

Generalized m -Regular α -Closed Set and Its Applications

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Abstract

This paper deals with a new type of generalized version of closed set via fuzzy m -regular open set. The class of this newly defined closed sets is strictly larger than that of fuzzy m -regular closed sets. Afterwards, new type of neighbourhood structure of a fuzzy point is introduced and studied some properties. Lastly a new type of continuous-like function is introduced and applications of this newly defined function is established in fuzzy m -regular and fuzzy m -normal spaces.

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1. INTRODUCTION

After introducing fuzzy set by Zadeh in [9], fuzzy topology was introduced by Chang [6]. A more general version of fuzzy topology was introduced in [2]. Fuzzy minimal structure was given in [1] as follows : A family \mathcal{F} of fuzzy sets on a non-empty set X is called fuzzy minimal structure on X if $\alpha 1_X \in \mathcal{F}$ for every $\alpha \in [0, 1]$. Afterwards, a more general version of it was given in [5, 7] as follows : A family \mathcal{M} of fuzzy sets on a non-empty set X is called a fuzzy minimal structure on X if $0_X \in \mathcal{M}$ and $1_X \in \mathcal{M}$. Throughout this paper we use the last concept. In [2], fuzzy minimal space was introduced as follows : Let X be a non-empty set and $m \subset I^X$. Then (X, m) is called a fuzzy minimal space (fuzzy m -space, for short) if $0_X, 1_X \in m$. The members of m are called fuzzy m -open [2] and the complement of a fuzzy m -open set is called fuzzy m -closed [2]. In [3], fuzzy m -regular open and fuzzy m - α -open sets are introduced whereas in [4], fgm -closed set is introduced.

2. PRELIMINARIES

Throughout the paper (X, m) or simply by X we shall mean a fuzzy m -space in the sense of Bhattacharyya [2]. In [9], fuzzy set is defined as follows : A fuzzy set A is a mapping from a non-empty set X into the closed interval $I = [0, 1]$, i.e., $A \in I^X$. The support [9] of a fuzzy set A , denoted by $suppA$ and is defined by $suppA = \{x \in X : A(x) \neq 0\}$. The fuzzy set with the singleton support $\{x\} \subseteq X$ and the value t ($0 < t \leq 1$) will be denoted by x_t . 0_X and 1_X are the constant fuzzy sets taking values 0 and 1 respectively in X . The complement [9] of a fuzzy set A is denoted by $1_X \setminus A$ and is defined by $(1_X \setminus A)(x) = 1 - A(x)$, for each $x \in X$. For any two fuzzy sets A, B in X , $A \leq B$ means $A(x) \leq B(x)$, for all $x \in X$ [9] while AqB means A is quasi-coincident (q-coincident, for short) [8] with B , i.e., there exists $x \in X$ such that $A(x) + B(x) > 1$. The negation of these two statements will be denoted by $A \not\leq B$ and $A \not q B$ respectively. For any two fuzzy sets $A, B \in I^X$, $A \vee B = \max\{A(x), B(x)\}$, for all $x \in X$ and $A \wedge B = \min\{A(x), B(x)\}$, for all $x \in X$. For a fuzzy point x_t and a fuzzy set A , $x_t \in A$ means $A(x) \geq t$.

For a fuzzy set A in a fuzzy m -space X , $mintA$ [2] and $mclA$ [2] are defined as follow :

$$mintA = \bigvee \{G : G \leq A, G \in m\}$$

$$mclA = \bigwedge \{F : A \leq F, 1_X \setminus F \in m\}$$

It can be observed that for a fuzzy set A in X , $mintA$ may not be fuzzy m -open and $mclA$ may not be fuzzy m -closed. But if m satisfies \mathcal{B} condition, i.e., m is closed under arbitrary union, then $mintA \in m$ and $mclA \in m^c$. $A \in I^X$ is called a fuzzy m -nbd of a fuzzy point x_t if there exists a fuzzy m -open set G such that $x_t \in G \leq A$ [2]. If, in addition, A is fuzzy m -open, then A is called fuzzy m -open nbd of x_t .

A fuzzy set A in a fuzzy m -space (X, m) is called fuzzy m -regular open [3] (resp., fuzzy m - α -open [3]) if $A = mint(mclA)$ (resp., $A \leq mint(mcl(mintA))$). The complement of this fuzzy set is called fuzzy m -regular closed [3] (resp., fuzzy m - α -closed [3]). The union (intersection) of fuzzy m - α -open (resp., fuzzy m - α -closed) sets contained in (resp., containing) a fuzzy set A in X is called fuzzy m - α -interior [3] (resp., fuzzy m - α -closure [3]) of A and is denoted by $maintA$ (resp., $maclA$). In general $maintA$ may not be fuzzy m - α -open and $maclA$ may not be fuzzy m - α -closed. But if m satisfies \mathcal{B} condition, then $maintA$ is fuzzy m - α -open and $maclA$ is fuzzy m - α -closed. For any fuzzy set A in a fuzzy m -space X , $macl(maclA) = maclA$ [3].

