCTION

In 1965, Zadeh [9] first introduced fuzzy set. After that paper, several aspects of fuzzy subsets were studied. In 1971, Rosenfeld [12] introduced fuzzy sets in the realm of group theory and formulated the concept of fuzzy subgroups of a group. An increasing number of properties from classical group theory have been generalized.

One of the most important problems of fuzzy subgroup theory is to determine the number of fuzzy subgroups of a finite group [13]. Several papers have treated the particular case of finite Abelian group. Laszlo [14] studied the construction of fuzzy subgroup of groups of orders one to six. Zhang and Zou [15] have determined the number of fuzzy subgroups of cyclic group of order p^n, where p is a prime number. Murali and Makamba [16, 17] determine the number of distinct fuzzy subgroup of a finite cyclic group of square-free order, while [18] deals with cyclic group of order $p^n q^m$, where p and q are different prime. In [10], Tărnăuceanu and Bentea established the recurrence relation verified by the number of fuzzy subgroups of a finite cyclic group. In [11] Ngcibi, Murali and Makamba obtain a formula for the group $Z_{p^m} \times Z_{p^n}$ when n=1, 2, 3 which was extended by A. Sehgal, S. Sehgal and P.K Sharma [1] for all values of n and further extended by A. Sehgal, S. Sehgal, P.K Sharma and M. Jakhar [19] , they obtained the formula for $Z_{p^m q^r} \times Z_{p^n}$. In [20] Isaac K. Appiah and B.B Makamba determine the number of distinct fuzzy subgroups of finite p-abelian group of rank three $Z_{p^m} \times Z_{p^1} \times Z_{p^1}$

In the present paper, we establish a recurrence relation for the number of all the fuzzy subgroups of a finite p-abelian group of rank three $Z_{p^m} \times Z_{p^n} \times Z_{p^r}$ and solve this recurrence relation by using the concept which was already used by authors in [1, 19].

2. PRELIMINARIES

Let X be a fixed non-empty set. A *Fuzzy Sets* μ on X is a function from X to [0,1]. A *Fuzzy subset* μ of a group G is called a *fuzzy subgroup* μ of G if

(i) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in G$, and

(ii) $\mu(x^{-1}) = \mu(x)$for all $x \in G$

Clearly, $\mu(e) = \max\ \mu(G)$. For each $\alpha \in [0,1]$, the *level subset* corresponding to α is defined as $\mu_\alpha = \{x \in G: \mu(x) \geq \alpha\}$. A fuzzy subset μ is a *fuzzy subgroup* of G if and only if its level subsets are subgroups of G.

Let $\sim$ be the natural equivalence relation on the set of all fuzzy subsets of G.Then $\mu \sim \rho$ iff $(\mu(x) > \mu(y) \Leftrightarrow \rho(x) > \rho(y)\ \forall x, y \in G)$. By the above equivalence relation, the fuzzy subgroups of G can be classified up to equivalence classes in such a way that two fuzzy subgroups $\mu\ and\ \rho$ of G are distinct if $\nsim \rho$. Suppose that G is a finite group and $\mu: G \to [0,1]$ is a fuzzy subgroup of G. Let $\mu(G) = \{\alpha_1, \alpha_2, \alpha_3, .., \alpha_n\}$ and assume that $\alpha_1 > \alpha_2 >\ldots> \alpha_n$. Then μ provides the following chain of subgroups of G ending with G

$$\mu_{\alpha_1} \subseteq \mu_{\alpha_2} \subseteq \mu_{\alpha_3} \subseteq \cdots \subseteq \mu_{\alpha_n} = G \qquad \ldots (1)$$

Then for any $x \in G$ and $i = 1,2,3, \ldots, n$, we have $\mu(x) = \alpha_i \Leftrightarrow x \in \mu_{\alpha_i} - \mu_{\alpha_{i-1}}$ with the assumption that $\mu_{\alpha_0} = \emptyset$. According to Volf [4], a necessary and sufficient condition for two fuzzy subgroups of G to be equivalent with respect to $\sim$ is that they are same level fuzzy subgroups, that is, they determine the same chain of subgroups of type (1). Hence, there exists a bijection between the equivalence classes of fuzzy subgroups of G and the set of chains of subgroups of the group G, which end in G. If F_G denotes the number of all distinct fuzzy subgroups of G, then

F_G is the number of chains of subgroups of length one of G ending in G plus the number of chains of subgroups of length more than one of the group G, which end in G. Hence

