Advances in Fuzzy Mathematics (AFM).
ISSN 0973-533X Volume 13, Number 1 (2018), pp. 49–58

http://www.ripublication.com/afm.htm

Some Basic Properties of D^*-fuzzy metric spaces and Cantor's Intersection Theorem

Arunima Majumder
Department of Mathematics, Visva Bharati,
Santiniketan-731235, West Bengal, India.

Tarapada Bag[1]
Department of Mathematics, Visva Bharati,
Santiniketan-731235, West Bengal, India.

Abstract

In this paper, ideas of open ball, closed ball, D^*-fuzzy bounded set, compact set have been introduced and some basic properties are studied. The concept of α-fuzzy diameter is defined and Cantor's Intersection Theorem is established in fuzzy setting.

AMS subject classification:
Keywords: Fuzzy bounded, Compact set, Fuzzy diameter, Cantor's Intersection Theorem.

1. Introduction

The theory of fuzzy sets was introduced by L. Zadeh in 1965 [12]. After that fuzzy mathematics has been developed in different directions and one such development is fuzzy metric space. Different authors have generalized the concept of fuzzy metric spaces in different ways [2, 3, 4] and studied various properties on such spaces [9, 10, 11].

On the other hand many authors have expansively developed the idea of fuzzy metric spaces and established fixed point theorems. Sedghi et al. [8] introduced the concept of M-fuzzy metric space which is a generalization of fuzzy metric space defined by George

[1]Corresponding author.

and Veeramani [5]. Recently Bag [1] modified the definition of M-fuzzy metric space introduced by Sedghi et al. [8] and termed as D^*-fuzzy metric space. In this paper, we have considered D^*-fuzzy metric space introduced by Bag [1] and also considered the definition of convergence, Cauchyness, completeness introduced by A. Majumder, T. Bag [6]. Exploring the results in fuzzy metric spaces by George and Veeramani [5] we have given a new idea of open ball, closed ball, fuzzy boundedness, compactness and established many results. We have defined a new concept of α-fuzzy diameter and established Cantor's Intersection Theorem in fuzzy setting.

The organization of this paper is as follows:
Section 2 is provided for preliminary results which are used in this paper. In Section 3, some basic properties of D^*-fuzzy metric spaces have been studied. In Section 4, Cantor's Intersection Theorem has been proved.

2. Preliminaries

In this Section some preliminary results are given which are used in paper

Definition 2.1. ([7]) A binary operation
$* : [0,1] \times [0,1] \to [0,1]$
is a continuous t-norm. if $*$ satisfies the following condition;

1. $*$ is associative and commutative,
2. $*$ is continuous,
3. $a * 1 = a$ for all $a \in [0,1]$,
4. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, $a, b, c, d \in [0,1]$.

Definition 2.2. ([1]) A 3-tuple $(X, D^*, *)$ is called D^*-fuzzy metric space if X is an arbitrary (non-empty) set and D^* is a fuzzy set on $X^3 \times [0,\infty)$, satisfying the following conditions for each $x, y, z, a \in X$ and $t, s \in [0,\infty)$;
$(FD^*1) D^*(x, y, z, 0) = 0$,
$(FD^*2) \forall t > 0, D^*(x, y, z, t) = 1$ iff $x = y = z$,
$(FD^*3) D^*(x, y, z, t) = D^*(p\{x, y, z\}, t)$ (symmetry), where p is a permutation function,

$$(FD^*4) D^*(x, y, a, t) * D^*(a, z, z, s) \leq D^*(x, y, z, t + s),$$

$$(FD^*5) \lim_{t\to\infty} D^*(x, y, z, t) = 1.$$

3. Some basic results

In this section some basic properties of D^*-fuzzy metric spaces have been studied. Throughout this paper we consider $*$ as a continuous t-norm.

Lemma 3.1. Let $(X, D^*, *)$ be a D^*-fuzzy metric space and $D^*(x, y, z, .)$ is continuous on $[0, \infty)$, $\forall x, y, z \in X$. Then
$D^*(x, y, y, t) = D^*(x, x, y, t)\ \forall t > 0, \forall x, y \in X$.

