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Fuzzy Linear Integral Equation and Its Application In Biomathematical Model

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Abstract

This paper deals with the fuzzy linear integral equations by using extension principle. Condition for existence of the solution has been derived uniquely. Also the conditions for which the solution does not exist by using the extension principle have been illustrated. Moreover some examples have been chosen to verify the validity of the existence theorem. The simplest mathematical model that describes the dynamics of diseases transmitted by direct contact between susceptible and infected individuals (SI model) has been modified to fuzzy SI Model and application of the proposed method is given in fuzzy SI model.

Keywords: Fuzzy integral; Fuzzy linear integral equations; Extension principle; Existence theorem; Fuzzy SI model.

1. INTRODUCTION

Fuzzy differential and fuzzy integral equations are very use full tool to model the dynamical systems whose characteristic is biased on vagueness or uncertainty due to partial information about the problem, or due to information which is not fully reliable, or due to inherent imprecision in the language with which the problem is defined, or due to receipt of information from more than one source about the problem which is conflicting [32]. So, the topic fuzzy differential and fuzzy integral equations have been a great area of interest and rapidly growing in recent years.

The concept of integration of fuzzy valued functions was first presented by Dubois and Prade [3,4]. The study of fuzzy integral equations begins with the investigations of Kaleva (see [5]). Kaleva [5] and Buckley [2] discussed the properties of differentiable fuzzy valued functions. S.Seikkala discusses the initial value problem in [20]. The existence and uniqueness theorems for Volterra and Fredholm integral equations involving fuzzy set valued functions of a real variable have been given by Park and Jeong [6, 9]. The existence theorem of fuzzy Volterra integral equation has been also given by P. V. Subrahmanyam and S. K. Sudarsanam [7]. A numerical algorithm for solving fuzzy Fredholm and Volterra integral equations of the second kind with arbitrary kernel using a parametric Riemann integral representation has been given by M. Friedman, M. Ming and A. Kandel [8]. Using a fixed point technique, a quadrature formula for the solution of nonlinear fuzzy Fredholm integral equations have been given by Bica [10]. Some results concerning the existence of the solution of fuzzy integro-differential equation have been presented in [31]. There are several numerical methods for fuzzy integral equations in [11-18, 21]. The numerical methods like Adomian decomposition, homotopy analysis and homotopy perturbation are used in [12, 17, 22-25]. There are other technique also like Quadrature rules and Nyström techniques [26, 27]. We have applied similar method as Buckley and Feuring use to solve fuzzy partial differential equations in [2].

In this paper we develop some results for the existence of the solution of linear integral equation. The paper is organized as follows: Section 2 contains some basic definitions of fuzzy sets and fuzzy number. Section 3 contains solution procedure of fuzzy lineal integral equations and the existence theorem of the solution. In section 4, the proposed method is illustrated by three numerical examples. Section 5 contains the fuzzy SI model as an application of the proposed method. Finally the conclusion is given in section 6.

2. PRELIMINARIES

Definition 2.1. (see [1]). If X is a collection of objects denoted by x then a fuzzy set $\tilde{A}$ in X is a set of ordered pairs denoted and defined by:

$\tilde{A} =\{(x, \mu_{\tilde{A}}(x))/x \in X\}$, were $\mu_{\tilde{A}}(x)$ is called membership function or grade of membership (also degree of compatibility or degree of truth) of x in $\tilde{A}$ which maps X to [0,1].

Definition 2.2. [1]. α-cut of a fuzzy $\tilde{A}$ set is a crisp set A_α and defined by

A_α or $\tilde{A}\,[\alpha]$ = $\{x/\mu_{\tilde{A}}(x) \geq \alpha\}$, where $\tilde{A} =\{(x, \mu_{\tilde{A}}(x))\}$.

Definition 2.3. [1] A fuzzy set $\tilde{A}$ is said to be convex fuzzy set if A_α is a convex set for all $\alpha \in (0,1]$.

Definition 2.4. [1] A fuzzy set $\tilde{A}$ is said to be normal fuzzy set if there exist an element $(a, 1) \in \tilde{A}$.

Definition 2.5. [1] If a fuzzy set is convex, normalized and its membership function, defined in $\mathbb{R}$, is piecewise continuous then it is called as fuzzy number.

A triangular fuzzy number $\tilde{A}$ is denoted by (a_1, a_2, a_3) and it is a fuzzy set $\{(x, \mu_{\tilde{A}}(x))\}$ where $\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, a_1 \leq x \leq a_2 \\ \frac{a_3-x}{a_3-a_2}, a_2 \leq x \leq a_3 \\ 0, \text{ otherwise} \end{cases}$. $\tilde{A}$ is called positive triangular fuzzy number if $a_1 > 0$ and negative triangular fuzzy number if $a_3 < 0$.

Definition 2.6. [18]. Let E be the set of all upper semicontinuous normal convex fuzzy numbers with bounded α-cut intervals. It means if $\tilde{v} \in$ E then the α-cutset is a closed bounded interval which is denoted by

$v_{\alpha} = [v_1, v_2]$.

For arbitrary $u_{\alpha} = [u_1, u_2]$, $v_{\alpha} = [v_1, v_2]$ and $k \geq 0$, addition $(u_{\alpha} + v_{\alpha})$ and multiplication by k are defined as $(u + v)_1(\alpha) = u_1(\alpha) + v_1(\alpha)$, $(u + v)_2(\alpha) = u_2(\alpha) + v_2(\alpha)$, $(ku)_1(\alpha) = ku_1(\alpha)$, $(ku)_2(\alpha) = ku_2(\alpha)$.

Since each $y \in$R can be regarded as a fuzzy number $\tilde{y}$ defined by

$$\mu_{\tilde{y}}(x) = \begin{cases} 1 \;\; if \;\; x = y, \\ 0 \;\; if \; x \neq y. \end{cases}$$

The Hausdorff distance between fuzzy numbers given by D :E×E→ $R_+ \cup \{0\}$,

$$D(\tilde{u}, \tilde{v}) = \sup_{\alpha \in [0,1]} \text{Max}\{|u_1(\alpha) - v_1(\alpha)|, |u_2(\alpha) - v_2(\alpha)|\}.$$

It is easy to see that D is a metric in E and has the following properties (see [19])

(i) $D(\tilde{u} \oplus \tilde{w}, \tilde{v} \oplus \tilde{w}) = D(\tilde{u}, \tilde{v}), \forall \, \tilde{u}, \tilde{v}, \tilde{w} \in$E,

(ii) $D(k \odot \tilde{u}, k \odot \tilde{v}) = |k| D(\tilde{u}, \tilde{v}), \forall k \in$R, $\tilde{u}, \tilde{v} \in$E,

(iii) $D(\tilde{u} \oplus \tilde{v}, \tilde{w} \oplus \tilde{e}) \leq D(\tilde{u}, \tilde{w}) + D(\tilde{v}, \tilde{e})$, $\forall \, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{e} \in$E,

(iv) (D,E) is a complete metric space.

