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Fuzzy Regular Generalized α-Closed Sets and Fuzzy Regular Generalized α-Continuous Functions

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Abstract

In this paper a new type of fuzzy generalized closed set viz., fuzzy regular generalized α-closed set is introduced and studied which is an independent notion of fuzzy generalized closed set defined in [2, 4]. Again a new type of fuzzy generalized continuity termed as fuzzy regular generalized α-continuity is introduced under which fuzzy normality remains invariant.

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1. Introduction and Preliminaries

After introducing fuzzy generalized closed sets in [2], many papers have been published by introducing different types of fuzzy generalized closed sets. In this regard [3, 4, 5, 6, 8] are to be mentioned. In this paper we introduce fuzzy regular generalized α-closed set and establish mutual relationship of this newly defined set with other fuzzy generalized type of closed sets defined earlier. Here also we introduce a new concept of fuzzy neighbourhood (nbd, for short) termed as fuzzy regular generalized α-open nbd of a fuzzy point and have seen that every fuzzy nbd of a fuzzy point is a fuzzy regular generalized α-open nbd but not conversely. Afterwards, fuzzy regular generalized α-continuous function is introduced and have found mutual relationship of this function with other types of fuzzy generalized continuity defined earlier.

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Throughout the paper (X, τ) or simply by X we shall mean a fuzzy topological space (fts, for short) in the sense of Chang [10]. A fuzzy set [18] A which is a mapping from a non-empty set X into the closed interval $I = [0, 1]$, i.e., $A \in I^X$. The support [18] of a fuzzy set A, denoted by $suppA$ and is defined by $suppA = \{x \in X : A(x) \neq 0\}$. The fuzzy set with the singleton support $\{x\} \subseteq X$ and the value t $(0 < t \leq 1)$ will be denoted by x_t. 0_X and 1_X are the constant fuzzy sets taking values 0 and 1 respectively in X. The complement [18] of a fuzzy set A is denoted by $1_X \setminus A$ and is defined by $(1_X \setminus A)(x) = 1 - A(x)$, for each $x \in X$. For any two fuzzy sets A, B in X, $A \leq B$ means $A(x) \leq B(x)$, for all $x \in X$ [18] while AqB means A is quasi-coincident (q-coincident, for short) [15] with B, i.e., there exists $x \in X$ such that $A(x) + B(x) > 1$. The negation of these two statements will be denoted by $A \not\leq B$ and $A \not{q} B$ respectively. For a fuzzy set A, clA and $intA$ will stand for fuzzy closure [10] and fuzzy interior [10] respectively. $A \in I^X$ is called a fuzzy nbd of a fuzzy point x_t if there exists a fuzzy open set G such that $x_t \leq G \leq A$ [15].

A fuzzy set A in an fts (X, τ) is called fuzzy regular open [1] (resp., fuzzy semiopen [1], fuzzy α-open [9], fuzzy β-open [11], fuzzy preopen [14]) if $A = intclA$ (resp., $A \leq clintA$, $A \leq intclintA$, $A \leq clintclA$, $A \leq intclA$). The complement of this fuzzy set in an fts (X, τ) is called fuzzy regular closed [1] (resp., fuzzy semiclosed [1], fuzzy α-closed [9], fuzzy β-closed [11], fuzzy preclosed [14]). The smallest fuzzy semiclosed (resp., fuzzy α-closed, fuzzy β-closed, fuzzy preclosed) set containing a fuzzy set A in an fts X is called fuzzy semiclosure [1] (resp., fuzzy α-closure [9], fuzzy β-closure [11], fuzzy preclosure [14]) and is denoted by $sclA$ (resp., αclA, βclA, $pclA$). A fuzzy set A in an fts X is called fuzzy π-open ($f\pi$-open, for short) [8] set if A can be expressed as the finite union of fuzzy regular open sets in X. The collection of all fuzzy regular open (resp., fuzzy semiopen, fuzzy α-open) sets in X is denoted by $FRO(X)$ (resp., $FSO(X)$, $F\alpha O(X)$) and of fuzzy regular closed (resp., fuzzy semiclosed, fuzzy α-closed) sets in X is denoted by $FRC(X)$ (resp., $FSC(X)$, $F\alpha C(X)$).

2. Fuzzy Generalized Closed Sets : Some Definitions

Let us now recall some definitions for ready references.

Definition 2.1. A fuzzy set A in an fts (X, τ) is called fuzzy

(i) generalized closed (fg-closed, for short) [2] if $clA \leq U$ whenever $A \leq U \in \tau$, The complement of an fg-closed set in an fts X is called an fg-open set in X.

(ii) semi generalized closed (fsg-closed, for short) [3] if $sclA \leq U$ whenever $A \leq U \in FSO(X)$,

(iii) generalized semiclosed (fgs-closed, for short) [3] if $sclA \leq U$ whenever $A \leq U \in \tau$,

(iv) generalized α-closed ($fg\alpha$-closed, for short) [3] if $\alpha clA \leq U$ whenever $A \leq U \in F\alpha O(X)$,

(v) α-generalized closed ($f\alpha g$-closed, for short) [3] if $\alpha clA \leq U$ whenever $A \leq U \in \tau$,

(vi) β-generalized closed ($f\beta g$-closed, for short) [7] if $\beta clA \leq U$ whenever $A \leq U \in \tau$,

(vii) regular generalized closed (frg-closed, for short) [8] if $clA \leq U$ whenever $A \leq U \in FRO(X)$,

(viii) generalized preclosed (fgp-closed, for short) [8] if $pclA \leq U$ whenever $A \leq U \in \tau$,

(ix) generalized preregular closed ($fgpr$-closed, for short) [3] if $pclA \leq U$ whenever $A \leq U \in FRO(X)$,

(x) weakly generalized closed (fwg-closed, for short) [6] if $clintA \leq U$ whenever $A \leq U \in \tau$,

(xi) strongly g-closed (fs^*g-closed, for short) [6] if $clA \leq U$ whenever $A \leq U$ where U is fg-open in X,

(xii) π-generalized closed ($f\pi g$-closed, for short) [6] if $clA \leq U$ whenever $A \leq U$ where U is $f\pi$-open set in X,

(xiii) weakly closed (fw-closed, for short) [6] if $clA \leq U$ whenever $A \leq U \in FSO(X)$.

The complements of the above mentioned closed sets are their respective open sets.

Definition 2.2. A fuzzy set A in an fts (X, τ) is called fuzzy

(i) mildly generalized closed (fmg-closed, for short) if $clintA \leq U$ whenever $A \leq U$ where U is fg-open in X,

(ii) semi weakly generalized closed ($fswg$-closed, for short) if $clintA \leq U$ whenever $A \leq U \in FSO(X)$,

(iii) regular weakly generalized closed ($frwg$-closed, for short) if $clintA \leq U$ whenever $A \leq U \in FRO(X)$.

The complements of the above mentioned closed sets are their respective open sets.

Definition 2.3. [13] A fuzzy set A in an fts (X, τ) is called fuzzy regular semiopen if there is a fuzzy regular open set U in X such that $U \leq A \leq clA$. The family of all fuzzy regular semiopen sets of X is denoted by $FRSO(X)$. The complement of fuzzy regular semiopen set is called fuzzy regular semiclosed.

3. $frg\alpha$-Closed Sets: Some Properties

In this section we have introduced and studied a new type of fuzzy generalized closed set, viz., $frg\alpha$-closed set and established mutual relationships of this closed set with the sets defined in Section 2.

Definition 3.1. A fuzzy set A in an fts (X, τ) is called fuzzy regular α-open ($fr\alpha$-open, for short) if there exists $U \in FRO(X)$ such that $U \leq A \leq \alpha clU$.

The family of all $fr\alpha$-open sets of X is denoted by $FR\alpha O(X)$.

Definition 3.2. A fuzzy set A in an fts (X, τ) is called fuzzy regular generalized α-closed ($frg\alpha$-closed, for short) if $\alpha clA \leq U$ whenever $A \leq U$ and $U \in FR\alpha O(X)$.