$F_G = 1 + \sum_{H<G} F_H$

$F_G = 1 + \sum_{distict\ H \in Iso(G)} (F_H \times n\_H)$, where Iso (G) is the set of representatives of isomorphism classes of subgroups of G and n_H denotes the size of the isomorphism class with representative H. … (2)

Using [20], we obtain the number of subgroups isomorphic to $Z_{p^{k_1}} \times Z_{p^{k_2}} \times Z_{p^{k_3}} (0 \le k_1 \le k_2 \le k_3)$ of $Z_{p^m} \times Z_{p^n} \times Z_{p^r}$ $(0 \le m \le n \le r)$

$$\text{is} \begin{cases} p^{2k_3-2k_1-3}(p+1)(p^2+p+1) & 0 \le k_1 < k_2 < k_3 \le m \le n \le r \\ p^{2k_3-2k_1-2}(p^2+p+1) & 0 \le k_1 = k_2 < k_3 \le m \le n \le r \\ p^{2k_3-2k_1-2}(p^2+p+1) & 0 \le k_1 < k_2 = k_3 \le m \le n \le r \\ 1 & 0 \le k_1 = k_2 = k_3 \le m \le n \le r \\ p^{m+k_3-2k_1-2}(p+1)^2 & 0 \le k_1 < k_2 \le m < k_3 \le n \le r \\ p^{m+k_3-2k_1-1}(p+1) & 0 \le k_1 = k_2 \le m < k_3 \le n \le r \\ p^{2m+k_3-2k_1-k_2-1}(p+1) & 0 \le k_1 \le m < k_2 < k_3 \le n \le r \\ p^{2m-2k_1} & 0 \le k_1 \le m < k_2 = k_3 \le n \le r \\ p^{2m+n-2k_1-k_2} & 0 \le k_1 \le m < k_2 \le n < k_3 \le r \\ p^{m+n-2k_1-1}(p+1) & 0 \le k_1 < k_2 \le m \le n < k_3 \le r \\ p^{m+n-2k_1} & 0 \le k_1 = k_2 \le m \le n < k_3 \le r \end{cases} \quad \ldots (3)$$

Theorem 2.1: - The number of all fuzzy subgroups of $\boldsymbol{Z_{p^m} \times Z_{p^n} \times Z_{p^r}}$ $\boldsymbol{(0 \le m \le n \le r)}$ is

$$\sum_{\substack{a-b+h+g=m \\ b+c=n}} \binom{b+c}{b,c}\binom{r}{h}\binom{r+g-1}{r-1,g} 2^{a+r} \frac{p^{2c+h+g}}{(1-p)^{b+c}} (-1)^{c+h} +$$
$$\sum_{\substack{a-b+d+f+g+i+k+1=m \\ b+c+e+f+h+j+k+1=n}} \left[\binom{b+c}{b,c}\binom{d+e+f}{d,e,f}\binom{r}{g}\binom{r}{h}\binom{r+i+j+k-1}{r-1,i,j,k} 2^{a+d+e+f+r+i+j+k+1} \times \frac{p^{2c+d+f+2g+h+2i+j+3k+1}(p^2-p+1)}{(1-p)^{b+c+1}} (-1)^{c+f+g+h+k}\right]$$

3. NUMBER OF FUZZY SUBGROUPS OF GROUP $Z_{p^1} \times Z_{p^1} \times Z_{p^1}$

Let us denote $A_{m,n,r}$ as number of fuzzy subgroups of group $Z_{p^m} \times Z_{p^n} \times Z_{p^r}$

By using (3), we have $A_{1,1,1} = 1 + (p^2 + p + 1)A_{0,0,1} + (p^2 + p + 1)A_{0,1,1} + A_{0,0,0}$ … (4)

By using corollary 5.2 of [19] , we have determine number of fuzzy subgroups of group Z_{p^n} which is isomorphic to $Z_{p^0} \times Z_{p^0} \times Z_{p^n}$ as $A_{0,0,n} = 2^n$. Hence $A_{0,0,1} = 2$ and $A_{0,0,0} = 2^0 = 1$ By using corollary 4.1 of [1] , we have determine number of fuzzy subgroups of group $Z_{p^1} \times Z_{p^n}$ which is isomorphic to $Z_{p^0} \times Z_{p^1} \times Z_{p^n}$ as $A_{0,1,n} = 2^{n+1} + np2^n$ Hence $A_{0,1,1} = 4 + 2p$

Hence (4) become $A_{1,1,1} = 2p^3 + 8p^2 + 8p+8$ … (5)