Proof. Let $(X, D^*, *)$ be a D^*-fuzzy metric space. Now $\forall x, y \in X$

$$D^*(x, y, y, t + \frac{1}{n}) \geq D^*(y, y, y, \frac{1}{n}) * D^*(x, x, y, t)\ \forall t > 0, n \in N$$

$$\Rightarrow D^*(x, y, y, t + \frac{1}{n}) \geq 1 * D^*(x, x, y, t)\ \forall x, y \in X\ \forall t > 0.$$

$$\Rightarrow \lim_{n \to \infty} D^*(x, y, y, t + \frac{1}{n}) \geq D^*(x, x, y, t)\ \forall x, y \in X\ \forall t > 0.$$

So, $D^*(x, y, y, t) \geq D^*(x, x, y, t)\ \forall x, y \in X\ \forall t > 0$.
Similarly, we can prove that $D^*(x, x, y, t) \geq D^*(x, y, y, t)\ \forall x, y \in X\ \forall t > 0$.
Thus we can write $D^*(x, x, y, t) = D^*(x, y, y, t)$. ■

Definition 3.2. Let $(X, D^*, *)$ be a D^*-fuzzy metric space. We define open ball $B_{D^*}(x, r, t)$ with centre x and radius $r (0 < r < 1)$ and $t > 0$ as
$B_{D^*}(x, r, t) = \{y \in X : D^*(x, y, y, t) > 1 - r\}$.

Theorem 3.3. Let $(X, D^*, *)$ be a D^*-fuzzy metric space.
Define $\tau = \{A \subset X : x \in A$ iff $\exists t > 0$ and $r,\ 0 < r < 1$ such that $B_{D^*}(x, r, t) \subset A\}$.
Thus τ is a topology on X.

Proof. (i) From definition, it is clear that $\emptyset,\ X \in \tau$.

(ii) Consider $A_1, A_2, ...A_n \in \tau$ and let $\bigcap_{i=1}^{n} A_i = A$.

Let $x \in A$. Then $x \in A_i$ for $i = 1, 2, ...n$.
So for each i, $\exists t_i > 0$ and $r_i > 0$ with $0 < r_i < 1$
such that $B_{D^*}(x, r_i, t_i) \subset A_i$ for $i = 1, 2, ...n$.

Let $t = \bigwedge_{i=1}^{n} t_i,\ r = \bigwedge_{i=1}^{n} r_i$

Then $B_{D^*}(x, r, t) \subset B_{D^*}(x, r_i, t_i) \subset A_i$ for $i = 1, 2, ...n$
$\Rightarrow B_{D^*}(x, r, t) \subset A_i$ for each $i = 1, 2, ...n$

$$\Rightarrow B_{D^*}(x, r, t) \subset \bigcap_{i=1}^{n} A_i = A$$

$\Rightarrow A \in \tau$.
(iii) Union of arbitrary number of members of τ is also a member of τ. Thus τ is a topology on $(X, D^*, *)$. ■

Proposition 3.4. Every D^*-fuzzy metric space $(X, D^*, *)$ is Hausdroff.

Proof. Let $(X, D^*, *)$ be a D^*-fuzzy metric space. Let x,y be two distinct points of X. Then $0 < D^*(x, y, y, t) < 1$. Let $D^*(x, y, y, t) = r$ for some $r,\ 0 < r < 1$. For each $r_0,\ r < r_0 < 1$, we can find a r_1, such that $r_1 * r_1 \geq r_0$. Now consider the open balls

$B_{D^*}(x, 1-r_1, \frac{1}{2}t)$ and $B_{D^*}(y, 1-r_1, \frac{1}{2}t)$. Then $B_{D^*}(x, 1-r_1, \frac{1}{2}t) \bigcap B_{D^*}(y, 1-r_1, \frac{1}{2}t) = \emptyset$.
If possible, suppose $\exists z$ such that
$z \in B_{D^*}(x, 1-r_1, \frac{1}{2}t) \bigcap B_{D^*}(y, 1-r_1, \frac{1}{2}t)$.
Then $r = D^*(x, y, y, t)$
$\geq D^*(y, y, z, \frac{t}{2}) * D^*(z, x, x, \frac{t}{2})$
$\geq r_1 * r_1 \geq r_0 > r$,
which is a contradiction.
Therefore $(X, D^*, *)$ is Hausdroff. ■

Proposition 3.5. Let $(X, D^*, *)$ be a D^*-fuzzy metric space where $D^*(x, y, z, \cdot)$ is continuous on $[0, \infty)$. Then every open ball in X is an open set.