The family of all $frg\alpha$-closed sets of X is denoted by $FRG\alpha C(X)$.

Proposition 3.3. $A \in FRO(X) \Rightarrow A \in FR\alpha O(X)$.

Proof. $A \in FRO(X) \Rightarrow A = intclA$. Then $A \leq A \leq \alpha clA$. ■

Proposition 3.4. $A \in FR\alpha O(X) \Rightarrow A \leq \alpha cl(intclA)$.

Proof. $A \in FR\alpha O(X) \Rightarrow$ there exists $U \in FRO(X)$ such that $U \leq A \leq \alpha clU$ (by Definition 3.1) $= \alpha cl(intclU) \leq \alpha cl(intclA) \Rightarrow A \leq \alpha cl(intclA)$. ■

Proposition 3.5. $A \in FR\alpha O(X) \Rightarrow A \in F\beta O(X)$.

Proof. $A \in FR\alpha O(X) \Rightarrow A \leq \alpha cl(intclA)$ (by Proposition 3.4) $\leq cl(intclA) \Rightarrow A \in F\beta O(X)$. ■

Remark 3.6. $frg\alpha$-closedness and $fg\alpha$-closedness are independent notions as it seen from the following examples.

Example 3.7. $fg\alpha$-closed set $\not\Rightarrow frg\alpha$-closed set.
(i) Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = 0.5$, $A(b) = 0.4$. Then (X, τ) is an fts. The collection of all $F\alpha O(X)$ as well as $FRO(X)$ is $\{0_X, 1_X, A\}$ and so that $F\alpha C(X)$ as well as $FRC(X)$ is $\{0_X, 1_X, 1_X \setminus A\}$. Consider the fuzzy sets C and D in X defined by $C(a) = C(b) = 0.5$, $D(a) = 0.5$, $D(b) = 0.45$. We claim that $C \in FR\alpha O(X)$. Indeed, $A \in FRO(X)$ such that $A < C < \alpha clA = 1_X \setminus A \Rightarrow C \in FR\alpha O(X)$. Now $D < C$ where $C \in FR\alpha O(X)$, but $\alpha clD = 1_X \setminus A \not< C \Rightarrow D \notin FRG\alpha C(X)$. But 1_X is the only fuzzy α-open set in X such that $D < 1_X$ and so $\alpha clD = 1_X \setminus A < 1_X \Rightarrow D$ is $fg\alpha$-closed.

(ii) $frg\alpha$-closed $\not\Rightarrow fg\alpha$-closed.
Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = 0.5$, $A(b) = 0.55$. Then (X, τ) is an fts. Here $FRO(X) = \{0_X, 1_X\}$ which implies that $FR\alpha O(X) = \{0_X, 1_X\}$. Now $F\alpha O(X) = \{0_X, 1_X, U\}$ where $U \geq A$ and so $F\alpha C(X) = \{0_X, 1_X, 1_X \setminus U\}$ where $1_X \setminus U \leq 1_X \setminus A$. Consider the fuzzy set B defined by $B(a) = B(b) = 0.6$. Then

$B \in F\alpha O(X)$ such that $B \leq B$, but $\alpha cl B = 1_X \nleq B \Rightarrow B$ is not $fg\alpha$-closed. But 1_X is the only fuzzy regular α-open set in X such that $B < 1_X \Rightarrow \alpha cl B = 1_X \leq 1_X \Rightarrow B$ is $frg\alpha$-closed.

Proposition 3.8. $A \in FR\alpha O(X) \Rightarrow A \in FSO(X)$.

Proof. $A \in FR\alpha O(X) \Rightarrow$ there exists $U \in FRO(X)$ such that $U \leq A \leq \alpha cl U \leq cl U = clint U$ (as $U \in FRO(X) \Rightarrow U$ is fuzzy open in X) $\leq clint A \Rightarrow A \in FSO(X)$. ■

Theorem 3.9. Every fw-closed set in X is $frg\alpha$-closed in X.

Proof. Let A be fw-closed set in X and $U \in FR\alpha O(X)$ such that $A \leq U$. Then by Proposition 3.8, $U \in FSO(X)$. By assumption, $cl A \leq U \Rightarrow \alpha cl A \leq cl A \leq U \Rightarrow A$ is $frg\alpha$-closed in X. ■

The converse of the above theorem need not be true, in general, as it seen from the following example.

Example 3.10. $frg\alpha$-closed set $\not\Rightarrow fw$-closed set.
Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = 0.5$, $A(b) = 0.6$. Then (X, τ) is an fts. Then $FSO(X) = \{0_X, 1_X, U\}$ where $U \geq A$ and 0_X and 1_X are the only fuzzy regular open sets in X. Consider the fuzzy set B in X defined by $B(a) = B(b) = 0.5$. We claim that B is $frg\alpha$-closed but not fw-closed. Indeed, $B < A \in FSO(X)$, but $cl B = 1_X \nleq A \Rightarrow B$ is not fw-closed. Next 1_X is the only fuzzy regular α-open set in X with $B < 1_X$ and so $\alpha cl B < 1_X \Rightarrow B$ is $frg\alpha$-closed.

Theorem 3.11. Every fuzzy closed (resp., fuzzy open) set in an fts X is $frg\alpha$-closed (resp., $frg\alpha$-open) in X.

Proof. Let A be fuzzy closed in X and $U \in FR\alpha O(X)$ with $A \leq U$. Then $\alpha cl A \leq cl A = A \leq U \Rightarrow A$ is $frg\alpha$-closed in X. Similarly we can prove that every fuzzy open set is $frg\alpha$-open. But the converse may not be true, as it seen from the following example. ■

Example 3.12. $frg\alpha$-closed set $\not\Rightarrow$ fuzzy closed set.
Consider Example 3.10. Here B is $frg\alpha$-closed in X but not fuzzy closed.

Since every fuzzy regular closed (resp., regular open) set is fuzzy closed (resp., open), we can easily state the following theorem.

Theorem 3.13. Every fuzzy regular closed (resp., fuzzy regular open) set in X is $frg\alpha$-closed (resp., $frg\alpha$-open) in X.

But the converse may not be true, as it seen from the following example.

Example 3.14. $frg\alpha$-closed set $\not\Rightarrow$ fuzzy regular closed set.

Consider Example 3.10. Here B is $frg\alpha$-closed but not fuzzy regular closed.
Since for any $A \in I^X$, $\beta clA \leq pclA \leq \alpha clA$ and $sclA \leq \alpha clA$, we can easily state the following theorem.

Theorem 3.15. Every $frg\alpha$-closed set in X is an $fgpr$-closed in X.
The converse of the above theorem need not be true, as it seen from the following example.

Example 3.16. $fgpr$-closed set $\not\Leftrightarrow frg\alpha$-closed set.
Consider Example 3.7(i). Here D is not $frg\alpha$-closed. But 1_X is the only fuzzy regular open set in X such that $D < 1_X$ and so $pclD \leq 1_X \Rightarrow D$ is $fgpr$-closed.

Theorem 3.17. Every $frg\alpha$-closed set in X is $f\alpha g$-closed in X.

Proof. Let A be $frg\alpha$-closed in X and U be fuzzy open set in X such that $A \leq U$. Then $U \in FR\alpha O(X)$ by Theorem 3.11 and so by hypothesis, $\alpha clA \leq U \Rightarrow A$ is $f\alpha g$-closed in X. ∎

But the converse may not be true, as it seen from the following example.

Example 3.18. $f\alpha g$-closed set $\not\Leftrightarrow frg\alpha$-closed set.
Consider Example 3.7(i). Here D is not $frg\alpha$-closed. But 1_X is the only fuzzy open set in X such that $D < 1_X$ and so $\alpha clD < 1_X \Rightarrow D$ is $f\alpha g$-closed.

Theorem 3.19. Every $frg\alpha$-closed set in X is fgp-closed in X.