4. constriction of recurrence relations for the number of fuzzy subgroups of group $Z_{p^m} \times Z_{p^n} \times Z_{p^r}$

Now we establish recurrence relation for $A_{m,n,r}$ by following cases:-

Case 1:- $0 < m < n < r$

By using (3), we have $A_{m,n,r} = 1 + \sum_{l=0}^{m}\sum_{k=l+1}^{m}\sum_{i=k+1}^{m} p^{2i-2l-3}(p+1)(p^2+p+1)\, A_{l,k,i} + \sum_{l=0}^{m}\sum_{i=l+1}^{m} p^{2i-2l-2}(p^2+p+1)\, A_{l,l,i} + \sum_{l=0}^{m}\sum_{i=l+1}^{m} p^{2i-2l-2}(p^2+p+1)\, A_{l,i,i} + \sum_{l=0}^{m} A_{l,l,l} + \sum_{l=0}^{m}\sum_{k=l+1}^{m}\sum_{i=m+1}^{n} p^{m+i-2l-2}(p+1)^2\, A_{l,k,i} + \sum_{l=0}^{m}\sum_{i=m+1}^{n} p^{m+i-2l-1}(p+1)\, A_{l,l,i} + \sum_{l=0}^{m}\sum_{k=m+1}^{n}\sum_{i=k+1}^{n} p^{2m+i-2l-k-1}(p+1)\, A_{l,k,i} + \sum_{l=0}^{m}\sum_{i=m+1}^{n} p^{2m-2l}\, A_{l,i,i} + \sum_{l=0}^{m}\sum_{k=m+1}^{n}\sum_{i=n+1}^{r-1} p^{2m+n-2l-k}\, A_{l,k,i} + \sum_{l=0}^{m}\sum_{k=m+1}^{n-1} p^{2m+n-2l-k}\, A_{l,k,r} + \sum_{l=0}^{m-1} p^{2m-2l}\, A_{l,n,r} + \sum_{l=0}^{m}\sum_{k=l+1}^{m}\sum_{i=n+1}^{r} p^{m+n-2l-1}(p+1)\, A_{l,k,i} + \sum_{l=0}^{m}\sum_{i=n+1}^{r} p^{m+n-2l}\, A_{l,l,i}$ … (6)

Change r to r-1 in (6), we get

$A_{m,n,r-1} = 1 + \sum_{l=0}^{m}\sum_{k=l+1}^{m}\sum_{i=k+1}^{m} p^{2i-2l-3}(p+1)(p^2+p+1)\, A_{l,k,i} + \sum_{l=0}^{m}\sum_{i=l+1}^{m} p^{2i-2l-2}(p^2+p+1)\, A_{l,l,i} + \sum_{l=0}^{m}\sum_{i=l+1}^{m} p^{2i-2l-2}(p^2+p+1)\, A_{l,i,i} + \sum_{l=0}^{m} A_{l,l,l} + \sum_{l=0}^{m}\sum_{k=l+1}^{m}\sum_{i=m+1}^{n} p^{m+i-2l-2}(p+1)^2\, A_{l,k,i} + \sum_{l=0}^{m}\sum_{i=m+1}^{n} p^{m+i-2l-1}(p+1)\, A_{l,l,i} + \sum_{l=0}^{m}\sum_{k=m+1}^{n}\sum_{i=k+1}^{n} p^{2m+i-2l-k-1}(p+1)\, A_{l,k,i} + \sum_{l=0}^{m}\sum_{i=m+1}^{n} p^{2m-2l}\, A_{l,i,i} + \sum_{l=0}^{m}\sum_{k=m+1}^{n}\sum_{i=n+1}^{r-2} p^{2m+n-2l-k}\, A_{l,k,i} + \sum_{l=0}^{m}\sum_{k=m+1}^{n-1} p^{2m+n-2l-k}\, A_{l,k,r-1} + \sum_{l=0}^{m-1} p^{2m-2l}\, A_{l,n,r-1} + \sum_{l=0}^{m}\sum_{k=l+1}^{m}\sum_{i=n+1}^{r-1} p^{m+n-2l-1}(p+1)\, A_{l,k,i} + \sum_{l=0}^{m}\sum_{i=n+1}^{r-1} p^{m+n-2l}\, A_{l,l,i}$ … (7)