Definition 2.7. (see [15]). Let f:R →E be a fuzzy valued function. If for arbitrary fixed t_0 ∈R and $\epsilon > 0$, a $\delta > 0$ such that

$|t - t_0| < \delta \Longrightarrow$ D$(f(t), f(t_0)) < \epsilon$,

f is said to be continuous.

Definition 2.8. (See [19]) Let $f: X \to Y$ be a mapping from a set X to a set Y. Then the extension principle allows us to define the fuzzy set $\tilde{B}$ in Y induced by the fuzzy set $\tilde{A}$ in X through f as follows:

$$\tilde{B} = \{(y, \mu_{\tilde{B}}(y)) : y = f(x), x \in A\}$$

$$\text{with } \mu_{\tilde{B}}(y) \triangleq \mu_{f(\tilde{A})}(y) = \begin{cases} \sup\limits_{y=f(x)} \mu_{\tilde{A}}(x) & f^{-1}(y) \neq \phi \\ 0 & f^{-1}(y) = \phi \end{cases}$$

where $f^{-1}(y)$ is the inverse image of y.

3. Fuzzy linear integral equations

Let us now first define the linear integral equation we are interested in and then fuzzify it to obtain our fuzzy linear integral equation. Let $I = [a, b]$, $I_1 = [a_1, b_1]$ for some $a, a_1 \geq 0$, be an interval. Let K(s, t, p) be continuous function for (s, t) ∈ $I_1 \times I$ and p = $(p_1, p_2, \dots\dots.., p_n)$ be a vector of constants with p_i in interval J_i, $1 \leq i \leq n$. And f(s, c) be continuous function for s ∈ I and c = $(c_1, c_2, \dots\dots, c_m)$ a vector of constants with c_j in interval L_j, $1 \leq j \leq m$.

The linear integral equation is

$$g(s) = \text{f(s, c)} + \lambda \int_a^{b \text{ or } s} \text{K(s, t, p)} g(\text{t})\text{dt} \tag{1}$$

where λ is a positive parameter. We assume the solution of the problem is

$$g(s) = \text{G(s, λ, p, c)} \tag{2}$$

where p ∈ $J = \Pi J_i$ and c ∈ $L = \Pi L_j$. The constants p_i and c_j are not known exactly so there will be uncertainty in their values. So we will model this uncertainty using fuzzy numbers. Then we will get a new equation which will be able to model this uncertainty. To get the fuzzy linear integral equation we replace the crisp parameter by fuzzy parameter. We get $\tilde{f}(\text{s}, \tilde{\text{c}})$ from f(s, c) where $\tilde{f}(\text{s}, \tilde{\text{c}})$ has $\tilde{c} = (\tilde{c}_1, \tilde{c}_2, \dots\dots, \tilde{c}_m)$

for $\tilde{c}_j$ a triangular fuzzy number in L_j, $1 \le j \le m$ and $\widetilde{K}(\mathrm{s,t},\tilde{p})$ from K(s, t, p) where $\widetilde{K}(\mathrm{s,t},\tilde{p})$ has $\tilde{p}$ = $(\tilde{p}_1, \tilde{p}_2, \dots\dots.., \tilde{p}_n)$ for $\tilde{p}_i$ a triangular fuzzy number in interval J_i, $1 \le i \le n$. The function $g(s)$ becomes $\tilde{g}$(s) where $\tilde{g}$(s) maps I into fuzzy numbers. The fuzzy integral equation is

$$\tilde{g}(s) = \tilde{f}(\mathrm{s},\tilde{c}) + \lambda \int_a^{b \text{ or } s} \widetilde{\mathrm{K}}(\mathrm{s,t,\tilde{p}})\tilde{g}(\mathrm{t})\mathrm{dt} \tag{3}$$

Now we want to solve the equation given in (3). So we will fuzzify the solution in (2), $\tilde{Y}(s) = \tilde{G}(s, \lambda, \tilde{p}, \tilde{c})$, where $\tilde{Y}(s)$ is computed using the extension principle. Now we can discuss about our solution strategy.

3.1. Solutions

Let $\mathrm{p}_\alpha = \Pi p_{i\alpha},\ c_\alpha = \Pi c_{j\alpha},\ Y_\alpha = [Y_1(s,\alpha),\ Y_2(s,\alpha)]$,

$f_\alpha = [f_1(s,\alpha),\ f_1(s,\alpha)]$, $\mathrm{K}_\alpha = [K_1(s,t,\alpha),\ K_1(s,t,\alpha)]$, for all $\alpha \epsilon (0,\ 1]$.

We know that

$$Y_1(s,\alpha) = \min\{G(s,\lambda,p,c)/\ p \in \mathrm{p}_\alpha,\ c \in c_\alpha\} \tag{4}$$

$$Y_2(s,\alpha) = \max\{G(s,\lambda,p,c)/\ p \in \mathrm{p}_\alpha,\ c \in c_\alpha\} \tag{5}$$

for all $(s,t) \in I_1 \times I$ and $\alpha \epsilon (0,\ 1]$. Now we assume that $K_i(s,t,\alpha)Y_j(t,\alpha)$ has continuous integral with respect to t. So, $\int_a^{b \text{ or } s} K_i(s,t,\alpha)Y_j(t,\alpha)\, dt$ is continuous for all i, j = 1, 2, $(s,t) \in I_1 \times I$ and $\alpha \epsilon (0,\ 1]$.

Now, Let $\widetilde{\Phi}(s,t) = \widetilde{\mathrm{K}}(\mathrm{s,t,\tilde{p}})\tilde{Y}(t)$ and $\Phi_\alpha = [\Phi_1(s,t,\alpha), \Phi_2(s,t,\alpha)]$

So, $\Phi_1(s,t,\alpha) = \min\{K(s,t,p)G(s,\lambda,p,c)/\ p \in \mathrm{p}_\alpha,\ c \in c_\alpha\}$ and

$\Phi_2(s,t,\alpha) = \max\{K(s,t,p)G(s,\lambda,p,c)/\ p \in \mathrm{p}_\alpha,\ c \in c_\alpha\}$.

Again let $\tau(s,\alpha) = [\int_a^{b \text{ or } s} \Phi_1(s,t,\alpha)\, dt, \int_a^{b \text{ or } s} \Phi_2(s,t,\alpha)\, dt\ ]$ for all $s \in I_1$ and $\alpha \epsilon (0,\ 1]$. Now if for each $s \in I_1$, $\tau(s,\alpha)$ defines the α-cut of a fuzzy number, then we will say that $\widetilde{\mathrm{K}}(\mathrm{s,t,\tilde{p}})\tilde{Y}(t)$ is integrable. Sufficient conditions for $\tau(s,\alpha)$ to define α-cut of a fuzzy number are

i) $\int_a^{b\ or\ s} \Phi_1(s,t,\alpha)\,dt$ is an increasing function of α for each $s \in I_1$

ii) $\int_a^{b\ or\ s} \Phi_2(s,t,\alpha)\,dt$ is a decreasing function of α for each $s \in I_1$.

iii) $\int_a^{b\ or\ s} \Phi_1(s,t,\alpha)\,dt \le \int_a^{b\ or\ s} \Phi_2(s,t,\alpha)\,dt$ for all $s \in I_1$ and $\alpha \in (0,\ 1]$.