Proof. Let A be $frg\alpha$-closed in X and U be fuzzy open in X such that $A \leq U$. By Theorem 3.11, $U \in FR\alpha O(X)$. By hypothesis, $\alpha clA \leq U \Rightarrow pclA \leq \alpha clA \leq U \Rightarrow$ A is fgp-closed. ∎

But the converse is not true, in general, as it seen from the following example.

Example 3.20. fgp-closed set $\not\Leftrightarrow frg\alpha$-closed set.
Consider Example 3.7(i). Here D is not $frg\alpha$-closed. But 1_X is the only fuzzy open set in X such that $D < 1_X$ and so $pclD < 1_X \Rightarrow D$ is fgp-closed.

Theorem 3.21. Every $frg\alpha$-closed set in X is fgs-closed in X.

Proof. Let A be $frg\alpha$-closed set in X and U be fuzzy open set in X such that $A \leq U$. Then $U \in FR\alpha O(X)$ by Theorem 3.11. By hypothesis, $\alpha clA \leq U \Rightarrow sclA \leq \alpha clA \leq U \Rightarrow A$ is fgs-closed in X. ∎

But the converse is not true, in general, as it seen from th following example.

Example 3.22. fgs-closed set $\not\Leftrightarrow frg\alpha$-closed set.
Consider Example 3.7(i). Here D is not $frg\alpha$-closed. But 1_X is the only fuzzy open set in X such that $D < 1_X$ and so $sclD \leq 1_X \Rightarrow D$ is fgs-closed.

Theorem 3.23. Every $frg\alpha$-closed set in X is $f\beta g$-closed in X.

Proof. Let $A(\in I^X)$ be $frg\alpha$-closed in X and U be a fuzzy open set in X such that $A \leq U$. By hypothesis, $\alpha cl A \leq U \Rightarrow \beta cl A \leq U \Rightarrow A$ is $f\beta g$-closed in X. ■

But the converse may not be true, as it seen from the following example.

Example 3.24. $f\beta g$-closed set $\not\Rightarrow frg\alpha$-closed set.
Consider Example 3.7(i). Here D is not $frg\alpha$-closed. But 1_X is the only fuzzy open set in X such that $D < 1_X$ and so $\beta cl D < 1_X \Rightarrow D$ is $f\beta g$-closed.

Remark 3.25. The following examples show that $frg\alpha$-closedness and fg-closedness, frg-closedness, fsg-closedness, fwg-closedness, $f\pi g$-closedness, $fswg$-closedness, fmg-closedness, fs^*g-closedness are independent notions.

Example 3.26. fg-closed set $\not\Rightarrow frg\alpha$-closed set.
Consider Example 3.7(i). Here D is not $frg\alpha$-closed. Here 1_X is the only fuzzy open set in X with $D < 1_X$ and so $cl D < 1_X \Rightarrow D$ is fg-closed.

Example 3.27. $frg\alpha$-closed set $\not\Rightarrow fg$-closed set.
Consider Example 3.10. Here B is $frg\alpha$-closed. But A being fuzzy open set in X with $B < A$. But $cl B = 1_X \not\leq A \Rightarrow B$ is not fg-closed.

Example 3.28. frg-closed set $\not\Rightarrow frg\alpha$-closed set.
Consider Example 3.7(i). Here D is not $frg\alpha$-closed. But 1_X is the only fuzzy regular open set in X with $D < 1_X \Rightarrow cl D \leq 1_X \Rightarrow D$ is frg-closed.

Example 3.29. $frg\alpha$-closed set $\not\Rightarrow frg$-closed set.
Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A, B\}$ where $A(a) = 0.5, A(b) = 0.4$, $B(a) = 0.7, B(b) = 0.5$. Then (X, τ) is an fts. $F\alpha O(X) = \{0_X, 1_X, A, B, U\}$ where $U \geq B$ and so $F\alpha C(X) = \{0_X, 1_X, 1_X \setminus A, 1_X \setminus B, 1_X \setminus U\}$ where $1_X \setminus U \leq 1_X \setminus B$. Consider the fuzzy set E defined by $E(a) = 0.3, E(b) = 0.4$. We claim that E is $frg\alpha$-closed but not frg-closed. Now $E < A \in FRO(X)$ and by Proposition 3.3, $A \in FR\alpha O(X)$. $\alpha cl E = E < A \Rightarrow E$ is $frg\alpha$-closed. But $cl E = 1_X \setminus B \not\leq A \Rightarrow E$ is not frg-closed.

Example 3.30. $f\pi g$-closed set $\not\Rightarrow frg\alpha$-closed set.
Consider Example 3.7(i). Here D is not $frg\alpha$-closed. But 1_X is the only $f\pi$-open set in X such that $D < 1_X$ and so $cl D < 1_X \Rightarrow D$ is $f\pi g$-closed.

Example 3.31. $frg\alpha$-closed set $\not\Rightarrow f\pi g$-closed.
Consider Example 3.29. Here E is $frg\alpha$-closed. But A is $f\pi$-open set in X with $E < A$, but $cl E = 1_X \setminus B \not\leq A \Rightarrow E$ is not $f\pi g$-closed.

Example 3.32. fwg-closed set $\not\Rightarrow frg\alpha$-closed set.
Consider Example 3.7(i). Here D is not $frg\alpha$-closed. But 1_X is the only fuzzy open set in X with $D < 1_X$ and so $clint D \leq 1_X \Rightarrow D$ is fwg-closed.

Example 3.33. $frg\alpha$-closed set $\not\Rightarrow fwg$-closed set.

Consider the fts defined in 3.29. Consider the fuzzy set C defined by $C(a) = 0.7$, $C(b) = 0.4$. Then 1_X is the only fuzzy regular α-open set in X such that $C < 1_X$ and so $\alpha clC = 1_X \Rightarrow C$ is $frg\alpha$-closed. But $C < B \in \tau$. $clintC = 1_X \setminus B \not\leq B \Rightarrow C$ is not fwg-closed.

Example 3.34. fsg-closed set $\not\Rightarrow frg\alpha$-closed set.
Consider Example 3.7(i). Here D is not $frg\alpha$-closed. $FSO(X) = \{0_X, 1_X, U\} = FSC(X)$ where $A \leq U \leq 1_X \setminus A$. So $D \leq D \in FSO(X)$ and $sclD = D \leq D \Rightarrow D$ is fsg-closed.

Example 3.35. $frg\alpha$-closed set $\not\Rightarrow fsg$-closed set.
Consider fts defined in Example 3.10. Let F be a fuzzy set in X defined by $F(a) = F(b) = 0.6$. As 1_X is the only fuzzy regular α-open set in X with $F < 1_X, \alpha clF \leq 1_X \Rightarrow F$ is $frg\alpha$-closed. Now $FSO(X) = \{0_X, 1_X, V\}$ where $V \geq A$ and so $FSC(X) = \{0_X, 1_X, 1_X \setminus V\}$ where $1_X \setminus V \leq 1_X \setminus A$. Now $F \leq F \in FSO(X)$, but $sclF = 1_X \not\leq F \Rightarrow F$ is not fsg-closed.

Example 3.36. $frg\alpha$-closed set $\not\Rightarrow fswg$-closed set.
Consider Example 3.35. Here F is $frg\alpha$-closed. Now $F \leq F \in FSO(X)$ and $cl(intF) = clA = 1_X \not\leq F \Rightarrow F$ is not $fswg$-closed.

Example 3.37. $fswg$-closed set $\not\Rightarrow frg\alpha$-closed set.
Consider fts defined in Example 3.7(i) and the fuzzy set E defined by $E(a) = 0.5$, $E(b) = 0.3$. Here $FSO(X) = \{0_X, 1_X, U\}$ where $A \leq U \leq 1_X \setminus A$. Now $E < A \in FSO(X)$ and $cl(intE) = 0_X < A \Rightarrow E$ is $fswg$-closed. But $E < A \in FR\alpha O(X)$ and $\alpha clE = 1_X \setminus A \not\leq A \Rightarrow E$ is not $frg\alpha$-closed.