Proof. Let $(X, D^*, *)$ be a D^*-fuzzy metric space.
For some $t > 0$ and $0 < r < 1$,
we can have $y \epsilon B_{D^*}(x, r, t) \Rightarrow D^*(x, x, y, t) > 1 - r$.
We find t_0, $0 < t_0 < t$ such that $D^*(x, x, y, t_0) > 1 - r$.
Let $r_0 = D^*(x, x, y, t_0) > 1 - r$. Since $r_0 > 1 - r$, we can find a $s, 0 < s < 1$, such that $r_0 > 1 - s > 1 - r$.
Now for a given r_0 and s such that $r_0 > 1 - s$, we find $r_1, 0 < r_1 < 1$ such that $r_0 * r_1 \geq 1 - s$.
Now we claim that $B_{D^*}(y, 1 - r_1, t - t_0) \subset B_{D^*}(x, r, t)$.
Now $z \in B_{D^*}(y, 1 - r_1, t - t_0)$
$\Rightarrow D^*(y, y, z, t - t_0) > r_1$.
So, $D^*(x, x, z, t) \geq D^*(x, x, y, t_0) * D^*(y, z, z, t - t_0)$
$= D^*(x, x, y, t_0) * D^*(y, y, z, t - t_0) \geq r_0 * r_1 \geq 1 - s \geq 1 - r$.
Therefore $z \in B_{D^*}(x, r, t)$ and hence
$B_{D^*}(y, 1 - r_1, t - t_0) \subset B_{D^*}(x, r, t)$. So every open ball in X is an open set. ■

Definition 3.6. Let $(X, D^*, *)$ be a D^*-fuzzy metric space. Then we define a closed ball with centre $x \epsilon X$ and radius r, $0 < r < 1, t > 0$ as
$B_{D^*}[x, r, t] = \{y \in X : D^*(x, y, y, t) \geq 1 - r\}$.

Definition 3.7. Let $(X, D^*, *)$ be a D^*-fuzzy metric space. A subset A of X is said to be closed if for any sequence $\{x_n\}$ in A such that $x_n \to x$ implies $x \in A$.

Definition 3.8. Let $(X, D^*, *)$ be a D^*-fuzzy metric space and $A(\subset X)$ be a nonempty subset of X. Then the closure of A denoted by $\bar{A}$ is a set such that for $x \in \bar{A}$, $\exists$ a sequence $\{x_n\}$ in A and $x_n \to x$.

Lemma 3.9. Every closed ball in a D^*-fuzzy metric space is a closed set if we assume that $D^*(x, y, z, \cdot)$ is continuous on $[0, \infty)$.

Proof. Let $(X, D^*, *)$ be a D^*-fuzzy metric space. Consider a closed ball $B_{D^*}[x, r, t]$.

Choose $y \in \overline{B_{D^*}[x,r,t]}$. Then $\exists\{y_n\}$ in
$B_{D^*}[x,r,t]$ such that $y_n \to y$. For a given $\epsilon > 0$,
$D^*(x,y,y,t+\epsilon) \geq D^*(y,y,y_n,\epsilon) * D^*(y_n,x,x,t) \geq D^*(y,y,y_n,\epsilon) * (1-r)$
Taking limit $n \to \infty$, we have,
$D^*(x,y,y,t+\epsilon) \geq \lim_{n\to\infty} D^*(y,y,y_n,\epsilon) * (1-r) \geq 1 * (1-r) = 1-r$
Take $\epsilon = \frac{1}{n}$ for $n \in N$
Now $D^*(x,y,y,t+\frac{1}{n}) \geq 1-r$
Letting $n \to \infty$, we have
$\lim_{n\to\infty} D^*(x,y,y,t+\frac{1}{n}) \geq 1-r$
So, $D^*(x,y,y,t) \geq 1-r$. Thus $y \in B[x,r,t]$. Therefore $B[x,r,t]$ is a closed set. ■

Definition 3.10. Let $(X, D^*, *)$ be a D^*-fuzzy metric space. A subset A of X is said to be D^*-fuzzy bounded if $\exists t > 0$ and $0 < r < 1$ such that
$D^*(x,y,y,t) > 1-r, \forall x,y \in A$.