We have already assumed that $\int_a^{b\ or\ s} \Phi_i(s,t,\alpha)\,dt$ is continuous for all i = 1, 2, $s \in I_1$ and $\alpha \in (0,\ 1]$. So if (i)-(iii) hold, $\widetilde{\mathrm{K}}(\mathrm{s},\mathrm{t},\tilde{\mathrm{p}})\tilde{Y}(t)$ is integrable. So for $\tilde{Y}(t)$ to be a solution to the fuzzy integral equation we need

I) $\widetilde{\mathrm{K}}(\mathrm{s},\mathrm{t},\tilde{\mathrm{p}})\tilde{Y}(t)$ is integrable (6)

II) Equation (3) holds for $\tilde{g}(s) = \tilde{Y}(s)$ (7)

i.e. $\tilde{Y}(s) = \tilde{f}(\mathrm{s},\tilde{c}) + \lambda \int_a^{b\ or\ s} \widetilde{\mathrm{K}}(\mathrm{s},\mathrm{t},\tilde{\mathrm{p}})\tilde{Y}(t)\mathrm{dt}$ (8)

i.e. The following equations must hold

Case 1: If $\tilde{Y}(t)$ and $\widetilde{\mathrm{K}}(\mathrm{s},\mathrm{t},\tilde{\mathrm{p}})$ are both positive fuzzy numbers

$$Y_1(s,\alpha) = f_1(s,\alpha) + \lambda \int_a^{b\ or\ s} K_1(s,t,\alpha) Y_1(t,\alpha)\,dt \tag{9}$$

$$and$$

$$Y_2(s,\alpha) = f_2(s,\alpha) + \lambda \int_a^{b\ or\ s} K_2(s,t,\alpha) Y_2(t,\alpha)\,dt \tag{10}$$

for s $\in I_1$ and $\alpha \in (0,\ 1]$.

Case 2: If $\tilde{Y}(t)$ is Negative and $\widetilde{\mathrm{K}}(\mathrm{s},\mathrm{t},\tilde{\mathrm{p}})$ is positive fuzzy number

$$Y_1(s,\alpha) = f_1(s,\alpha) + \lambda \int_a^{b\ or\ s} K_2(s,t,\alpha) Y_1(t,\alpha)\,dt \tag{11}$$

$$and$$

$$Y_2(s,\alpha) = f_2(s,\alpha) + \lambda \int_a^{b\ or\ s} K_1(s,t,\alpha) Y_2(t,\alpha)\,dt \tag{12}$$

for s $\in I_1$ and $\alpha \in (0,\ 1]$.

Case 3: If $\tilde{Y}(t)$ is positive and $\widetilde{\mathrm{K}}(\mathrm{s},\mathrm{t},\tilde{\mathrm{p}})$ is negative fuzzy number

$$Y_1(s,\alpha) = f_1(s,\alpha) + \lambda \int_a^{b \, or \, s} K_1(s,t,\alpha) Y_2(t,\alpha)\, dt \tag{13}$$

$$and$$

$$Y_2(s,\alpha) = f_2(s,\alpha) + \lambda \int_a^{b \, or \, s} K_2(s,t,\alpha) Y_1(t,\alpha)\, dt \tag{14}$$

for s $\in I_1$ and $\alpha \epsilon$ (0, 1].

Case 4: If $\tilde{Y}(t)$ and $\widetilde{\mathrm{K}}(\mathrm{s,t,\tilde{p}})$ are both negative fuzzy numbers

$$Y_1(s,\alpha) = f_1(s,\alpha) + \lambda \int_a^{b \, or \, s} K_2(s,t,\alpha) Y_2(t,\alpha)\, dt \tag{15}$$

$$and$$

$$Y_2(s,\alpha) = f_2(s,\alpha) + \lambda \int_a^{b \, or \, s} K_1(s,t,\alpha) Y_1(t,\alpha)\, dt \tag{16}$$

for s $\in I_1$ and $\alpha \epsilon$ (0, 1].

Now from equations (13) and (15) we can see that the left hand sides contain only $Y_1(s,\alpha)$ and the right hand sides do not contain $Y_1(s,\alpha)$ but those contain $Y_2(s,\alpha)$. And from equations (14) and (16) we can see that the left hand sides contain only $Y_2(s,\alpha)$ and the right hand sides do not contain $Y_2(s,\alpha)$ but those contain $Y_1(s,\alpha)$. Therefore from (4) and (5) it is clear equations (13) to (16) cannot be true except $\alpha = 1$. Hence for Case 3 and Case 4 solutions do not exist when $\alpha \neq 1$.

Now we will present sufficient conditions for the existence of the solutions for the Cases 1 and 2.

3.2. Theorem 1

Let us assume $\widetilde{\mathrm{K}}(\mathrm{s,t,\tilde{p}})\tilde{Y}(t)$ is integrable.

(a) If $\tilde{Y}(t)$ and $\widetilde{\mathrm{K}}(\mathrm{s,t,\tilde{p}})$ are both positive fuzzy numbers and for all $i \in \{1, 2, \dots \dots, n\}$, $G(s,\lambda,p,c)$ and $K(s,t,p)$ are both increasing (or both decreasing) in p_i for $(s,t) \in I_1 \times I$ and $p \in J$ and for all $j \in \{1, 2, \dots \dots, m\}$, $G(s,\lambda,p,c)$ and $f(s,c)$ are both increasing (or both decreasing) in c_j for $s \in I$ and $c \in L$ then $\tilde{Y}(t)$ is a solution.

(b) If $\tilde{Y}(t)$ is Negative and $\widetilde{\mathrm{K}}(\mathrm{s,t,\tilde{p}})$ is positive fuzzy number and for all $i \in \{1, 2, \dots \dots, n\}$, $G(s,\lambda,p,c)$ is increasing and $K(s,t,p)$ is decreasing (or $G(s,\lambda,p,c)$ is decreasing and $K(s,t,p)$ is increasing) in p_i for $(s,t) \in I_1 \times I$ and $p \in J$ and for all $j \in \{1, 2, \ldots\ldots, m\}$, $G(s, \lambda, p, c)$ and $f(s, c)$ are both increasing (or both decreasing) in c_j for $s \in I_1$ and $c \in L$ then $\tilde{Y}(t)$ is a solution.