Example 3.38. fmg-closed set $\not\Rightarrow frg\alpha$-closed set.
Consider the fts defined in Example 3.29 and consider the fuzzy set F defined by $F(a) = F(b) = 0.4$. Now A is fg-open in X such that $F < A$. Then $cl(intF) = cl0_X = 0_X < A \Rightarrow F$ is fmg-closed. Again $F < A \in FR\alpha O(X)$, but $\alpha clF = 1_X \setminus A \not\leq A \Rightarrow F$ is not $frg\alpha$-closed.

Example 3.39. $frg\alpha$-closed set $\not\Rightarrow fmg$-closed set.
Consider Example 3.10 and the fuzzy set A. Here A is fg-open in X and $A \leq A$. But $cl(intA) = clA = 1_X \not\leq A \Rightarrow A$ is not fmg-closed. But 1_X is the only fuzzy regular α-open set in X with $A < 1_X$ and so $\alpha clA < 1_X \Rightarrow A$ is $frg\alpha$-closed.

Example 3.40. $frg\alpha$-closed set $\not\Rightarrow fs^*g$-closed set.
Consider the fts defined in Example 3.29 and the fuzzy set C defined by $C(a) = 0.3, C(b) = 0.4$. Here $C < A \in FR\alpha O(X)$ and $\alpha clC = C < A \Rightarrow C$ is $frg\alpha$-closed. Now A is fg-open in X with $C < A$, but $clC = 1_X \setminus B \not\leq A \Rightarrow C$ is not fs^*g-closed.

Example 3.41. fs^*g-closed set $\not\Rightarrow frg\alpha$-closed set.
Consider Example 3.7(i). Here the collection of fg-open sets in X is $\{0_X, 1_X, U\}$ where

$U \leq A$. 1_X is the only fg-open set in X with $D < 1_X$ and so $clD \leq 1_X \Rightarrow D$ is fs^*g-closed in X. But D is not $frg\alpha$-closed.

Example 3.42. $frwg$-closed set $\not\Rightarrow frg\alpha$-closed set.
Consider Example 3.7(i). Here D is not $frg\alpha$-closed. Here 1_X is the only fuzzy regular open set in X with $D < 1_X$ and so $cl(intD) \leq 1_X \Rightarrow D$ is $frwg$-closed.

Theorem 3.43. Union of two $frg\alpha$-closed sets in an fts X is $frg\alpha$-closed.

Proof. Let A, B be two $frg\alpha$-closed sets in X and $A \bigvee B \leq U \in FR\alpha O(X)$. Then $A \leq U, B \leq U$. By hypothesis, $\alpha clA \leq U, \alpha clB \leq U \Rightarrow \alpha cl(A \bigvee B) = \alpha clA \bigvee \alpha clB \leq U \Rightarrow A \bigvee B$ is $frg\alpha$-closed in X. ■

Remark 3.44. From the above theorem it can be stated that "Intersection of two $frg\alpha$-open sets in X is $frg\alpha$-open in X". But intersection of two $frg\alpha$-closed sets may not be so as it seen from the following example.

Example 3.45. Let $X = \{a, b\}, \tau = \{0_X, 1_X, A\}$ where $A(a) = 0.5$, $A(b) = 0.4$. Then (X, τ) is an fts. Then $F\alpha O(X) = FRO(X) = \{0_X, 1_X, A\}$ and that of $F\alpha C(X) = FRC(X) = \{0_X, 1_X, 1_X \setminus A\}$. Consider two fuzzy sets F_1 and F_2 in X defined by $F_1(a) = 0.6, F_1(b) = 0.45, F_2(a) = 0.5, F_2(b) = 0.7$. Then 1_X is the only fuzzy regular α-open set in X with $F_1 \leq 1_X, F_2 \leq 1_X$ and so $sclF_1 = 1_X, sclF_2 = 1_X \Rightarrow F_1$ and F_2 are $frg\alpha$-closed in (X, τ). But $D = F_1 \bigwedge F_2$ is not $frg\alpha$-closed set in (X, τ) Example 3.7(i).

Theorem 3.46. If a fuzzy set A is fuzzy regular open and $frg\alpha$-closed in an fts X, then A is fuzzy α-clopen in X.

Proof. $A \in FRO(X) \Rightarrow A$ is fuzzy open in $X \Rightarrow A \in F\alpha O(X)$. Let $U \in FR\alpha O(X)$ with $A \leq U$. Now by Proposition 3.3, $A \in FR\alpha O(X)$. Then $A \leq A \leq U \Rightarrow \alpha clA \leq A \leq U \Rightarrow \alpha clA \leq A \Rightarrow A = \alpha clA \Rightarrow A \in F\alpha C(X) \Rightarrow A$ is fuzzy α-clopen in X. ■

Theorem 3.47. If A is $frg\alpha$-closed in X such that $A \leq B \leq \alpha clA$, then B is $frg\alpha$-closed in X.

Proof. Let $B \leq U \in FR\alpha O(X)$. Then $A \leq B \leq U$. By hypothesis, $\alpha clA \leq U$. So $B \leq \alpha clA \Rightarrow \alpha clB \leq \alpha clA \leq U \Rightarrow B$ is $frg\alpha$-closed in X. ■

The converse of the above theorem need not be true as it seen from the following example.

Example 3.48. Let $X = \{a, b\}, \tau = \{0_X, 1_X, A\}$ where $A(a) = 0.5$, $A(b) = 0.4$. Then (X, τ) is an fts. $F\alpha O(X) = FRO(X) = \{0_X, 1_X, A\}$ and $F\alpha C(X) = \{0_X, 1_X, 1_X \setminus A\}$. Consider two fuzzy sets C and D defined by $C(a) = 0.5, C(b) = 0.6, D(a) = D(b) = 0.6$. We first prove that C and D are $frg\alpha$-closed in X. Now as $A < C < \alpha clA = C$,

$C \in FR\alpha O(X)$. Then $C \leq C$ and $\alpha clC = 1_X \setminus A = C \leq C \Rightarrow C$ is $frg\alpha$-closed in X. Again 1_X is the only fuzzy regular α-open set in X with $D < 1_X$ and so $\alpha clD = 1_X \Rightarrow D$ is $frg\alpha$-closed in X. Again $C < D$, but $\alpha clC = 1_X \setminus A \not\geq D$, i.e., $C < D \not\leq \alpha clC$.

Theorem 3.49. If a fuzzy set A is fuzzy regular open and frg-closed in X, then A is $frg\alpha$-closed in X.

Proof. $A \in FRO(X)$, $A \leq A \Rightarrow clA \leq A$ (as A is frg-closed) $\Rightarrow \alpha clA \leq clA \leq A \Rightarrow \alpha clA = A$. Let $U \in FR\alpha O(X)$ be such that $A \leq U$. Then $\alpha clA = A \leq U \Rightarrow A$ is $frg\alpha$-closed in X. ∎

Theorem 3.50. If $A \in I^X$ be both fuzzy regular α-open and $frg\alpha$-closed in X, then $A \in F\alpha C(X)$.

Proof. $A \leq A \in FR\alpha O(X) \Rightarrow \alpha clA \leq A$ (as A is $frg\alpha$-closed) $\Rightarrow A \in F\alpha C(X)$. ∎

Theorem 3.51. Let $A \in FR\alpha O(X)$ and $frg\alpha$-closed in X, $B \in F\alpha C(X)$. Then $A \bigwedge B \in FRG\alpha C(X)$.

Proof. By Theorem 3.50, $A \in F\alpha C(X)$. Since intersection of two fuzzy α-closed sets is fuzzy α-closed, $A \bigwedge B \in F\alpha C(X)$. Since every fuzzy α-closed set is $frg\alpha$-closed, $A \bigwedge B \in FRG\alpha C(X)$. ∎

Theorem 3.52. If $A(\in I^X)$ be fuzzy open and fg-closed in X, then $A \in FRG\alpha C(X)$.