Definition 3.11. Let $(X, D^*, *)$ be a D^*-fuzzy metric space. $A(\subset X)$ be a nonempty subset of X. Then A is said to be compact if every open cover of A has a finite subcover. i.e. if $A \subset \bigcup \{G_i : i \in I\}$ be an open cover of A in X where G_i is open subset of X for each i, then $A \subset \bigcup \{G_i : i = 1, 2, ...n\}$.

Theorem 3.12. Let $(X, D^*, *)$ is a D^*-fuzzy metric space where $D^*(x,y,z,\cdot)$ is continuous on $[0,\infty)$ and $A(\subset X)$ be a non-empty compact subset of X. Then A is D^*-fuzzy bounded.

Proof. Let $(X, D^*, *)$ is a D^*-fuzzy metric space and A is a compact subset of X. Fix $t > 0$ and $0 < r < 1$. Consider an open cover $\{B_{D^*}(x,r,t) : x \in A\}$ of A. Since A is compact, $\exists x_1, x_2, ..., x_n \in A$ such that

$$A \subseteq \bigcup_{i=1}^{n} B_{D^*}(x_i,r,t).$$

Let $x, y \in A$. Then $x \in B_{D^*}(x_i,r,t)$ and $y \in B_{D^*}(x_j,r,t)$ for some i, j. Let $\alpha = min\{D^*(x_i,x_i,x_j,t);\ 1 \leq i\ j \leq n\}$.
Now $D^*(x,y,y,4t) \geq D^*(x,y,x_j,3t) * D^*(x_j,y,y,t) \geq D^*(y,x_j,x_i,2t) * D^*(x_i,x,x,t) * D^*(x_j,y,y,t) \geq D^*(y,x_j,x_j,t) * D^*(x_j,x_i,x_i,t) * D^*(x_j,x,x,t) * D^*(x_j,y,y,t)$.
Then $D^*(x,y,y,4t) \geq (1-r) * \alpha * (1-r) * (1-r)$.
Taking $t' = 4t$ and choose s such that $(1-r) * \alpha * (1-r) * (1-r) > 1-s,\ 0 < s < 1$.
We have $D^*(x,y,y,t') \geq 1-s\ \forall x,y \in A$. Hence A is D^*- fuzzy bounded. ■

4. Cantor's Intersection Theorem

In this section Cantor's Intersection Theorem is established in fuzzy setting.

Lemma 4.1. Let $(X, D^*, *)$ is a D^*-fuzzy metric space and $\{x_n\}, \{y_n\}$ be two sequences in X, such that, $x_n \to x_0$ and $y_n \to y_0$, for some $x_0, y_0 \in X$. Then $\lim_{n\to\infty} D^*(x_n, x_n, y_n, t) = D^*(x_0, x_0, y_0, t)\ \forall t > 0$.