Proof

(a) For simplicity assume $m = n = 2$ and $G(s, \lambda, p, c)$ and $K(s, t, p)$ are both increasing in p_1; $G(s, \lambda, p, c)$ and $K(s, t, p)$ are both decreasing in p_2; $G(s, \lambda, p, c)$ and $f(s, c)$ are both increasing in c_1; $G(s, \lambda, p, c)$ and $f(s, c)$ are both decreasing in c_2. The others cases are similar and so we will omit them. So, we have

$$Y_1(s, \alpha) = G\big(s, \lambda, p_{11}(\alpha), p_{22}(\alpha), c_{11}(\alpha), c_{22}(\alpha)\big) \tag{17}$$

$$Y_2(s, \alpha) = G\big(s, \lambda, p_{12}(\alpha), p_{21}(\alpha), c_{12}(\alpha), c_{21}(\alpha)\big) \tag{18}$$

$$K_1(s, t, \alpha) = K\big(s, t, p_{11}(\alpha), p_{22}(\alpha)\big) \tag{19}$$

$$K_2(s, t, \alpha) = K\big(s, t, p_{12}(\alpha), p_{21}(\alpha)\big) \tag{20}$$

$$f_1(s, \alpha) = f\big(s, c_{11}(\alpha), c_{22}(\alpha)\big) \tag{21}$$

$$f_2(s, \alpha) = f\big(s, c_{12}(\alpha), c_{21}(\alpha)\big) \tag{22}$$

for all α where $p_{i\alpha} = [p_{i1}(\alpha),\ p_{i2}(\alpha)]$ and $c_{i\alpha} = [c_{i1}(\alpha),\ c_{i2}(\alpha)]$, $i = 1,2$.

Now $G(s, \lambda, p_1, p_2, c_1, c_2)$ is the solution of the linear integral equation (1) which means

$$G(s, \lambda, p_1, p_2, c_1, c_2) = \mathrm{f}(\mathrm{s}, c_1, c_2) + \lambda \int_a^{b\ or\ s} \mathrm{K}(\mathrm{s}, \mathrm{t}, p_1, p_2) G(s, \lambda, p_1, p_2, c_1, c_2)\mathrm{dt} \tag{23}$$

for all $s \in I_1$, $p_i \in J_i$, and $c_i \in L_i$, $i = 1,2$. But $p_{ij}(\alpha) \in J_i$ and $c_{ij}(\alpha) \in L_i$ for all $\alpha \epsilon\ (0,\ 1]$ and $i, j = 1,2$. Hence,

$$G\big(s, \lambda, p_{11}(\alpha), p_{22}(\alpha), c_{11}(\alpha), c_{22}(\alpha)\big) = f\big(s, c_{11}(\alpha), c_{22}(\alpha)\big) +$$

$$\lambda \int_a^{b\ or\ s} K\big(s, t, p_{11}(\alpha), p_{22}(\alpha)\big) G\big(s, \lambda, p_{11}(\alpha), p_{22}(\alpha), c_{11}(\alpha), c_{22}(\alpha)\big)\mathrm{dt}$$

and

$$G\big(s, \lambda, p_{12}(\alpha), p_{21}(\alpha), c_{12}(\alpha), c_{21}(\alpha)\big) = f\big(s, c_{12}(\alpha), c_{21}(\alpha)\big) +$$

$\lambda \int_a^{b\ or\ s} K\big(s, t, p_{12}(\alpha), p_{21}(\alpha)\big) G\big(s, \lambda, p_{12}(\alpha), p_{21}(\alpha), c_{12}(\alpha), c_{21}(\alpha)\big)\mathrm{dt}$ for all $s \in I_1$ and $\alpha \epsilon\ (0,\ 1]$. Therefore equations (13) and (14) are true. Hence $\tilde{Y}(s)$ is a solution.

(b) For simplicity assume $m = n = 2$ and $G(s, \lambda, p, c)$ is increasing and $K(s, t, p)$ is decreasing in p_1; $G(s, \lambda, p, c)$ is decreasing and $K(s, t, p)$ is increasing in p_2; $G(s, \lambda, p, c)$ and $f(s, c)$ are both increasing in c_1; $G(s, \lambda, p, c)$ and $f(s, c)$ are both decreasing in c_2. The others cases are similar and so we will omit them. Then equations (17), (18), (21) and (22) are still true and equations (19) and (20) became

$$K_1(s,t,\alpha) = K\big(s,t,p_{12}(\alpha),p_{21}(\alpha)\big) \tag{24}$$

$$K_2(s,t,\alpha) = K\big(s,t,p_{11}(\alpha),p_{22}(\alpha)\big) \tag{25}$$

for all α where $p_{i\alpha} = [p_{i1}(\alpha), p_{i2}(\alpha)]$ and $c_{i\alpha} = [c_{i1}(\alpha), c_{i2}(\alpha)],\ i = 1,2.$

Now $G(s,\lambda,p_1,p_2,c_1,c_2)$ is the solution of the linear integral equation (1) which means

$G(s,\lambda,p_1,p_2,c_1,c_2)=$ f(s,c_1,c_2) $+ \lambda \int_a^{b\ or\ s}$ K(s, t, p_1,p_2)$G(s,\lambda,p_1,p_2,c_1,c_2)$dt, for all $s \in I_1, p_i \in J_i$, and $c_i \in L_i,\ i = 1,2$. But $p_{ij}(\alpha) \in J_i$ and $c_{ij}(\alpha) \in L_i$ for all $\alpha \epsilon (0,\ 1]$ and $i,j = 1,2$.

Hence,

$$G\big(s,\lambda,p_{11}(\alpha),p_{22}(\alpha),c_{11}(\alpha),c_{22}(\alpha)\big) = f\big(s,c_{11}(\alpha),c_{22}(\alpha)\big) +$$

$$\lambda \int_a^{b\ or\ s} K\big(s,t,p_{11}(\alpha),p_{22}(\alpha)\big)G\big(s,\lambda,p_{11}(\alpha),p_{22}(\alpha),c_{11}(\alpha),c_{22}(\alpha)\big)\text{dt}$$

and

$$G\big(s,\lambda,p_{12}(\alpha),p_{21}(\alpha),c_{12}(\alpha),c_{21}(\alpha)\big) = f\big(s,c_{12}(\alpha),c_{21}(\alpha)\big) +$$

$\lambda \int_a^{b\ or\ s} K\big(s,t,p_{12}(\alpha),p_{21}(\alpha)\big)G\big(s,\lambda,p_{12}(\alpha),p_{21}(\alpha),c_{12}(\alpha),c_{21}(\alpha)\big)$dt for all $s \in I_1$ and $\alpha \epsilon (0,\ 1]$. Therefore equations (15) and (16) are true. Hence, $\tilde{Y}(s)$ is a solution.

So, we can present the above theorem in the following form

3.2.1. Corollary 1

Let $\widetilde{\text{K}}(\text{s},\text{t},\tilde{\text{p}})\tilde{Y}(t)$ be integrable.

(a)Let $\tilde{Y}(t)$ and $\widetilde{\text{K}}(\text{s},\text{t},\tilde{\text{p}})$ are both positive fuzzy numbers then $\tilde{Y}(s)$ is a solution if $\frac{\partial G}{\partial p_i}\frac{\partial K}{\partial p_i} \geq 0$ and $\frac{\partial G}{\partial c_j}\frac{\partial f}{\partial c_j} \geq 0$ for all $i = 1,2,\ldots,n,\ j = 1,2,\ldots,m,\ (s,t) \in I_1 \times I,$ $p \in J$ and $c \in L$.