Proof. Let $A \leq U \in FR\alpha O(X)$. As A is fuzzy open and fg-closed, $clA \leq A \Rightarrow \alpha clA \leq clA \leq A \leq U \Rightarrow A \in FRG\alpha C(X)$. ∎

But the converse is not true, in general, as it seen from the following example.

Example 3.53. Consider Example 3.10. Here A is fuzzy open and $frg\alpha$-closed in X. Indeed, 1_X is the only fuzzy regular α-open set in X with $A < 1_X$ and so $\alpha clA \leq 1_X \Rightarrow A$ is $frg\alpha$-closed in X. But $A \leq A \Rightarrow clA = 1_X \not\leq A \Rightarrow A$ is not fg-closed in X.

Remark 3.54. Infact in an fts (X, τ), if $FR\alpha O(X) = \{0_X, 1_X\}$, then any $A \in I^X$ is $frg\alpha$-closed. But the converse may not be true, in general, as it seen from the following example.

Example 3.55. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, B\}$ where $B(a) = B(b) = 0.5$. Then (X, τ) is an fts. Now $FRO(X) = F\alpha O(X) = \{0_X, 1_X, B\}$ and that of $F\alpha C(X) = \{0_X, 1_X, 1_X \setminus B\}$. Then $B \in FR\alpha O(X)$. Let $A \in I^X$ be arbitrary. Then either $A \leq B$ or $A \not\leq B$. If $A \leq B$, then $\alpha clA = 1_X \setminus B = B \Rightarrow A \in FRG\alpha C(X)$. If $A \not\leq B$, then 1_X is the only fuzzy regular α-open set in X such that $A \leq 1_X \Rightarrow \alpha clA \leq 1_X \Rightarrow A \in FRG\alpha C(X)$.

Theorem 3.56. In an fts (X, τ), $FR\alpha O(X) \subset \{G \in I^X : G^c \in F\alpha O(X)\}$ iff every $A \in I^X$ is $frg\alpha$-closed in X.

Proof. Let $FR\alpha O(X, \tau) \subset \{G \in I^X : G^c \in F\alpha O(X)\}$. Let $A \in I^X$ be such that $A \leq U \in FR\alpha O(X)$. Then $U \in \{G \in I^X : G^c \in F\alpha O(X)\} \Rightarrow U \notin F\alpha C(X)$ and so $\alpha clA \leq \alpha clU = U \Rightarrow A$ is $frg\alpha$-closed in X.

Conversely, suppose that every $A \in I^X$ be $frg\alpha$-closed in X. Let $U \in FR\alpha O(X, \tau)$. Then $U \leq U \Rightarrow \alpha clU \leq U$ (by hypothesis) $\Rightarrow U \in F\alpha C(X) \Rightarrow U^c \in F\alpha O(X) \Rightarrow U \in \{G \in I^X : G^c \in F\alpha O(X)\}$. ■

Remark 3.57. It is clear from definitions that the notions $F\alpha O(X)$ and $FR\alpha O(X)$ are independent follows from next two examples.

Example 3.58.

(i) $F\alpha O(X) \not\Leftrightarrow FR\alpha O(X)$
Let $X = \{a\}, \tau = \{0_X, 1_X, A\}$ where $A(a) = 0.6$. Then (X, τ) is an fts. Consider the fuzzy set B in X defined by $B(a) = 0.7$. Then $intclintB = intclA = int1_X = 1_X \geq B \Rightarrow B \in F\alpha O(X)$. Clearly $FRO(X) = \{0_X, 1_X\}$ and so $B \notin FR\alpha O(X)$.

(ii) $FR\alpha O(X) \not\Leftrightarrow F\alpha O(X)$
Let $X = \{a, b\}, \tau = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.4$. Then (X, τ) is an fts. Clearly 0_X and 1_X are the only fuzzy α-open sets in X. Now $FRO(X) = \{0_X, 1_X, A\}$. Consider the fuzzy set B defined by $B(a) = B(b) = 0.5$. We claim that $B \in FR\alpha O(X)$ but not in $F\alpha O(X)$. It is clear that B is not fuzzy α-open in X. Now $A \in FRO(X)$ such that $A \leq B \leq \alpha clA = 1_X \setminus A$ and hence B is fuzzy regular α-open set in X.

4. $frg\alpha$-nbd of a Fuzzy Point and a Fuzzy Set

Definition 4.1. A fuzzy set A in an fts (X, τ) is called an $frg\alpha$-nbd of a fuzzy point x_α in X if there exists an $frg\alpha$-open set G in X such that $x_\alpha \in G \leq A$.

Definition 4.2. A fuzzy set A in an fts (X, τ) is called an $frg\alpha$-nbd of a fuzzy set B in X if there exists an $frg\alpha$-open set G in X such that $B \leq G \leq A$.

Remark 4.3. The $frg\alpha$-nbd of a fuzzy point x_α need not be an $frg\alpha$-open set in X, as it seen from the following example.

Example 4.4. Consider Example 3.7(i). Here $1_X \setminus D$ is not $frg\alpha$-open in X.
Let E be a fuzzy set in X defined by $E(a) = 0.5, E(b) = 0.7$. We claim that E is $frg\alpha$-closed in X. Now 1_X is the only fuzzy regular α-open set in X with $E < 1_X$. Then $\alpha clE < 1_X \Rightarrow E$ is $frg\alpha$-closed in X and so $1_X \setminus E$ is $frg\alpha$-open in X. Now consider the fuzzy point $a_{0.5}$. Then $(1_X \setminus E)(a) = 0.5, (1_X \setminus E)(b) = 0.3$ and so $a_{0.5} \in 1_X \setminus E < 1_X \setminus D \Rightarrow 1_X \setminus D$ is $frg\alpha$-nbd of $a_{0.5}$.

It is obvious from definition that

Theorem 4.5. Every nbd of a fuzzy point x_α is an $frg\alpha$-nbd of it.
But the converse may not be true, as it seen from the following example.

Example 4.6. Consider Example 4.4. Here $1_X \setminus E$ being an *frg*α-open set is *frg*α-nbd of $a_{0.5}$, but $1_X \setminus E$ is not a fuzzy nbd of $a_{0.5}$.

Remark 4.7. An *frg*α-open set is an *frg*α-nbd of each of its points. But the converse may not be true, in general, as it seen from the following example.

Example 4.8. Consider Example 4.4. Here $1_X \setminus D$ is not *frg*α-open. We claim that $1_X \setminus D$ is an *frg*α-nbd of each of its points. The points of $1_X \setminus D$ are either of the form a_t where $0 < t \leq 0.5$ or of the form $b_{t'}$ where $0 < t' \leq 0.55$. For the points a_t, $0 < t < 0.5$, $1_X \setminus E$ is the *frg*α-open set in X such that $a_t \leq 1_X \setminus E < 1_X \setminus D \Rightarrow 1_X \setminus D$ is an *frg*α-nbd of a_t. For the points $b_{t'}$, $0 < t' \leq 0.55$, consider the fuzzy set F defined by $F(a) = 0.6$, $F(b) = 0.45$. Then 1_X is the only fuzzy regular α-open set with $F < 1_X$ and so $\alpha clF \leq 1_X \Rightarrow F$ is *frg*α-closed in $X \Rightarrow 1_X \setminus F$ is *frg*α-open in X. Now $b_{t'} \in 1_X \setminus F < 1_X \setminus D \Rightarrow 1_X \setminus D$ is an *frg*α-nbd of $b_{t'}$. Consequently, $1_X \setminus D$ is an *frg*α-nbd of each of its points.

Theorem 4.9. Let $F \in I^X$ be *frg*α-closed set in X and $x_\alpha \in 1_X \setminus F$. Then there exists an *frg*α-nbd G of x_α such that $G \not{q} F$.