Proof. Choose $\epsilon > 0$ arbitrary. Now, $\forall t > 0$,
$D^*(x_0, x_0, y_0, t + \epsilon) \geq D^*(x_0, x_0, x_n, \frac{\epsilon}{2}) * D^*(x_n, y_0, y_0, t + \frac{\epsilon}{2})$
$\geq D^*(x_0, x_0, x_n, \frac{\epsilon}{2}) * D^*(y_0, y_0, y_n, \frac{\epsilon}{2}) * D^*(y_n, x_n, x_n, t)$
Now, letting $n \to \infty$, we have,
$D^*(x_0, x_0, y_0, t+\epsilon) \geq \lim_{n\to\infty} D^*(x_0, x_0, x_n, \frac{\epsilon}{2}) * \lim_{n\to\infty} D^*(y_n, y_0, y_0, \frac{\epsilon}{2}) * \lim_{n\to\infty} D^*(x_n, x_n, y_n, t)$
$= 1 * 1 * \lim_{n\to\infty} D^*(x_n, x_n, y_n, t)$
$= \lim_{n\to\infty} D^*(x_n, x_n, y_n, t)$
We have,
$\lim_{\epsilon\to\infty} D^*(x_0, x_0, y_0, t + \epsilon) = D^*(x_0, x_0, y_0, t) \geq \lim_{n\to\infty} D^*(x_n, x_n, y_n, t)$
So, $D^*(x_0, x_0, y_0, t) \geq \lim_{n\to\infty} D^*(x_n, x_n, y_n, t)\ \forall t > 0$....... (1)
Similarly, for any $t > 0$, we choose $\epsilon(0 < \frac{\epsilon}{2} < t)$.
We can write,
$D^*(x_n, x_n, y_n, t) \geq D^*(x_n, x_n, x_0, \frac{\epsilon}{4}) * D^*(x_0, y_n, y_n, t - \frac{\epsilon}{4})$
$\geq D^*(x_n, x_n, x_0, \frac{\epsilon}{4}) * D^*(y_n, y_n, y_0, \frac{\epsilon}{4}) * D^*(y_0, x_0, x_0, t - \frac{\epsilon}{2})$
Now, letting $n \to \infty$, we have,
$\lim_{n\to\infty} D^*(x_n, x_n, y_n, t) \geq \lim_{n\to\infty} D^*(x_n, x_n, x_0, \frac{\epsilon}{4}) * \lim_{n\to\infty} D^*(y_n, y_n, y_0, \frac{\epsilon}{4})$
$* D^*(x_0, x_0, y_0, t - \frac{\epsilon}{2})$
$\Rightarrow \lim_{n\to\infty} D^*(x_n, x_n, y_n, t) \geq 1 * 1 * D^*(x_0, x_0, y_0, t - \frac{\epsilon}{2})$
$= D^*(x_0, x_0, y_0, t - \frac{\epsilon}{2})$.
As $\epsilon \to 0$, we write,
$\lim_{n\to\infty} D^*(x_n, x_n, y_n, t) \geq \lim_{\epsilon\to 0} D^*(x_0, x_0, y_0, t - \frac{\epsilon}{2})$
$= D^*(x_0, x_0, y_0, t)$
i.e $\lim_{n\to\infty} D^*(x_n, x_n, y_n, t) \geq D^*(x_0, x_0, y_0, t)\ \forall t > 0$.......(2)
So, by (1) and (2), we can write
$\lim_{n\to\infty} D^*(x_n, x_n, y_n, t) = D^*(x_0, x_0, y_0, t)\ \forall t > 0$. ■

Definition 4.2. Let $(X, D^*, *)$ be a D^*-fuzzy metric space where $*$ is a continuous t-norm. Then we define α-fuzzy diameter of A where A is an nonempty subset of X as

$$\alpha - \delta(A) = \bigvee_{x,y\in A} \bigwedge \{t > 0 : D^*(y, y, x, t) \geq \alpha\},\ \alpha \in (0, 1)$$
$$= \bigvee_{x,y\in A} \bigwedge \{t > 0 : D^*(x, x, y, t) \geq \alpha\},\ \alpha \in (0, 1)$$

Lemma 4.3. Let $(X, D^*, *)$ be a D^*-fuzzy metric space and $A(\subset X)$ be a nonempty subset of X and $D^*(x, y, z, \cdot)$ is continuous in $[0, \infty)$. Then $\alpha - \delta(A) = \alpha - \delta(\bar{A})\ \forall \alpha \in (0, 1)$.