(b) Let $\tilde{Y}(t)$ is Negative and $\widetilde{\text{K}}(\text{s},\text{t},\tilde{\text{p}})$ is positive fuzzy number then $\tilde{Y}(s)$ is a solution if $\frac{\partial G}{\partial p_i}\frac{\partial K}{\partial p_i} \leq 0$ and $\frac{\partial G}{\partial c_j}\frac{\partial f}{\partial c_j} \geq 0$ for all $i = 1,2,\ldots,n,\ j = 1,2,\ldots,m, (s,t) \in I_1 \times I,\ p \in J$ and $c \in L$.

3.3. Theorem 2

Let us assume $\widetilde{K}(s,t,\tilde{p})\tilde{Y}(t)$ is integrable.

(a)Let $\tilde{Y}(t)$ and $\widetilde{K}(s,t,\tilde{p})$ are both positive fuzzy numbers and there is an $i \in \{1,2,\ldots\ldots,n\}$ so that for variable $p_i, G(s,\lambda,p,c)$ is strictly increasing and $K(s,t,p)$ is strictly decreasing (or $G(s,\lambda,p,c)$ is strictly decreasing and $K(s,t,p)$ is strictly increasing) for $(s,t) \in I_1 \times I$and $p \in J$ or there is a $j \in \{1,2,\ldots\ldots,m\}$ so that for variable $c_j, G(s,\lambda,p,c)$ is strictly increasing and $f(s,c)$ is strictly decreasing (or $G(s,\lambda,p,c)$ is strictly decreasing and $f(s,c)$ is strictly increasing) for $(s,t) \in I_1 \times I$ and $c \in L$, then $\tilde{Y}(s)$ is not a solution.

(b) Let $\tilde{Y}(t)$ is Negative and $\widetilde{K}(s,t,\tilde{p})$ is positive fuzzy number and there is an $i \in \{1,2,\ldots\ldots,n\}$ so that for variable $p_i, G(s,\lambda,p,c)$ and $K(s,t,p)$ are both strictly increasing (or both strictly decreasing) for $(s,t) \in I_1 \times I$ and $p \in J$ or there is a $j \in \{1,2,\ldots\ldots,m\}$ so that for variable $c_j, G(s,\lambda,p,c)$ is strictly increasing and $f(s,c)$ is strictly decreasing (or $G(s,\lambda,p,c)$ is strictly decreasing and $f(s,c)$ is strictly increasing) for $(s,t) \in I_1 \times I$ and $c \in L$, then $\tilde{Y}(s)$ is not a solution.

Proof

(a)Case 1: Assume there is an $i = l, l \in \{1,2,\ldots\ldots,n\}$ such that $G(s,\lambda,p,c)$ is strictly increasing in p_l and $K(s,t,p)$ is strictly decreasing in p_l(the proof for the case $G(s,\lambda,p,c)$ is strictly decreasing in p_l and $K(s,t,p)$ is strictly increasing in p_l will be similar).

So, we have

$$Y_1(s,\alpha) = G\left(s,\lambda,(p_{1min}(\alpha),p_{2min}(\alpha),\ldots,p_{(l-1)min}(\alpha),p_{l1}(\alpha),p_{(l+1)min}(\alpha),\ldots,p_{nmin}(\alpha)),c_{min}(\alpha)\right), \quad (26)$$

where $p_{imin}(\alpha) \in p_{i\alpha}, i = 1,2,\ldots\ldots,n, p_{lmin}(\alpha) = p_{l1}(\alpha)$ and $c_{min}(\alpha) \in c_\alpha$ for which $G(s,\lambda,p,c)$ is minimum.

$$Y_2(s,\alpha) = G\left(s,\lambda,(p_{1max}(\alpha),p_{2max}(\alpha),\ldots,p_{(l-1)max}(\alpha),p_{l2}(\alpha),p_{(l+1)max}(\alpha),\ldots,p_{nmax}(\alpha)),c_{max}(\alpha)\right), \quad (27)$$

where $p_{imax}(\alpha) \in p_{i\alpha}, i = 1,2,\ldots\ldots,n, p_{lmax}(\alpha) = p_{l2}(\alpha)$ and $c_{max}(\alpha) \in c_\alpha$ for which $G(s,\lambda,p,c)$ is maximum.

$K_1(s,t,\alpha) =$

$$K\big(s,t,(\,p'_{1min}(\alpha),p'_{2min}(\alpha),\ldots,p'_{(l-1)min}(\alpha),p_{l2}(\alpha),p'_{(l+1)min}(\alpha),\ldots,p'_{nmin}(\alpha))\big), \qquad (28)$$

where $p'_{imin}(\alpha) \in p_{i\alpha}, i = 1,2,\ldots.,n, p'_{lmin}(\alpha) = p_{l2}(\alpha)$ for which $K(s,t,p)$ is minimum.

$K_2(s,t,\alpha) =$

$$K\big(s,t,(\,p'_{1max}(\alpha),p'_{2max}(\alpha),\ldots,p'_{(l-1)max}(\alpha),p_{l1}(\alpha),p'_{(l+1)max}(\alpha),\ldots,p'_{nmax}(\alpha))\big), \qquad (29)$$

where $p'_{imax}(\alpha) \in p_{i\alpha}, i = 1,2,\ldots.,n, p'_{lmax}(\alpha) = p_{l1}(\alpha)$ for which $K(s,t,p)$ is maximum.

$$f_1(s,\alpha) = f\big(s,c'_{min}(\alpha)\big), \qquad (30)$$

where $c'_{min}(\alpha) \in c_\alpha$ for which $f(s,c)$ is minimum.

$$f_2(s,\alpha) = f\big(s,c'_{max}(\alpha)\big), \qquad (31)$$

where $c'_{max}(\alpha) \in c_\alpha$ for which $f(s,c)$ is maximum.

Therefore, equations (9) and (10) are not true. Hence, $\tilde{Y}(s)$ is not a solution.

<u>Case 2</u>: Assume there is a $j = r,\ r \in \{1,2,\ldots\ldots.,m\}$, such that $G(s,\lambda,p,c)$ is strictly increasing in c_r and $f(s,p)$ is strictly decreasing in c_r (the proof for the case $G(s,\lambda,p,c)$ is strictly decreasing in c_r and $f(s,p)$ is strictly increasing in c_r will be similar).

So, we have

$Y_1(s,\alpha) =$

$$G\big(s,\lambda,,p_{min}(\alpha)(\,c_{1min}(\alpha),c_{2min}(\alpha),\ldots,c_{(r-1)min}(\alpha),c_{r1}(\alpha),c_{(r+1)min}(\alpha),\ldots,c_{mmin}(\alpha))\big), \qquad (32)$$

where $c_{jmin}(\alpha) \in c_{j\alpha},\ j = 1,2,\ldots.,m,\ c_{rmin}(\alpha) = c_{r1}(\alpha)$ and $p_{min}(\alpha) \in p_\alpha$

for which $G(s,\lambda,p,c)$ is minimum.