Proof. $1_X \setminus F$ is *frg*α-open in X. By Remark 4.7, there exists an *frg*α-nbd G of x_α such that $x_\alpha \in G \leq 1_X \setminus F \Rightarrow G \not{q} F$. ■

Definition 4.10. The set of all *frg*α-nbds of a fuzzy point x_t $(0 < t \leq 1)$ in an fts (X, τ) is called the *frg*α-nbd system at x_t, to be denoted by *frg*α $-N(x_t)$.

Theorem 4.11. For a fuzzy point x_t in an fts (X, τ), the following statements hold:

(i) *frg*α-$N(x_t) \neq \phi$,

(ii) $G \in$ *frg*α-$N(x_t) \Rightarrow x_t \in G$,

(iii) $G \in$ *frg*α-$N(x_t)$, $F \geq G \Rightarrow F \in frg\alpha - N(x_t)$,

(iv) $F, G \in$ *frg*α-$N(x_t) \Rightarrow F \bigwedge G \in frg\alpha - N(x_t)$,

(v) $G \in$ *frg*α-$N(x_t) \Rightarrow$ there exists $F \in frg\alpha - N(x_t)$ such that $F \leq G$ and $F \in frg\alpha - N(y_{t'})$ for every $y_{t'} \in F$.

Proof.

(i) Since 1_X is $frg\alpha$-open set, it is an *frg*α-nbd of any fuzzy point x_t $(0 < t \leq 1)$ and so *frg*α-$N(x_t) \neq \phi$.

(ii) and (iii) follow from definition of $frg\alpha$-$N(x_t)$.

(iv) Follows from Remark 3.44.

(v) Follows from Definition 4.10 and Remark 4.7. ■

Theorem 4.12. Let x_t be a fuzzy point in an fts (X, τ). Let $frg\alpha$-$N(x_t)$ be a non-empty collection of fuzzy sets in X satisfying the following conditions:

(1) $G \in frg\alpha\text{-}N(x_t) \Rightarrow x_t \in G$,

(2) $F, G \in frg\alpha\text{-}N(x_t) \Rightarrow F \bigwedge G \in frg\alpha\text{-}N(x_t)$.

Let τ consist of 0_X and all those non zero fuzzy sets G of X having the property that $x_t \in G \Rightarrow$ there exists an $F \in frg\alpha\text{-}N(x_t)$ such that $x_t \in F \leq G$. Then τ is a fuzzy topology on X.

Proof.

(i) By hypothesis, $0_X \in \tau$.

(ii) It is clear from the given property of τ that $1_X \in \tau$ as 1_X is an $frg\alpha$-$N(x_t)$ for every fuzzy point $x_t (0 < t < 1)$ in an fts X by (1).

(iii) Let $G_1, G_2 \in \tau$. If $G_1 \bigwedge G_2 = 0_X$, then $G_1 \bigwedge G_2 \in \tau$. If $G_1 \bigwedge G_2 \neq 0_X$, let $x_t \in G_1 \bigwedge G_2$ (where $0 < t < 1$). Then $G_1(x) \geq t, G_2(x) \geq t$. Since $G_1, G_2 \in \tau$, there exist $F_1, F_2 \in frg\alpha\text{-}N(x_t)$ such that $x_t \in F_1 \leq G_1, x_t \in F_2 \leq G_2$. Then $x_t \in F_1 \bigwedge F_2 \leq G_1 \bigwedge G_2$ and by (2), $F_1 \bigwedge F_2 \in frg\alpha - N(x_t) \Rightarrow G_1 \bigwedge G_2 \in \tau$ by construction of τ.

(iv) Let $\mathcal{G} = \{G_\alpha : \alpha \in \Lambda\}$ where $G_\alpha \in \tau$, for all $\alpha \in \Lambda$. Let $x_t \in \bigvee_{\alpha \in \Lambda} G_\alpha$. Then $x_t \in G_\beta$, for some $\beta \in \Lambda$. By construction of τ, there exists $F_\beta \in frg\alpha\text{-}N(x_t)$ such that $x_t \in F_\beta \leq G_\beta \leq \bigvee_{\alpha \in \Lambda} G_\alpha \Rightarrow \bigvee_{\alpha \in \Lambda} G_\alpha \in \tau$.

It follows that τ is a fuzzy topology on X. ■

5. *frg*α-Continuous Function: Some Properties

In this section we first introduce some spaces in which the converses of Theorem 3.6, Theorem 3.9, Theorem 3.11, Theorem 3.13, Theorem 3.15, Theorem 3.17, Theorem 3.19, Theorem 3.21 and Theorem 3.23 are true. After that $frg\alpha$-continuous function is introduced and studied and establish mutual relationships of this function with functions defined earlier in [3, 8, 16].

Definition 5.1. An fts (X, τ) is said to be an

(i) fT_g-space [6] if every fg-closed set is fuzzy closed,

(ii) fT_{g^*}-space [6] if every fs^*g-closed set is fuzzy closed,

(iii) $f\beta T_b$-space [7] if every $f\beta g$-closed set is fuzzy closed,

(iv) $f\alpha T_b$-space [3] if every $f\alpha g$-closed set is fuzzy closed,

(v) fT_b-space [3] if every fgs-closed set is fuzzy closed,

(vi) fT_α-space if every $fg\alpha$-closed set is fuzzy closed,

(vii) frT_g-space if every $frwg$-closed set is fuzzy closed,

(viii) fT_r-space if every *frg*-closed set is fuzzy closed,

(ix) fsT_g-space if every $fswg$-closed set is fuzzy closed,

(x) fT_w-space if every fwg-closed set is fuzzy closed,

(xi) fT_p-space if every fgp-closed set is fuzzy closed,

(xii) fmT_g-space if every fmg-closed set is fuzzy closed,

(xiii) fT_{sg}-space [6] if every fsg-closed set is fuzzy closed,

(xiv) fT_π-space if every $f\pi g$-closed set is fuzzy closed,

(xv) fgT_p-space if every $fgpr$-closed set is fuzzy closed,

(xvi) frT_α-space if every $frg\alpha$-closed set is fuzzy closed,

Using Definition 5.1, we can easily state the following Theorem.

Theorem 5.2. In an

(i) fT_g-space every fg-closed set is $frg\alpha$-closed,

(ii) fT_{g^*}-space every fs^*g-closed set is $frg\alpha$-closed,

(iii) $f\beta T_b$-space every $f\beta g$-closed set is $frg\alpha$-closed,

(iv) $f\alpha T_b$-space every $f\alpha g$-closed set is $frg\alpha$-closed,

(v) fT_b-space every fgs-closed set is $frg\alpha$-closed,

(vi) fT_α-space every $fg\alpha$-closed set is $frg\alpha$-closed,

(vii) frT_g-space every $frwg$-closed set is $frg\alpha$-closed,

(viii) fT_r-space every frg-closed set is $frg\alpha$-closed,

(ix) fsT_g-space every $fswg$-closed set is $frg\alpha$-closed,

(x) fT_w-space every fwg-closed set is $frg\alpha$-closed,

(xi) fT_p-space every fgp-closed set is $frg\alpha$-closed,

(xii) fmT_g-space every fmg-closed set is $frg\alpha$-closed,

(xiii) fT_π-space every $f\pi g$-closed set is $frg\alpha$-closed,

(xiv) fT_{sg}-space every fsg-closed set is $frg\alpha$-closed,

(xv) fgT_p-space every $fgpr$-closed set is $frg\alpha$-closed.

Definition 5.3. A function $f : (X, \tau) \to (Y, \tau_1)$ is said to be fuzzy

(i) continuous [16] if $f^{-1}(V) \in \tau^c$ for every $V \in \tau_1^c$,

(ii) open (resp., closed [17]) [17] function if $f(V) \in \tau_1$ (resp., $f(V) \in \tau_!^c$) for every $V \in \tau$ (resp., $V \in \tau^c$),

(iii) fgs-continuous [3] if $f^{-1}(V)$ is fgs-open in X for every $V \in \tau_1$,

(iv) fgp-continuous [8] if $f^{-1}(V)$ is fgp-closed in X for every $V \in \tau_1^c$.