From (6) and (7), we have

$$A_{m,n,r} - 2A_{m,n,r-1} = \sum_{l=0}^{m}\sum_{k=m+1}^{n-1} p^{2m+n-2l-k} A_{l,k,r} + \sum_{l=0}^{m-1} p^{2m-2l} A_{l,n,r} + \sum_{l=0}^{m}\sum_{k=l+1}^{m} p^{m+n-2l-1}(p+1) A_{l,k,r} + \sum_{l=0}^{m} p^{m+n-2l} A_{l,l,r} \quad \ldots (8)$$

Change n to n-1 in (8), we have

$$A_{m,n-1,r} - 2A_{m,n-1,r-1} = \sum_{l=0}^{m}\sum_{k=m+1}^{n-2} p^{2m+n-1-2l-k} A_{l,k,r} + \sum_{l=0}^{m-1} p^{2m-2l} A_{l,n-1,r} + \sum_{l=0}^{m}\sum_{k=l+1}^{m} p^{m+n-1-2l-1}(p+1) A_{l,k,r} + + \sum_{l=0}^{m} p^{m+n-1-2l} A_{l,l,r} \quad \ldots (9)$$

From (8) and (9), we have

$$A_{m,n,r} - 2A_{m,n,r-1} - 2pA_{m,n-1,r} + 2pA_{m,n-1,r-1} = \sum_{l=0}^{m-1} p^{2m-2l} A_{l,n,r} \quad \ldots (10)$$

Change m to m-1 in (10), we have

$$A_{m-1,n,r} - 2A_{m-1,n,r-1} - 2pA_{m-1,n-1,r} + 2pA_{m-1,n-1,r-1} = \sum_{l=0}^{m-2} p^{2m-2-2l} A_{l,n,r} \quad \ldots (11)$$

From (10) and (11), we get

$$A_{m,n,r} - 2A_{m,n,r-1} - 2pA_{m,n-1,r} + 2pA_{m,n-1,r-1} - 2p^2A_{m-1,n,r} + 2p^2A_{m-1,n,r-1} + 2p^3A_{m-1,n-1,r} - 2p^3A_{m-1,n-1,r-1} = 0 \quad \ldots (12)$$

Case 2: $0 < m < n = r$

Similarly, we can derive

$$A_{m,n,n} - 2(1+p)A_{m,,n-1,n} + 2pA_{m,n-1,n-1} - 2p^2A_{m-1,n,n} + 2p^2(1+p)A_{m-1,n-1,n} - 2p^3A_{m-1,n-1,n-1} = 0 \quad \ldots (13)$$

Case 3: $0 < m = n < r$

Similarly, we can derive

$$A_{m,m,r} - 2A_{m,m,r-1} - 2p(1+p)A_{m-1,m,r} + 2p(1+p)A_{m-1,m,r-1} + 2p^3A_{m-1,m-1,r} - 2p^3A_{m-1,m-1,r-1} = 0 \quad \ldots (14)$$

Case 4:- m=n=r

Similarly, we can derive

$$A_{m,m,m} - 2(1+p+p^2)A_{m-1,m,m} + 2p(1+p+p^2)A_{m-1,m-1,m} - 2p^3A_{m-1,m-1,m-1} = 0 \quad \ldots (15)$$

Case 5:- $m = 0, 1 \leq n < r$

From equation (7) of [1], we have

$$A_{0,n,r} - 2A_{0,n,r-1} - 2pA_{0,n-1,r} + 2pA_{0,n-1,r-1} = 0 \ \ where\ n < r \quad \ldots (16)$$

Case 6:- $\boldsymbol{m = 0, 1 \leq n = r}$

From equation (12) of [1], we have

$$A_{0,n,n} - 2(1+p)A_{0,n-1,n} + 2pA_{0,n-1,n-1} = 0 \qquad \dots (17)$$

5. Solution of recurrence relation

Now we find the solution of recurrence relation

$$A_{m,n,r} - 2A_{m,n,r-1} - 2pA_{m,n-1,r} + 2pA_{m,n-1,r-1} - 2p^2A_{m-1,n,r} + 2p^2A_{m-1,n,r-1} + 2p^3A_{m-1,n-1,r} - 2p^3A_{m-1,n-1,r-1} = 0$$