$Y_2(s,\alpha) =$

$$G\big(s,\lambda,p_{max}(\alpha),(\,c_{1max}(\alpha),c(\alpha),\ldots,c_{(r-1)max}(\alpha),c_{r2}(\alpha),c_{(r+1)max}(\alpha),\ldots,c_{mmax}(\alpha))\big), \qquad (33)$$

where $c_{jmax}(\alpha) \in c_{j\alpha}, j = 1,2,\ldots.,m, c_{rmax}(\alpha) = c_{r2}(\alpha)$ and $p_{max}(\alpha) \in p_\alpha$ for which $G(s,\lambda,p,c)$ is maximum.

$$K_1(s,t,\alpha) = K\big(s,t,p'_{min}(\alpha)\big), \tag{34}$$

where $p'_{min}(\alpha) \in p_\alpha$ for which $K(s,t,p)$ is minimum.

$$K_2(s,t,\alpha) = K\big(s,t,p'_{max}(\alpha)\big), \tag{35}$$

where $p'_{max}(\alpha) \in p_\alpha$ for which $K(s,t,p)$ is maximum.

$$f_1(s,\alpha) = f\big(s,(\,c'_{1min}(\alpha), c'_{2min}(\alpha),\dots, c'_{(r-1)min}(\alpha), c_{r2}(\alpha), c'_{(r+1)min}(\alpha),\dots, c'_{mmin}(\alpha))\big), \tag{36}$$

where $c'_{jmin}(\alpha) \in c_{j\alpha}, j = 1,2,\dots, m, c_{rmin}(\alpha) = c_{r2}(\alpha)$ for which $f(s,c)$ is minimum.

$$f_2(s,\alpha) = f\big(s,(\,c'_{1max}(\alpha), c'_{2max}(\alpha),\dots, c'_{(r-1)max}(\alpha), c_{r1}(\alpha), c'_{(r+1)max}(\alpha),\dots, c'_{mmax}(\alpha))\big), \tag{37}$$

where $c'_{jmax}(\alpha) \in c_{j\alpha}, j = 1,2,\dots, m, c'_{rmin}(\alpha) = c_{r1}(\alpha)$ for which $f(s,c)$ is maximum.

Therefore, equations (9) and (10) are not true. Hence, $\tilde{Y}(s)$ is not a solution.

(b) <u>Case-1</u>: Assume there is an $i = l,\ l \in \{1,2,\dots\dots, n\}$ such that $G(s,\lambda,p,c)$and $K(s,t,p)$ are both strictly increasing in p_l(the proof for the case $G(s,\lambda,p,c)$ and $K(s,t,p)$ are both strictly decreasing in p_l will be similar). Then equations (30), (31), (34) and (35) are still true and equations (32) and (33) became

$$K_1(s,t,\alpha) = K\big(s,t,(\,p'_{1min}(\alpha), p'_{2min}(\alpha),\dots, p'_{(l-1)min}(\alpha), p_{l1}(\alpha), p'_{(l+1)min}(\alpha),\dots, p'_{nmin}(\alpha))\big), \tag{38}$$

where $p'_{imin}(\alpha) \in p_{i\alpha}, i = 1,2,\dots, n, p'_{lmin}(\alpha) = p_{l1}(\alpha)$ for which $K(s,t,p)$ is minimum.

$$K_2(s,t,\alpha) = K\big(s,t,(\,p'_{1max}(\alpha), p'_{2max}(\alpha),\dots, p'_{(l-1)max}(\alpha), p_{l2}(\alpha), p'_{(l+1)max}(\alpha),\dots, p'_{nmax}(\alpha))\big), \tag{39}$$

where $p'_{imax}(\alpha) \in p_{i\alpha}, i = 1,2,\dots, n, p'_{lmax}(\alpha) = p_{l2}(\alpha)$ for which $K(s,t,p)$ is maximum.

Therefore, equations (11) and (12) are not true. Hence, $\tilde{Y}(s)$ is not a solution.

Case- 2: The proof is the same of the proof of (a) case 2.

We can present the above theorem in the following form

3.3.1. Corollary 2

Let $\widetilde{\mathrm{K}}(\mathrm{s},\mathrm{t},\tilde{\mathrm{p}})\tilde{Y}(t)$ be integrable.

(a)Let $\tilde{Y}(t)$ and $\widetilde{\mathrm{K}}(\mathrm{s},\mathrm{t},\tilde{\mathrm{p}})$ are both positive fuzzy numbers then $\tilde{Y}(s)$ is not a solution if $\frac{\partial G}{\partial p_i}\frac{\partial K}{\partial p_i} < 0$ for some $i \in \{1, 2, \dots \dots, n\}$ or $\frac{\partial G}{\partial c_j}\frac{\partial f}{\partial c_j} < 0$ for some $j \in \{1, 2, \dots \dots, m\}$.

(b) Let $\tilde{Y}(t)$ is Negative and $\widetilde{\mathrm{K}}(\mathrm{s},\mathrm{t},\tilde{\mathrm{p}})$ is positive fuzzy number then $\tilde{Y}(s)$ is not a solution if $\frac{\partial G}{\partial p_i}\frac{\partial K}{\partial p_i} > 0$ for some $i \in \{1, 2, \dots \dots, n\}$ or $\frac{\partial G}{\partial c_j}\frac{\partial f}{\partial c_j} < 0$ for some $j \in \{1, 2, \dots \dots, m\}$.

Now for the case when λ is a negative eigen value i.e. $\lambda = -\lambda_1$, where $\lambda_1 > 0$, the original integral equation is

$$g(s) = \mathrm{f(s, c)} - \lambda_1 \int_a^{b\ or\ s} \mathrm{K}_1(\mathrm{s},\mathrm{t},\mathrm{p})g(\mathrm{t})\mathrm{dt}$$

$$\text{or, } g(s) = \mathrm{f(s, c)} + \lambda_1 \int_a^{b\ or\ s} -\mathrm{K}_1(\mathrm{s},\mathrm{t},\mathrm{p})g(\mathrm{t})\mathrm{dt}$$

Therefore, $\lambda = \lambda_1, \lambda_1 > 0$ and $\mathrm{K}(\mathrm{s},\mathrm{t},\mathrm{p}) = -\mathrm{K}_1(\mathrm{s},\mathrm{t},\mathrm{p})$ are in equation (1).

Now, if we want to fuzzify the eigen value λ i.e. $\lambda = \tilde{\lambda}_2$ (fuzzy) then the original integral equation is

$$g(s) = \mathrm{f(s, c)} + \lambda_2 \int_a^{b\ or\ s} \mathrm{K}_1(\mathrm{s},\mathrm{t},\mathrm{p}')g(\mathrm{t})\mathrm{dt}$$

$$\text{or, } g(s) = \mathrm{f(s, c)} + \int_a^{b\ or\ s} \lambda_2\mathrm{K}_1(\mathrm{s},\mathrm{t},\mathrm{p}')g(\mathrm{t})\mathrm{dt}$$

Therefore, $\lambda = 1, \mathrm{K}(\mathrm{s},\mathrm{t},\mathrm{p}) = \lambda_2\mathrm{K}_1(\mathrm{s},\mathrm{t},\mathrm{p}')$ and $p = (\mathrm{p}', \lambda_2)$ are in equation (1).