Definition 5.4. A function $f : (X, \tau) \to (Y, \tau_1)$ is said to be

(i) $frg\alpha$-open (resp., $frg\alpha$-closed) if $f(U) \in FRG\alpha O(Y)$ (resp., $f(U) \in FRG\alpha C(Y)$) for all $U \in \tau$ (resp., $U \in \tau^c$)

(ii) $frg\alpha$-continuous if $f^{-1}(V) \in FRG\alpha O(X)$ (resp., $f^{-1}(U) \in FRG\alpha C(X)$) for all $U \in \tau_1$ (resp., $U \in \tau_1^c$),

(iii) $frg\alpha^*$-continuous if $f^{-1}(U) \in FRG\alpha O(X)$ (resp., $f^{-1}(U) \in FRG\alpha C(X)$) for all $U \in FRG\alpha O(Y)$ (resp., $U \in FRG\alpha C(Y)$).

Theorem 5.5. Every fuzzy continuous function is $frg\alpha$-continuous.

Proof. The proof follows from the fact that every fuzzy closed set is $frg\alpha$-closed. ∎

The converse of the above theorem need not be true, as it seen from the following example.

Example 5.6. $frg\alpha$-continuity $\not\Rightarrow$ fuzzy continuity.
Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$, $\tau_1 = \{0_X, 1_X, B\}$ where $A(a) = 0.5$, $A(b) = 0.6$, $B(a) = B(b) = 0.5$. Then (X, τ) and (X, τ_1) are fts's. Consider the identity function $i : (X, \tau) \to (X, \tau_1)$. Now $B \in \tau_1^c$, but $i^{-1}(B) = B \notin \tau^c \Rightarrow i$ is not fuzzy continuous.

Now 1_X is the only fuzzy regular α-open set in (X, τ) such that $B < 1_X$ for all $B \in I^X$ and so $\alpha clB \leq 1_X \Rightarrow B$ is $frg\alpha$-closed in (X, τ). This is true for any fuzzy set B in $(X, \tau) \Rightarrow i$ is $frg\alpha$-continuous.

Theorem 5.7. Every $frg\alpha^*$-continuous function is $frg\alpha$-continuous.

Proof. The proof follows from Theorem 3.11. ■

But the converse may not be true, as it seen from the following example.

Example 5.8. $frg\alpha$-continuous $\not\Rightarrow$ $frg\alpha^*$-continuous.
$X = \{a, b\}$, $\tau = \{0_X, 1_X, A, B\}$, $\tau_1 = \{0_X, 1_X, C\}$ where $A(a) = 0.5, A(b) = 0.4$, $B(a) = 0.6, B(b) = 0.5$ and $C(a) = C(b) = 0.6$. Then (X, τ) and $X, \tau_1)$ are fts's. Then $F\alpha O(X, \tau) = \{0_X, 1_X, A, B, U\}$ where $U \geq B$ and that of $F\alpha C(X, \tau) = \{0_X, 1_X, 1_X \setminus A, 1_X \setminus B, 1_X \setminus U\}$ where $1_X \setminus U \leq 1_X \setminus B$ and $FRO(X, \tau) = \{0_X, 1_X, A\}$. Consider the identity function $i : (X, \tau) \to (X, \tau_1)$. Here $1_X \setminus C \in \tau_1^c$, $i^{-1}(1_X \setminus C) = 1_X \setminus C < A \in FR\alpha O(X, \tau) \Rightarrow \alpha cl(1_X \setminus C) = 1_X \setminus C < A \Rightarrow 1_X \setminus C \in FRG\alpha C(X, \tau) \Rightarrow i$ is $frg\alpha$-continuous.
Again every fuzzy set in (X, τ_1) is $frg\alpha$-closed in (X, τ_1). Let us consider the fuzzy set D in (X, τ_1) defined by $D(a) = 0.5, D(b) = 0.3$. Then D is $frg\alpha$-closed in (X, τ_1). $i^{-1}(D) = D < A \in FR\alpha O(X, \tau)$, but $\alpha clD = 1_X \setminus A \not\leq A \Rightarrow D$ is not $frg\alpha$-closed in (X, τ).

Theorem 5.9. Every fuzzy open (resp., fuzzy closed) function is $frg\alpha$-open (resp., $frg\alpha$-closed).

Proof. The proof follows from Theorem 3.11. ■

But the converse need not be true, as it seen from the following example.

Example 5.10. $frg\alpha$-closed function $\not\Rightarrow$ fuzzy closed function. Consider Example 5.8. Since every fuzzy set in (X, τ_1) is $frg\alpha$-closed, so i is $frg\alpha$-closed function. Now $1_X \setminus A \in \tau^c$. $i(1_X \setminus A) = 1_X \setminus A \notin \tau_!^c \Rightarrow i$ is not fuzzy closed function.

Theorem 5.11. Composition of two $frg\alpha^*$-continuous functions is $frg\alpha^*$-continuous.

Proof. Obvious. ■

Theorem 5.12. Let $f : (X, \tau) \to (Y, \tau_1)$ be $frg\alpha^*$-continuous and $g : (Y, \tau_1) \to (Z, \tau_2)$ be $frg\alpha$-continuous functions. Then $g \circ f : (X, \tau) \to (Z, \tau_2)$ is $frg\alpha$-continuous.

Proof. Let $V \in \tau_2^c$. Then $g^{-1}(V) \in FRG\alpha C(Y, \tau_1)$. By hypothesis, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V) \in FRG\alpha C(X, \tau)$ and as a result, $g \circ f$ is $frg\alpha$-continuous. ■

Remark 5.13. The composition of two $fr g\alpha$-continuous functions need not so, as it seen from the following example.

Example 5.14. Let $X = \{a, b\}, \tau = \{0_X, 1_X, A, B\}, \tau_1 = \{0_X, 1_X, C\}, \tau_2 = \{0_X, 1_X, D\}$ where $A(a) = 0.5$, $A(b) = 0.4$, $B(a) = 0.6$, $B(b) = 0.5$, $C(a) = C(b) = 0.6$ and $D(a) = 0.5, D(b) = 0.7$. Then (X, τ), (X, τ_1) and (X, τ_2) are fts's. Consider two identity functions $i : (X, \tau) \to (X, \tau_1)$ and $i_1 : (X, \tau_1) \to (X, \tau_2)$. In (X, τ_1) and (X, τ_2) every fuzzy set is $frg\alpha$-closed (as 0_X and 1_X are the only fuzzy regular open sets in (X, τ_1) and (X, τ_2)). In Example 5.8, it is seen that the fuzzy closed set $1_X \setminus C$ in (X, τ_1) is $frg\alpha$-closed in (X, τ). So i and i_1 are $frg\alpha$-continuous functions. Now $1_X \setminus D \in \tau_2^c$. But $(i_1 \circ i)^{-1}(1_X \setminus D) = 1_X \setminus D$ is not $frg\alpha$-closed in $(X, \tau) \Rightarrow i_1 \circ i$ is not $frg\alpha$-continuous function.

Remark 5.15. Let $f : (X, \tau) \to (Y, \tau_1)$, $g : (Y, \tau_1) \to (Z, \tau_2)$ be $frg\alpha$-continuous functions where (Y, τ_1) is frT_α-space. Then $g \circ f : (X, \tau) \to (Z, \tau_2)$ is $frg\alpha$-continuous function.

Remark 5.16. Composition of two $frg\alpha$-open ($frg\alpha$-closed) functions need not be so, as it seen from the following example.