Proof. Choose $\alpha_0 \in (0, 1)$ arbitrary.
We have $\alpha_0 - \delta(A) = \bigvee_{x,y\in A} \bigwedge \{t > 0 : D^*(x, x, y, t) \geq \alpha_0\}$
Since $A \subset \bar{A}$ we have
$\bigvee_{x,y\in \bar{A}} \bigwedge \{t > 0 : D^*(x, x, y, t) \geq \alpha_0\} \geq \bigvee_{x,y\in A} \bigwedge \{t > 0 : D^*(x, x, y, t) \geq \alpha_0\}$
$\Rightarrow \alpha_0 - \delta(\bar{A}) \geq \alpha_0 - \delta(A)$......(i)
Next suppose $\alpha_0 - \delta(A) < t_0$
$\Rightarrow \bigvee_{x,y\in A} \bigwedge \{t > 0 : D^*(x, x, y, t) \geq \alpha_0\} < t_0$
$\Rightarrow \bigwedge \{t > 0 : D^*(x, x, y, t) \geq \alpha_0\} < t_0\ \forall x, y \in A$ ....(ii)
Choose $x_0, y_0 \in \bar{A}$. Then $\exists$ sequences $\{x_n\}, \{y_n\}$ in A, such that
$\lim_{n\to\infty} D^*(x_n, x_n, y_n, t) = D^*(x_0, x_0, y_0, t)\ \forall t > 0$ (by Lemma 4.1)............(iii)
From (ii) we have, $\bigwedge \{t > 0 : D^*(x_n, x_n, y_n, t) \geq \alpha_0\} < t_0\ \forall n$
Then by proposition, it follow that
$D^*(x_n, x_n, y_n, t_0) \geq \alpha_0\ \forall n$
$\Rightarrow \lim_{n\to\infty} D^*(x_n, x_n, y_n, t_0) \geq \alpha_0$
$D^*(x_0, x_0, y_0, t_0) \geq \alpha_0$ by (iii)
$\Rightarrow \bigwedge \{t > 0 : D^*(x_0, x_0, y_0, t) \geq \alpha_0\} \leq t_0$
$\Rightarrow \bigvee_{x,y\in \bar{A}} \bigwedge \{t > 0 : D^*(x, x, y, t) \geq \alpha_0\} \leq t_0$
$\Rightarrow \alpha_0 - \delta(\bar{A}) \leq t_0$
$\Rightarrow \alpha_0 - \delta(A) \geq \alpha_0 - \delta(\bar{A})$ (iv)
From (i) and (iv) we get $\alpha_0 - \delta(A) = \alpha_0 - \delta(\bar{A})$
Since $\alpha \in (0, 1)$ is arbitrary,
we have $\alpha_0 - \delta(A) = \alpha_0 - \delta(\bar{A})\ \forall \alpha \in (0, 1)$. ■

Theorem 4.4. (Cantor's Intersection Theorem) A necessary and sufficient condition that D^*- fuzzy metric space (X,$D^*, *$) be complete is that every nested sequence of non-empty closed subsets F_i with α-fuzzy diameter tending to 0 for each $\alpha \in (0, 1)$ as

$i \to \infty$ be such that $\bigcap_{i=1}^{\infty} F_i$ contains exactly one point.

Proof. First we suppose that (X,$D^*, *$) is a complete D^*- fuzzy metric space. Consider a sequence of closed subsets F_i such that $F_1 \supset F_2 \supset F_3$..... with $\alpha - \delta(F_n) \to 0$ as $n \to \infty$ $\forall \alpha \in (0,1)$.
Choose $x_n \in F_n$ for each n=1,2,3,..... We have obtained a sequence $\{x_n\}$ in X. Now we verify that $\{x_n\}$ is a Cauchy sequence.
We have $x_n \in F_n$ and $x_{n+p} \in F_{n+p} \subset F_n$ $\forall n$ and p=1,2,3,......
Now $\bigwedge \{t > 0 : D^*(x_n, x_n, x_{n+p}, t) \geq \alpha\} \leq \alpha - \delta(F_n)$ $\forall n$ and p=1,2,3,.and $\forall \alpha \in (0,1)$.
$\Rightarrow \lim_{n\to\infty} \bigwedge \{t > 0 : \ D^*(x_n, x_n, x_{n+p}, t) \geq \alpha_0\} = 0$ p=1,2,3,...and $\forall \alpha \in (0,1)$
$\Rightarrow$ for each $\epsilon > 0$, $\exists N(\epsilon, \alpha)$ such that
$\bigwedge \{t > 0 : D^*(x_n, x_n, x_{n+p}, t) \geq \alpha\} < \epsilon$ $\forall n \geq N(\alpha, \epsilon)$ and p=1,2,3,.and $\forall \alpha \in (0,1)$.
$\Rightarrow D^*(x_n, x_n, x_{n+p}, \epsilon) \geq \alpha$ $\forall \alpha \in (0,1)$, $\forall n \geq N(\alpha, \epsilon)$ and for p=1,2,.....
$\Rightarrow \lim_{n\to\infty} D^*(x_n, x_n, x_{n+p}, \epsilon) \geq \alpha$ $\forall \alpha \in (0,1)$ and for p=1,2,3.....
$\Rightarrow \lim_{n\to\infty} D^*(x_n, x_n, x_{n+p}, \epsilon) = 1$ for p=1,2,3,....
Since $\epsilon > 0$ is arbitrary, it follows that
$\lim_{n\to\infty} D^*(x_n, x_n, x_{n+p}, t) = 1$ $\forall t > 0$ for p=1,2,3,....
Thus $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, $\{x_n\}$ converges to a point $x \in X$.
Let k be an arbitrary positive integer and the set F_k. Then each member of the sequence x_k, x_{k+1}, x_{k+2}...... lies in F_k. Since F_k is closed, if follows that $x \in F_k$ and as k is arbitrary, we have $x \in \bigcap_{i=1}^{\infty} F_i$.