4. NUMERIAL RESULTS

Here we have presented three examples showing the situations where the solution will and will not exist.

4.1. Example 1

Solve the fuzzy linear integral equation $\tilde{g}(s) = \tilde{c}_1 + \frac{1}{2}\int_0^1 (\mathrm{t} + \tilde{\mathrm{p}}_1)\tilde{g}(\mathrm{t})\mathrm{dt}$, for $\tilde{\mathrm{p}}_1$ a constant in $J_1 = [0,1]$ and a constant $\tilde{c}_1$ in $L_1 = [0, M_1]$, for some $M_1 > 0$.

Solution

The crisp integral equation is

$g(s) = c_1 + \frac{1}{2}\int_0^1 (\mathrm{t} + \mathrm{p}_1)g(\mathrm{t})\mathrm{dt}$, for p_1 a constant in $J_1 = [0,1]$ and a constant c_1 in $L_1 = [0, M_1]$, for some $M_1 > 0$.

And the solution of the equation is $g(s) = \mathrm{G}(\mathrm{s}, \mathrm{p}_1, c_1)$

$= \frac{4c_1}{3-2p_1}$.

Therefore $\lambda = \frac{1}{2}$, $\mathrm{f}(\mathrm{s}, c_1) = c_1$ and $\mathrm{K}(\mathrm{s}, \mathrm{t}, \mathrm{p}_1) = (\mathrm{t} + \mathrm{p}_1)$ are in equation (1). Now for the corresponding fuzzy integral equation $\mathrm{f}(\mathrm{s}, \tilde{c}_1) = \tilde{c}_1, \mathrm{K}(\mathrm{s}, \mathrm{t}, \tilde{\mathrm{p}}_1) = (\mathrm{t} + \tilde{\mathrm{p}}_1) \geq 0$ for $t \in [0,1]$, $\tilde{\mathrm{p}}_1 \in [0,1]$ and $\tilde{Y}(s) = \frac{4\tilde{c}_1}{3-2\tilde{p}_1} \geq 0$ for $\tilde{c}_1 \in L_1$, $\tilde{\mathrm{p}}_1 \in [0,1]$.

Now, $\frac{\partial G}{\partial p_1} > 0$ for all $c_1 \in L_1$ and $\mathrm{p}_1 \in J_1$, $\frac{\partial K}{\partial p_1} > 0$ for all $\mathrm{p}_1 \in J_1$, $\frac{\partial f}{\partial c_1} > 0$ for all $c_1 \in L_1$ and $\frac{\partial G}{\partial c_1} > 0$ for all $c_1 \in L_1$ and $\mathrm{p}_1 \in J_1$.

Hence, $\frac{\partial G}{\partial p_1}\frac{\partial K}{\partial p_1} > 0$ and $\frac{\partial G}{\partial c_1}\frac{\partial f}{\partial c_1} > 0$ are for all $c_1 \in L_1$, $\mathrm{p}_1 \in J_1, s$ and t.

Therefore, from Corollary 1, the solution $\tilde{Y}(s)$, of the fuzzy linear integral equation

$\tilde{g}(s) = \tilde{c}_1 + \frac{1}{2}\int_0^1 (\mathrm{t} + \tilde{\mathrm{p}}_1)\tilde{g}(\mathrm{t})\mathrm{dt}$, exists and $\tilde{Y}(s)[\alpha] = [\frac{4c_{11}(\alpha)}{3-2p_{11}(\alpha)}, \frac{4c_{12}(\alpha)}{3-2p_{12}(\alpha)}]$.

4.2. Example 2

Solve the fuzzy linear integral equation $\tilde{g}(s) = \tilde{c}_1 s + \int_0^2 \tilde{g}(\mathrm{t})\mathrm{dt}$, for a constant $\tilde{c}_1$ in $L_1 = [0, M_1]$, for some $M_1 > 0$.

Solution

The crisp integral equation is

$g(s) = c_1 s + \int_0^2 g(\mathrm{t})\mathrm{dt}$, for a constant c_1 in $L_1 = [0, M_1]$, for some $M_1 > 0$.

And the solution of this equation is $g(s) = \mathrm{G}(\mathrm{s}, c_1)$

$$= (s-2)c_1$$

Therefore $\lambda = 1, \mathrm{f}(\mathrm{s}, c_1) = c_1 \mathrm{s}$ and $\mathrm{K}(\mathrm{s}, \mathrm{t}) = 1 > 0$ are in equation (1). Now for the corresponding fuzzy integral equation $\mathrm{f}(\mathrm{s}, \tilde{c}_1) = \tilde{c}_1 s$ and $\tilde{Y}(s) = (s-2)\tilde{c}_1 \leq 0$ for $c_1 \in L_1, 0 < s < 2$. Now $\frac{\partial f}{\partial c_1} > 0$ for all $c_1 \in L_1$ and $s > 0$. And $\frac{\partial G}{\partial c_1} < 0$ for all $c_1 \in L_1$ and $0 < s < 2$. So, $\frac{\partial G}{\partial c_1}\frac{\partial f}{\partial c_1} < 0$ are for all $c_1 \in L_1$ and $0 < s < 2$. Therefore, from Corollary 2, for $0 < s < 2$, the solution $\tilde{Y}(s)$ of the fuzzy linear integral equation

$\tilde{g}(s) = \tilde{c}_1 s + \int_0^2 \tilde{g}(\mathrm{t})\mathrm{dt}$, does not exist.

4.3 Example 3

Solve the fuzzy linear integral equation $\tilde{g}(s) = 1 + \tilde{p}_1 \int_0^1 \tilde{g}(t)dt$, for $\tilde{\mathrm{p}}_1$ a constant in $J_1 = [0, N_1]$ such that $p_{1\alpha} = [p_{11}, p_{12}]$, for some $N_1 > 0$.