Multiply both sides by x^m and summation m from 1 to ∞, we get

$$\sum_{m=1}^{\infty} A_{m,n,r}\, x^m - 2\sum_{m=1}^{\infty} A_{m,n,r-1}\, x^m - 2p\sum_{m=1}^{\infty} A_{m,n-1,r}\, x^m + 2p\sum_{m=1}^{\infty} A_{m,n-1,r-1}\, x^m - 2p^2\sum_{m=1}^{\infty} A_{m-1,n,r}\, x^m + 2p^2\sum_{m=1}^{\infty} A_{m-1,n,r-1}\, x^m + 2p^3\sum_{m=1}^{\infty} A_{m-1,n-1,r}\, x^m - 2p^3\sum_{m=1}^{\infty} A_{m-1,n-1,r-1}\, x^m = 0$$

Take $\sum_{m=0}^{\infty} A_{m,n,r}\, x^m = B_{n,r}$, we get

$$\left(B_{n,r} - A_{0,n,r}\right) - 2\left(B_{n,r-1} - A_{0,n,r-1}\right) - 2p\left(B_{n-1,r} - A_{0,n-1,r}\right) + 2p\left(B_{n-1,r-1} - A_{0,n-1,r-1}\right) - 2p^2xB_{n,r} + 2p^2xB_{n,r-1} + 2p^3xB_{n-1,r} - 2p^3xB_{n-1,r-1} = 0$$

$$(1-2xp^2)B_{n,r} - 2(1-xp^2)B_{n,r-1} - 2p(1-xp^2)B_{n-1,r} + 2p(1-xp^2)B_{n-1,r-1} = A_{0,n,r} - 2A_{0,n,r-1} - 2pA_{0,n-1,r} + 2pA_{0,n-1,r-1} \qquad \dots (18)$$

By using (16) and (18), we have

$$(1-2xp^2)B_{n,r} - 2(1-xp^2)B_{n,r-1} - 2p(1-xp^2)B_{n-1,r} + 2p(1-xp^2)B_{n-1,r-1} = 0$$

Multiply both sides by y^n and summation n from 1 to ∞, we get

$$(1-2xp^2)\sum_{n=1}^{\infty} B_{n,r}\, y^n - 2(1-xp^2)\sum_{n=1}^{\infty} B_{n,r-1}\, y^n - 2p(1-xp^2)\sum_{n=1}^{\infty} B_{n-1,r}\, y^n + 2p(1-xp^2)\sum_{n=1}^{\infty} B_{n-1,r-1}\, y^n = 0$$

Take $\sum_{n=0}^{\infty} B_{n,r}\, y^n = C_r$, we get

$$(1-2xp^2)(C_r - B_{0,r}) - 2(1-xp^2)(C_{r-1} - B_{0,r-1}) - 2p(1-xp^2)yC_r + 2p(1-xp^2)yC_{r-1} = 0$$

$$\left(1-2xp^2-2py(1-xp^2)\right)C_r - 2(1-xp^2)(1-py)C_{r-1} = (1-2xp^2)B_{0,r} - 2(1-xp^2)B_{0,r-1} \qquad \dots (19)$$

Now we calculate the value of $B_{0,r} = \sum_{m=0}^{\infty} A_{m,0,r} x^m$

We know that number of fuzzy subgroups of group $Z_{p^m} \times Z_{p^r}$ is same as $Z_{p^m} \times Z_{p^0} \times Z_{p^r}$

From equation (7) of [1], we have $A_{m,0,r} - 2A_{m,0,r-1} - 2pA_{m-1,0,r} + 2pA_{m-1,0,r-1} = 0$

Multiply both sides by x^m and summation m from 1 to ∞, we get

$$\sum_{m=1}^{\infty} A_{m,0,r}\, x^m - 2\sum_{m=1}^{\infty} A_{m,0,r-1}\, x^m - 2p\sum_{m=1}^{\infty} A_{m-1,0,r}\, x^m + 2p\sum_{m=1}^{\infty} A_{m-1,0,r-1}\, x^m = 0$$

We have $\sum_{m=0}^{\infty} A_{m,0,r}\, x^m = B_{0,r}$

$$(B_{0,r} - A_{0,0,r}) - 2(B_{0,r-1} - A_{0,0,r-1}) - 2pxB_{0,r} + 2pxB_{0,r-1} = 0$$

$$(1-2px)B_{0.r} - (2-2px)B_{0,r-1} = A_{0,0,r} - 2A_{0,0,r-1}$$

From [8], we know that number of fuzzy subgroups of group $Z_{p^0} \times Z_{p^0} \times Z_{p^r}$ are 2^r , hence $A_{0,0,r} = 2^r$, hence above equation can be written as $(1-2px)B_{0.r} - (2-2px)B_{0,r-1} = 0$