Example 5.17. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$, $\tau_1 = \{0_X, 1_X, B\}$ and $\tau_2 = \{0_X, 1_X, C, D\}$ where $A(a) = 0.5, A(b) = 0.7$, $B(a) = B(b) = 0.6$, $C(a) = 0.5, C(b) = 0.4$, $D(a) = 0.6, D(b) = 0.5$. Then (X, τ), (X, τ_1) and (X, τ_2) are fts's. Consider the identity functions $i : (X, \tau) \to (X, \tau_1)$ and $i_1 : (X, \tau_1) \to (X, \tau_2)$. Now 0_X and 1_X are the only fuzzy regular open sets in $X, \tau_1)$ and so every fuzzy set is $frg\alpha$-closed in (X, τ_1) and so i is $frg\alpha$-open (resp., $frg\alpha$-closed) function. Now $1_X \setminus B \in \tau_1^c$ and $i_1(1_X \setminus B) = 1_X \setminus B$ is $frg\alpha$-closed in (X, τ_2) (as shown in Example 5.8) $\Rightarrow i_1$ is $frg\alpha$-closed function. Again $B \in \tau_1$, $i_1(B) = B$ is $frg\alpha$-open in $(X, \tau_2) \Rightarrow i_1$ is $frg\alpha$-open function. Now $1_X \setminus A \in \tau^c$, $(i_1 \circ i)(1_X \setminus A) = 1_X \setminus A < C \in FR\alpha O(X, \tau_2)$. Indeed, $F\alpha O(X, \tau_2) = \{0_X, 1_X, A, B, U\}$ where $U \geq B$ and $F\alpha C(X, \tau_2) = \{0_X, 1_X, 1_X \setminus A, 1_X \setminus B, 1_X \setminus U\}$ where $1_X \setminus U \leq 1_X \setminus B$ and $FRO(X, \tau_2) = \{0_X, 1_X, C\}$. But $\alpha cl(1_X \setminus A) = 1_X \setminus C \not\leq C \Rightarrow i_1 \circ i$ is not $frg\alpha$-closed (resp., $frg\alpha$-open) function.

Remark 5.18. Let $f : (X, \tau) \to (Y, \tau_1)$ and $g : (Y, \tau_1) \to (Z, \tau_2)$ be $frg\alpha$-open (resp., $frg\alpha$-closed) functions where (Y, τ_1) is frT_α-space. Then $g \circ f : (X, \tau) \to (Z, \tau_2)$ is $frg\alpha$-open (resp., $frg\alpha$-closed) function.

Theorem 5.19. Every $frg\alpha$-continuous function is fgs-continuous.

Proof. The proof follows from Theorem 3.21. ■

But the converse it not true, in general, as it seen from the following example.

Example 5.20. fgs-continuity $\not\Rightarrow frg\alpha$-continuity.
Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$, $\tau_1 = \{0_X, 1_X, B\}$ where $A(a) = 0.5, A(b) = 0.4$,

$B(a) = 0.5, B(b) = 0.55$. Then (X, τ) and (X, τ_1) are fts's. Consider the identity function $i : (X, \tau) \to (X, \tau_1)$. Now $1_X \setminus B \in \tau_1^c$, $i^{-1}(1_X \setminus B) = 1_X \setminus B < A \in FR\alpha O(X, \tau)$, but $\alpha cl(1_X \setminus B) = 1_X \setminus A \not\leq A \Rightarrow 1_X \setminus B \notin FRG\alpha C(X, \tau)$ and so i is not $frg\alpha$-continuous function. But 1_X is the only fuzzy open set in (X, τ) such that $1_X \setminus B < 1_X \Rightarrow scl(1_X \setminus B) = 1_X \setminus B < 1_X \Rightarrow 1_X \setminus B$ is fgs-closed in (X, τ) and so i is fgs-continuous.

Theorem 5.21. Every $frg\alpha$-continuous function is fgp-continuous.

Proof. The proof follows from Theorem 3.19. ■

But the converse is not true, in general, as it seen from the following example.

Example 5.22. fgp-continuity $\not\Rightarrow frg\alpha$-continuity.
Consider Example 5.20. Here 1_X is the only fuzzy open set in (X, τ) with $i^{-1}(1_X \setminus B) = 1_X \setminus B < 1_X$ and so $pcl(1_X \setminus B) \leq 1_X \Rightarrow i$ is fgp-continuous, but not $frg\alpha$-continuous as it is shown in Example 5.20.

Remark 5.23.

(i) Let $f : (X, \tau) \to (Y, \tau_1)$ be an $frg\alpha$-continuous function where (X, τ) is frT_α-space. Then f is fuzzy continuous function.

(ii) Let $f : (X, \tau) \to (Y, \tau_1)$ be an $frg\alpha$-continuous function where (Y, τ_1) is frT_α-space. Then f is $frg\alpha^*$-continuous function.

(iii) Let $f : (X, \tau) \to (Y, \tau_1)$ be an $frg\alpha$-closed (resp., $frg\alpha$-open) function where (Y, τ_1) is frT_α-space. Then f is fuzzy closed (resp., fuzzy open) function.

(iv) Let $f : (X, \tau) \to (Y, \tau_1)$ be fgp-continuous function where (X, τ) is fT_p-space. Then f is fuzzy continuous and $frg\alpha$-continuous function.

6. Applications

Definition 6.1. [12] An fts (X, τ) is called fuzzy normal if for any two fuzzy closed sets A, B with $A \not{q} B$, there exist fuzzy open sets U, V in X such that $A \leq U, B \leq V$ and $U \not{q} V$.

Theorem 6.2. Let $f : (X, \tau) \to (Y, \tau_1)$ be an $frg\alpha$-continuous, fuzzy open, bijective function. If (X, τ) is fuzzy normal and frT_α-space, then Y is fuzzy normal.

Proof. Let $A, B \in \tau_1^c$ with $A \not{q} B$. Then $f^{-1}(A), f^{-1}(B) \in FRG\alpha C(X, \tau)$ with $f^{-1}(A) \not{q} f^{-1}(B)$. As X is frT_α-space, $f^{-1}(A), f^{-1}(B) \in \tau^c$ also. As X is fuzzy normal space, there exist fuzzy open sets $U, V \in \tau$ such that $f^{-1}(A) \leq U$, $f^{-1}(B) \leq V$ and $U \not{q} V$. As f is surjective, $A = f(f^{-1}(A)) \leq f(U), B = f(f^{-1}(B)) \leq f(V)$ where $f(U), f(V) \in \tau_1$ as f is an open function. We claim that $f(U) \not{q} f(V)$.

Indeed, $f(U)qf(V) \Rightarrow$ there exists $y \in Y$ such that $[f(U)](y) + [f(V)](y) > 1 \Rightarrow U(f^{-1}(y)) + V(f^{-1}(y)) > 1$ (as f is bijective) $\Rightarrow U q V$, a contradiction. Consequently, Y is fuzzy normal space. ■

Theorem 6.3. Let $f : (X, \tau) \rightarrow (Y, \tau_1)$ be an $frg\alpha$-continuous, open, bijective function where (X, τ) is fuzzy normal and fT_p-space. Then Y is fuzzy normal.

Proof. The proof follows from Theorem 3.19 and Theorem 6.2. ■

Theorem 6.4. Let $f : (X, \tau) \rightarrow (Y, \tau_1)$ be an $frg\alpha$-continuous, open, bijective function where (X, τ) is fuzzy normal and fT_b-space. Then Y is fuzzy normal.

Proof. The proof follows from Theorem 3.21 and Theorem 6.2. ■

Theorem 6.5. Let $f : (X, \tau) \rightarrow (Y, \tau_1)$ be an $frg\alpha$-continuous, open, bijective function where (X, τ) is fuzzy normal and fgT_p-space. Then Y is fuzzy normal.

Proof. The proof follows from Theorem 3.15 and Theorem 6.2. ■

Theorem 6.6. Let $f : (X, \tau) \rightarrow (Y, \tau_1)$ be an $frg\alpha$-continuous, open, bijective function where (X, τ) is fuzzy normal and fT_α-space. Then Y is fuzzy normal.

Proof. The proof follows from Theorem 3.6 and Theorem 6.2. ■

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