Uniqueness. If possible suppose that $\exists y \in X$ such that $y \in \bigcap_{i=1}^{\infty} F_i$
Now for $x, y, \in F_k$ for k=1,2,3,.... we have
$\bigwedge \{t > 0 : D^*(x, x, y, t) \geq \alpha\} \leq \alpha - \delta(F_k)$ $\forall \alpha \in (0,1)$ k=1,2,3,...
$\Rightarrow \bigwedge \{t > 0 : D^*(x, x, y, t) \geq \alpha\} = 0 \because \alpha - \delta(F_k) \to 0$ as $k \to \infty$.
$\Rightarrow D^*(x, x, y, t) \geq \alpha$ $\forall \alpha \in (0,1)$, $\forall t > 0$
$\Rightarrow D^*(x, x, y, t) = 1$ $\forall t > 0$
$\Rightarrow x = y$.
Conversely suppose that the condition of the theorem is satisfied. we shall show that X is complete.
Let $\{x_n\}$ be a Cauchy sequence of points of X.
Let $H_n = \{x_n, x_{n+1}, x_{n+2}, ...\}$
We have $\lim_{n\to\infty} D^*(x_n, x_n, x_{n+p}, t) = 1$ $\forall t > 0$ for p=1,2,3,....
So $\lim_{n\to\infty} D^*(x_n, x_n, x_{n+p}, t) > \alpha$ $\forall t > 0$ for p=1,2,3,.... and $\forall \alpha \in (0,1)$

Choose $t_0 > 0$ be arbitrary. Then for each $\alpha, \exists$ a positive integer $N(\alpha)$ such that
$D^*(x_n, x_n, x_{n+p}, t_0) > \alpha \ \forall n \geq N(\alpha)$ p=1,2,3,... $\forall \alpha \in (0, 1)$
$\Rightarrow \bigwedge \{t > 0 : D^*(x_n, x_n, x_{n+p}, t) > \alpha\} \leq t_0 \ \forall n \geq N(\alpha)$ and p=1,2,3,... $\forall \alpha \in (0, 1)$
$\Rightarrow \bigwedge \{t > 0 : D^*(x_n, x_n, x_{n+p}, t) \geq \alpha\} \leq t_0 \ \forall n \geq N(\alpha)$ and p=1,2,3,... $\forall \alpha \in (0, 1)$
$\Rightarrow \bigvee_{x_n \in Hn} \bigwedge \{t > 0 : \ D^*(x_n, x_n, x_{n+p}, t) \geq \alpha\} \leq t_0 \ \forall n \geq N(\alpha) \ \forall \alpha \in (0, 1)$
$\Rightarrow \alpha - \delta(H_n) \leq t_0 \ \forall n \geq N(\alpha) \ \forall \alpha \in (0, 1)$
$\Rightarrow \alpha - \delta(\bar{H_n}) \leq t_0 \ \forall n \geq N(\alpha) \ \forall \alpha \in (0, 1)$
Since t_0 is arbitrary, we have $\alpha - \delta(\bar{H_n}) = 0$ as $n \to \infty \ \forall \alpha \in (0, 1)$.
We have $H_{n+1} \subset H_n$ for each n and so $\bar{H_{n+1}} \subset \bar{H_n} \ \forall n$.
Thus $\{\bar{H_n}\}$ constitutes a closed, nested sequence of non-empty sets in X where $\alpha - \bar{H_n} \to 0$ as $n \to \infty$. Thus by hypothesis
$\exists$ a unique $x \in \bigcap_{i=1}^{\infty} \bar{H_n}$.
Since $x_n \in H_n \subset \bar{H_n}$ and $x \in \bar{H_n}$, so
$\bigwedge \{t > 0 : D^*(x_n, x_n, x, t) \geq \alpha\} \leq \alpha - \delta(\bar{H_n}) \ \forall \alpha \in (0, 1)$
$\Rightarrow \lim_{n \to \infty} \bigwedge \{t > 0 : \ D^*(x_n, x_n, x, t) \geq \alpha\} = 0$
Choose $\epsilon > 0$. The $\exists N(\alpha, \epsilon)$ such that
$\bigwedge \{t > 0 : D^*(x_n, x_n, x, t) \geq \alpha\} < \epsilon \ \forall \alpha \in (0, 1) \ \forall n \geq N(\alpha, \epsilon)$
$\Rightarrow D^*(x_n, x_n, x, \epsilon) \geq \alpha \ \forall \alpha \in (0, 1) \ \forall n \geq N(\alpha, \epsilon)$
$\Rightarrow \lim_{n \to \infty} D^*(x_n, x_n, x, \epsilon) \geq \alpha \ \forall \alpha \in (0, 1)$
$\Rightarrow \lim_{n \to \infty} D^*(x_n, x_n, x, \epsilon) = 1$
Since $\epsilon > 0$ is arbitrary. So
$\lim_{n \to \infty} D^*(x_n, x_n, x, t) = 1 \ \forall t > 0$
$\Rightarrow x_n \to x$.
$\Rightarrow X$ is complete. ■