=> $B_{0.r} - \frac{(2-2px)}{(1-2px)} B_{0.r-1} = 0$

Solution of above recurrence relation is $B_{0,r} = A[\frac{2-2px}{1-2px}]^r$ … (20)

Put r=0 in $\sum_{m=0}^{\infty} A_{m,0,r}\, x^m = B_{0,r}$, we get $\sum_{m=0}^{\infty} A_{m,0,0}\, x^m = B_{0,0}$

=> $B_{0,0} = \sum_{m=0}^{\infty}(2x)^m = \frac{1}{1-2x}$ [∴ $A_{m,0,0} = 2^m$]

Put r=0 in (20), we get $B_{0,0} = A$

Put value of A in (20), we get $B_{0,r} = \frac{1}{1-2x}[\frac{2-2px}{1-2px}]^r$ … (21)

By use of (21) in (20), we get

$$(1-2xp^2)B_{0,r} - 2(1-xp^2)B_{0,r-1} = \frac{(1-2xp^2)}{1-2x}[\frac{2-2px}{1-2px}]^r - \frac{2(1-xp^2)}{1-2x}[\frac{2-2px}{1-2px}]^{r-1}$$

$$(1-2xp^2)B_{0,r} - 2(1-xp^2)B_{0,r-1} = \frac{xp(1-p)}{(1-px)(1-2x)}[\frac{2-2px}{1-2px}]^r$$

$$\left(1-2xp^2-2py(1-xp^2)\right)C_r - 2(1-xp^2)(1-py)C_{r-1} = \frac{xp(1-p)}{(1-px)(1-2x)}[\frac{2-2px}{1-2px}]^r$$

$$C_r - \frac{2(1-xp^2)(1-py)}{(1-2xp^2-2py(1-xp^2))}C_{r-1} = \frac{xp(1-p)}{(1-px)(1-2x)(1-2xp^2-2py(1-xp^2))}[\frac{2-2px}{1-2px}]^r \quad \ldots (22)$$

The C.S of recurrence relation in equation is $C_r = A\,\frac{2^r\left(1-xp^2\right)^r(1-py)^r}{\left(1-2xp^2-2py(1-xp^2)\right)^r}$

The P.S of recurrence relation in equation is $C_r = B[\frac{2-2px}{1-2px}]^r$

$$B(1-2x)[\left(1-2xp^2-2py(1-xp^2)\right)(1-px)-(1-xp^2)(1-py)(1-2px)] = xp(1-p)$$

$$B = \frac{1}{(1-2x)(1-\frac{y}{x(1-p)}+\frac{p^2y}{(1-p)})}$$

The P.S of recurrence relation in equation is $C_r = \frac{1}{(1-2x)(1-\frac{y}{x(1-p)}+\frac{p^2y}{(1-p)})}[\frac{2-2px}{1-2px}]^r$

The general solution $C_r = A\,\frac{2^r\left(1-xp^2\right)^r(1-py)^r}{\left(1-2xp^2-2py(1-xp^2)\right)^r} + \frac{1}{(1-2x)(1-\frac{y}{x(1-p)}+\frac{p^2y}{(1-p)})}[\frac{2-2px}{1-2px}]^r$

$$C_0 = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{m,n,0}\, x^m\, y^n = \sum_{r=0}^{\infty}\sum_{m=0}^{\infty} A_{m,0,r}\, x^m\, y^r = \sum_{r=0}^{\infty} B_{0,r}\, y^r$$

From (21), we have $B_{0,r} = \frac{1}{1-2x}[\frac{2-2px}{1-2px}]^r$

$$C_0 = \sum_{r=0}^{\infty}\frac{1}{1-2x}[\frac{2-2px}{1-2px}]^r\, y^r = \frac{1-2px}{(1-2px-2y+2pxy)(1-2x)}$$

$$C_0 = A + \frac{1}{(1-2x)(1-\frac{y}{x(1-p)}+\frac{p^2y}{(1-p)})}$$

$$A = \frac{1-2px}{(1-2px-2y+2pxy)(1-2x)} - \frac{1}{(1-2x)(1-\frac{y}{x(1-p)}+\frac{p^2y}{(1-p)})} = \frac{-\frac{y}{x(1-p)}+\frac{(p^2+2)y}{(1-p)}-\frac{2p(p^2-p+1)xy}{1-p}}{(1-2x)(1-\frac{y}{x(1-p)}+\frac{p^2y}{(1-p)})(1-2px-2y+2pxy)}$$