Solution

The crisp integral equation is

$g(s) = 1 + p_1 \int_0^2 g(\mathrm{t})\mathrm{dt}$, for a constant p_1 in $J_1 = [0, N_1]$, for some $N_1 > 0$. The solution of this equation is $g(s) = \mathrm{G}(\mathrm{s}, p_1)$

$= \frac{1}{1-p_1}$, for $p_1 \neq 1$

Therefore $\lambda = 1, \mathrm{f}(\mathrm{s}, c_1) = 1$ and $\mathrm{K}(\mathrm{s}, \mathrm{t}) = p_1 > 0$ are in equation (1). Now for the corresponding fuzzy integral equation $\widetilde{\mathrm{K}}(\mathrm{s}, \mathrm{t}) = \tilde{p}_1$ and $\widetilde{Y}(s) = \begin{cases} \frac{1}{1-\tilde{p}_1} \geq 0, \text{ for } 0 \leq p_{12} < 1 \\ \frac{1}{1-\tilde{p}_1} \leq 0, \text{for } 1 < p_{11} \leq N_1 \end{cases}$. Now $\frac{\partial K}{\partial p_1} > 0$ and $\frac{\partial G}{\partial p_1} > 0$, for all $p_{12} \in [0,1)$ and $\frac{\partial K}{\partial p_1} > 0$ and $\frac{\partial G}{\partial p_1} > 0$, for all $p_{11} \in (1, N_1]$. Therefore, $\frac{\partial G}{\partial p_1}\frac{\partial f}{\partial p_1} > 0$, for all $0 \leq p_{12} < 1$

Hence, from Corollary 1, the solution $\widetilde{Y}(s)$ exists for $0 \leq p_{12} < 1$ and from Corollary 2, the solution $\widetilde{Y}(s)$ does not exist for $1 < p_{11} \leq N_1$.

The examples show that using the Theorem 1 and 2 we can say, the solution of fuzzy integral equation will exist or not even before solving it. Therefore use of this method and the condition of existence of the solution is time saving than the usual extension principle when the solution does not exist.

5. The Fuzzy SI Model

The simplest mathematical model that describes the dynamics of diseases transmitted by direct contact between susceptible and infected individuals is called SI [28-30]. In this model the individual does not recover from the disease. A typical example is AIDS, wherein once infected by the virus, the individual will remain with it for the rest of his life [30]. The differential equations which describe SI-model are given by

$$\frac{dS}{dt} = -mSI, S(0) = S_0 \geq 0$$

$$\frac{dI}{dt} = mSI, I(0) = I_0 > 0$$

where $S(t)$ is the proportion of susceptible individuals, $I(t)$ is the proportion of infected individuals at time t and m is the transmission coefficient of the disease.

If we consider there is no variation of the population then

$$S(t) + I(t) = 1, \forall\, t \geq 0.$$

Then the system becomes

$$\frac{dI}{dt} = m(1 - I)I, I(0) = I_0 > 0$$

which gives

$$I = I_0 + m\int_0^t (1 - I)Idt \quad (40)$$

whose solution is given by

$$I(t) = \frac{I_0 e^{mt}}{1 - I_0 + I_0 e^{mt}} = G(t, I_0, m)(say) \text{ and } (t) = 1 - I(t) = \frac{S_0}{1 - I_0 + I_0 e^{mt}}.$$

Now, focusing on incorporating the population diversity in the model, we consider the parameter m as fuzzy number $\widetilde{m}$. In reality, most of the cases, it is impossible to find out the exact fraction of infected and susceptible individuals at time $t = 0$. So, we consider S_0 and I_0 as fuzzy numbers $\tilde{S}_0$ and $\tilde{I}_0$ respectively.

Therefore our modified model becomes

$$\tilde{I} = \tilde{I}_0 + \widetilde{m}\int_0^t (1 - \tilde{I})\tilde{I}dt \quad (41)$$

with

$$\tilde{S}(t) + \tilde{I}(t) = 1, \forall\, t \geq 0 \quad (42)$$

Here, in equation (45), $\widetilde{K}(\tilde{I}, t, \widetilde{m}) = \widetilde{m}(1 - \tilde{I})\tilde{I} > 0$, $\tilde{f}(\tilde{I}, \tilde{I}_0) = \tilde{I}_0$ and $\tilde{Y}(\tilde{I}, t, \widetilde{m}, \tilde{I}_0) = \frac{\tilde{I}_0 e^{\widetilde{m}t}}{1 - \tilde{I}_0 + \tilde{I}_0 e^{\widetilde{m}t}}$.

Now, $\frac{\partial G}{\partial m} > 0$, $\frac{\partial K}{\partial m} > 0$, $\frac{\partial G}{\partial I_0} > 0$ and $\frac{\partial f}{\partial I_0} > 0 \quad \forall\, t \geq 0$.

So, $\frac{\partial G}{\partial m}\frac{\partial K}{\partial m} > 0$ and $\frac{\partial G}{\partial I_0}\frac{\partial f}{\partial I_0} > 0 \quad \forall\, t \geq 0$.

Therefore, from Corollary 1, the solution of (41) exists and by using extension principle the solution is

$$I_\alpha = \left[\frac{I_{01} e^{m_1 t}}{1 - I_{01} + I_{01} e^{m_1 t}}, \frac{I_{02} e^{m_2 t}}{1 - I_{02} + I_{02} e^{m_2 t}}\right] \text{ and } S_\alpha = 1 - I_\alpha = \left[\frac{1 - I_{01}}{1 - I_{01} + I_{01} e^{m_1 t}}, \frac{1 - I_{02}}{1 - I_{02} + I_{02} e^{m_2 t}}\right] \quad (43)$$

From, $\tilde{I}_0 + \tilde{S}_0 = 1$ we get $I_{01} + S_{01} = 1$ and $I_{02} + S_{02} = 1$.

So (43) becomes

$$I_\alpha = [\frac{I_{01}e^{m_1 t}}{S_{01}+I_{01}e^{m_1 t}}, \frac{I_{02}e^{m_2 t}}{S_{02}+I_{02}e^{m_2 t}}] \text{ and } S_\alpha = 1 - I_\alpha = [\frac{S_{01}}{S_{01}+I_{01}e^{m_1 t}}, \frac{S_{02}}{S_{02}+I_{02}e^{m_2 t}}] \quad (44)$$

Here if we take $\tilde{m}$, $\tilde{I}_0$ and $\tilde{S}_0$ as a particular triangular fuzzy number (a_0, a_1, a_1), a crisp constant I_0 and a crisp constant S_0 respectively, the given fuzzy SI model will be equivalence to the fuzzy SI model given in [28]. If we take $\tilde{I}_0$, $\tilde{S}_0$ as fuzzy numbers and $\tilde{m}$, as a crisp constant m, the given fuzzy SI model will be equivalence to the fuzzy SI model given in [30].

6. CONCLUSIONS

Solutions of the fuzzy linear integral equations have been presented. The main results (Theorem 1) proved in this paper gives that the solution of the fuzzy linear integral equation can be found directly from the crisp solution without going through the complexity of fuzziness if the conditions of the theorem 1 have been satisfied. We have shown when the solution by extension principle will not exist which is another main result (Theorem 2), proved in this paper. We have considered four cases depending on the kernel and the solution of the crisp integral equation. We have seen that if the kernel $\tilde{\mathrm{K}}(\mathrm{s}, \mathrm{t}, \tilde{\mathrm{p}})$ is negative and the solution $\tilde{Y}(t)$ is positive (or negative), the solution of the fuzzy equation does not exist. We have presented three examples. The usefulness of the proposed method has been shown in one application to fuzzy SI model which is more general and realistic than the fuzzy model given in [28, 30].

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