5. Conclusion

In this paper, some basic results on D^*-fuzzy metric spaces have been established. We also define a new concept of α-fuzzy diameter and establish Cantor's Intersection Theorem in fuzzy setting. The results of the paper can significantly contribute for the development of new ideas in D^*-fuzzy metric spaces.

Acknoledgements

The present work is partially supported by Special Assistance Programme (SAP) of UGC, New Delhi, India [Grant No. F. 510/4/DRS/2009 (SAP-I)].

References

[1] T. Bag, Some results on D^*-metric spaces, International Journal of Mathematics and Scientific Computing. 2(1) (2012).

[2] T. Bag, Fuzzy Cone Metric Spaces and Fixed Point Theorems of Contractive Mappings, Annals of Fuzzy Mathematics and Information, 6(3) (2013), 657–668.

[3] Li. Changqing, Li. Kedian, On Topological Properties of the Hausdorff Fuzzy Metric Spaces, Filomat, 31(5) (2017), 1167–1173.

[4] N. R. Das and M. L. Saha, On fixed points in fuzzy metric spaces, Annals of Fuzzy Mathematics and Informatics, Vol. 7, No. 2, (2014), 313–318.

[5] A. George and P. Veeramani, On Some results in fuzzy metric spaces, *Fuzzy Sets and Systems. vol.* 64, (1994), 395–399.

[6] A. Majumder, T. Bag, Some fixed point theorems on D^*-fuzzy metric spaces, Annals of Fuzzy Mathematics and Informatics, Vol. 12, no. 3 (2016), 411–417.

[7] B. Schweizer, A. Sklar, Statistical metric spaces, Pacific J. Maths., 10(1960), 313–334.

[8] S. Sedghi and N. Shobe, Fixed point Theorem in M-fuzzy metric spaces with property (E), *Advances in Fuzzy Mathematics*, vol. 1, no. 1, (2006), 55–65.

[9] S. Sedghi, N. Shobe and H. Zhou, A common Fixed Point Theorem in D*-metric Spaces, *Fixed Point Theory and Applications*, vol. 2007, Article ID 27906, (2007), 1–13.

[10] D. Singh, M. Sharma and R. Sharma, Fixed point theorems in M- fuzzy metric space, Annals of Fuzzy Mathematics and Informatics, Vol. 5, No. 1, (2013), 147–155.

[11] T. Veerapandi and Aji. M. Pillai, A common fixed point theorem and some fixed point theorems in D*-metric spaces, African Journal of Mathematics and Computer Science Research, vol. 4(8), (2011), 273–280.

[12] L. A. Zadeh, Fuzzy Sets, Inform. and Control. Vol. 8 (1965), 338–353.