$$C_r = \frac{\left[-\frac{y}{x(1-p)}+\frac{(p^2+2)y}{(1-p)}-\frac{2p(p^2-p+1)xy}{1-p}\right]}{(1-2x)(1-\frac{y}{x(1-p)}+\frac{p^2y}{(1-p)})(1-2px-2y+2pxy)}\frac{2^r\left(1-xp^2\right)^r(1-py)^r}{\left(1-2xp^2-2py(1-xp^2)\right)^r} + \frac{1}{(1-2x)(1-\frac{y}{x(1-p)}+\frac{p^2y}{(1-p)})}[\frac{2-2px}{1-2px}]^r$$

$$C_r = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{m,n,r}\, x^m\, y^n$$

$$A_{m,n,r} = coefficient\ of\ x^m y^n\ in\ C_r =$$

$$\sum_{\substack{b+c+e+f+h+j+k+1=n\\ a-b+d+f+g+i+k-1=m}}\binom{b+c}{b,c}\binom{d+e+f}{d,e,f}\binom{r}{g}\binom{r}{h}\binom{r+i+j+k-1}{r-1,i,j,k}2^{a+d+e+f+r+i+j+k}\frac{p^{2c+d+f+2g+h+2i+j+3k}}{(1-p)^{b+c+1}}(-1)^{c+f+g+h+k+1} + \sum_{\substack{a-b+d+f+g+i+k=m\\ b+c+e+f+h+j+k+1=n}}\binom{b+c}{b,c}\binom{d+e+f}{d,e,f}\binom{r}{g}\binom{r}{h}\binom{r+i+j+k-1}{r-1,i,j,k}2^{a+d+e+f+r+i+j+k}\frac{p^{2c+d+f+2g+h+2i+j+3k}(p^2+2)}{(1-p)^{b+c+1}}(-1)^{c+f+g+h+k} -$$

$$= -\sum_{\substack{a-b+d+f+g+i+k+1=m \\ b+c+e+f+h+j+k+1=n}} \binom{b+c}{b,c}\binom{d+e+f}{d,e,f}\binom{r}{g}\binom{r}{h}\binom{r+i+j+k-1}{r-1,i,j,k} 2^{a+d+e+f+r+i+j+k+1} \frac{p^{2c+d+f+2g+h+2i+j+3k+1}(p^2-p+1)}{(1-p)^{b+c+1}} (-1)^{c+f+g+h+k}$$
$$+\sum_{\substack{a-b+h+g=m \\ b+c=n}} \binom{b+c}{b,c}\binom{r}{h}\binom{r+g-1}{r-1,g} 2^{a+r} \frac{p^{2c+h+g}}{(1-p)^{b+c}} (-1)^{c+h} \quad \ldots (23)$$

Corollary 5.1 [6]: - Prove that number of all distinct fuzzy subgroups of $\boldsymbol{Z_{p^r}}$ are $\boldsymbol{2^r}$ where $\boldsymbol{r \geq 0}$.

Proof: - We know that $Z_{p^r} \sim Z_{p^0} \times Z_{p^0} \times Z_{p^r}$

Put n=0 and m=0 in (23), We get $A_{0,0,r} = 2^r$ where $r \geq 0$.

Corollary 5.2 [6]: - Prove that number of all distinct fuzzy subgroups of $\boldsymbol{Z_{p^1} \times Z_{p^r}}$ are $\boldsymbol{2^{r+1} + rp2^r}$ where $\boldsymbol{r \geq 1}$.

Proof: - We know that $\boldsymbol{Z_{p^1} \times Z_{p^r}} \sim Z_{p^1} \times Z_{p^0} \times Z_{p^m}$

Put n=0 and m=1 in (23), we get $A_{1,0,r} = 2^{r+1} + rp2^r$ where $r \geq 1$

Corollary 5.3[1]: - Prove that number of all distinct fuzzy subgroups of $\boldsymbol{Z_{p^m} \times Z_{p^r}}$ are $\sum_{a+h+g=m} \binom{r}{h}\binom{r+g-1}{r-1,g} 2^{a+r} p^{h+g} (-1)^h$ where $\boldsymbol{r \geq m}$

Proof: - We know that $\boldsymbol{Z_{p^m} \times Z_{p^r}} \sim Z_{p^m} \times Z_{p^0} \times Z_{q^r}$

Put n=0 in (23), then we get $A_{m,0,r} = \sum_{a+h+g=m} \binom{r}{h}\binom{r+g-1}{r-1,g} 2^{a+r} p^{h+g} (-1)^h$

6. CONCLUSION

This research work is extended works of the number of all distinct fuzzy subgroups of an abelian p-group of rank two to the number of all distinct fuzzy subgroups of an abelian p-group of rank three. This generalized formula verified the result for cyclic group whose order is power of prime, internal direct product of two cyclic groups whose order is power of prime. In future, this works can be extended for the internal direct of an abelian group of rank three.

REFERENCES

[1] Amit Sehgal, Sarita Sehgal and P.K. Sharma, The Number of Fuzzy Subgroups of a finite Abelian p-Group $Z_{p^m} \times Z_{p^n}$, Advances in Fuzzy Sets and Systems, 21(1) , 2016, pp. 49-57.

[2] Amit Sehgal, Sarita Sehgal and P.K. Sharma, The number of subgroups of a finite abelian p-group of rank two, Journal for Algebra and Number theory Academia, 5(1), February 2015, 23-31.

[3] Amit Sehgal and P.K. Sharma, On Number of Fuzzy Subgroup of a finite Cyclic Group, Proceeding of National Conference on Advances in Mathematics & its Application 25-27th June, 2013 NIT, Hamirpur (H.P) 293-298

[4] A. C. Volf, Counting fuzzy subgroups and chains of subgroups, Fuzzy Syst. Artif. Intell., 10 (2004), 191–200

[5] Amit Sehgal, Sarita Sehgal and P.K. Sharma, The Number of Fuzzy Subgroups of a finite Dihedral $D_{p^m q^n}$, International Journal of Fuzzy Mathematical Archive, 8(1) , 2015, pp. 51-57.

[6] Chen Yanheng, Jiang Youyi and Jai Songfang, On the number of fuzzy subgroups of finite abelian p-group, International Journal of Algebra, Vol. 6, 2012, no. 5, 233-238.

[7] H. Dabari, F. Saeedi and M. Farrokhi D. G., The number of fuzzy Subgroups of some non-abelian groups, Iranian Journal of Fuzzy Systems 10(6) (2013), 101-107

[8] J. M. OH ,An explicit formula for the number of fuzzy subgroups of a finite abelian p-group of rank two, Iranian Journal of Fuzzy Systems 10(6) 2013, 125-133

[9] L.A. Zadeh, Fuzzy Sets, *Information and Control* 8(1965), 338-353

[10] M. Tărnăuceanu and L. Bentea On the number of fuzzy subgroups of finite abelian groups, *Fuzzy Sets and System*, 159(2008), 1084-1096

[11] Ngcibi, Murali and Makamba, Fuzzy subgroups of rank two abelian p-group, Iranian Journal of Fuzzy Systems 7(2010), 149-153

[12] A. Rosenfeld, Fuzzy Groups, *Journal of Math. Anal. And Appl.* 35(3) (1971), 512-517

[13] M. Tărnăuceanu, The number of fuzzy subgroup of finite cyclic groups and Delannoy number, *European Journal of Combinatorics*, 30(2009), 283-286

[14] F. Lazlo, Structure and construction of fuzzy subgroup of finite group, *Fuzzy Sets and System*, 51(1992), 105-109

[15] Y. Zhang and K. Zou, A note on an equivalence relation on fuzzy subgroups, *Fuzzy Sets and System*, 95(1992), 243-247

[16] V. Murali and B.B. Makamba, On an equivalence of fuzzy subgroup I, *Fuzzy Sets and System*, 123(2001), 259-264

[17] V. Murali and B.B. Makamba, On an equivalence of fuzzy subgroup II, *Fuzzy Sets and System*, 136(2003), 93-104

[18] V. Murali and B.B. Makamba, Counting the number of fuzzy subgroup of an abelian group of order $p^n q^m$, *Fuzzy Sets and System*, 144(2004), 459-470

[19] Amit Sehgal, Sarita Sehgal , P.K. Sharma and Manjeet Jakhar , The Number of Fuzzy Subgroups of a finite Abelian p-Group $Z_{p^m q^r} \times Z_{p^n}$, Advances in Fuzzy Sets and Systems, 21(4) , 2016, pp. 291-302.

[20] Amit Sehgal, Sarita Sehgal and P.K. Sharama, The number of subgroups of a finite abelian p -group of rank three, J. Calcutta Math. Soc. 12, No. 2, 137-152 (2016).

[21] Isaac K. Appiah and B. B. Makamba, Counting distinct fuzzy subgroups of some rank-3 Abelian groups, Iranian Journal of Fuzzy Systems 14(1) 2017, 163-181