The collection of all fuzzy m -regular open (resp., fuzzy m - α -open) sets in X is denoted by $FmRO(X)$ (resp., $Fm\alpha O(X)$) and that of fuzzy m -regular closed (resp., fuzzy m - α -closed) sets in X is denoted by $FmRC(X)$ (resp., $Fm\alpha C(X)$).

3. $fgmr\alpha$ -CLOSED SET : SOME PROPERTIES

In this section a new type of generalized version of closed set is introduced in fuzzy m -space. Afterwards, a new type of neighbourhood structure of a fuzzy point is introduced and studied.

Definition 3.1. A fuzzy set A in a fuzzy m -space (X, m) is called fuzzy m -regular α -open ($fmr\alpha$ -open, for short) if there exists $U \in FmRO(X)$ such that $U \leq A \leq m\alpha clU$.

The family of all $fmr\alpha$ -open sets of X is denoted by $FmR\alpha O(X)$.

Definition 3.2. A fuzzy set A in a fuzzy m -space (X, m) is called fuzzy generalized m -regular α -closed ($fgmr\alpha$ -closed, for short) if $m\alpha clA \leq U$ whenever $A \leq U$ and $U \in FmR\alpha O(X)$.

The complement of a fuzzy generalized m -regular α -closed set is called fuzzy generalized m -regular α -open ($fgmr\alpha$ -open, for short) set.

The family of all $fgmr\alpha$ -closed (resp., $fgmr\alpha$ -open) sets of X is denoted by $FGmR\alpha C(X)$ (resp., $FGmR\alpha O(X)$).

Result 3.3. Let (X, m) be a fuzzy m -space and $A \in I^X$. Then $A \in FmRO(X) \Rightarrow A \in FmR\alpha O(X)$ and $A \in Fm\alpha O(X) \Rightarrow A \in FmR\alpha O(X)$.

Proof. $A \in FmRO(X) \Rightarrow A = \text{mint}(mclA)$. Then $A \leq A \leq m\alpha clA \Rightarrow A \in FGmR\alpha O(X)$.

Next $A \in Fm\alpha O(X) \Rightarrow A \leq \text{mint}(mcl(\text{mint}A)) \leq \text{mint}(mclA) \leq m\alpha clA(\text{mint}(mclA)) \dots(1)$. Now $A \in FmR\alpha O(X) \Rightarrow$ there exists $U \in FmRO(X)$ such that $U \leq A \leq m\alpha clU = m\alpha cl(\text{mint}(mclU)) \Rightarrow A \leq m\alpha cl(\text{mint}(mclA))$ which is true in (1). Hence the proof.

Note 3.4. Let (X, m) be a fuzzy m -space and $A \in I^X$. Then A is fuzzy m -closed set in X implies A is $fgmr\alpha$ -closed set in X . Infact, $A \in m^c$ and $U \in FmR\alpha O(X)$ with $A \leq U$. Then $m\alpha clA \leq mclA = A \leq U \Rightarrow A \in FGmR\alpha C(X)$.

Similarly we can show that $A \in m \Rightarrow A \in FGmR\alpha O(X)$.

But the converse is not necessarily true, as it is seen from the next example.

Example 3.5. Let $X = \{a, b\}$, $m = \{0_X, 1_X, S\}$ where $S(a) = 0.5, S(b) = 0.6$. Then (X, m) is a fuzzy m -space. Here $FmRO(X) = \{0_X, 1_X\}$. Consider the fuzzy set T defined by $T(a) = T(b) = 0.5$. Clearly $T \notin m^c$, but $T \in FGmR\alpha C(X)$.

Note 3.6. Let (X, m) be a fuzzy m -space where m satisfies \mathcal{B} condition and $A \in I^X$. Then $A \in FmRC(X)$ (resp., $A \in FmRO(X)$) implies $A \in FGmR\alpha C(X)$ (resp., $A \in FGmR\alpha O(X)$). But the converse may not be true follows from Example 3.5. Here $T \in FGmR\alpha C(X)$, but $T \notin FmRC(X)$. Also $T \in FGmR\alpha O(X)$, but $T \notin FmRO(X)$.

Let us recall the following definition from [4] for ready references.

Definition 3.7 [4]. A fuzzy set A in a fuzzy m -space (X, m) is called fgm -closed if $mclA \leq U$ whenever $A \leq U \in m$.

Remark 3.8. The next two examples show that fgm -closedness and $fgmr\alpha$ -closedness are independent concepts.

Example 3.9. $fgmr\alpha$ -closed set $\not\Rightarrow fgm$ -closed set

Consider Example 3.5. Here T is $fgmr\alpha$ -closed set. Now $T \leq S \in m$. But $mclT = 1_X \not\leq S \Rightarrow T$ is not fgm -closed set in X .

Example 3.10. fgm -closed set $\not\Rightarrow fgm\alpha$ -closed set

Let $X = \{c, d\}$, $m = \{0_X, 1_X, U\}$ where $U(c) = 0.5, U(d) = 0.4$. Then (X, m) is a fuzzy m -space. Consider the fuzzy set V defined by $V(c) = 0.5, V(d) = 0.45$. Then $V < 1_X \in m$ only and so V is fgm -closed set in X . Next consider the fuzzy set W defined by $W(c) = W(d) = 0.5$. We claim that $W \in FmR\alpha O(X)$. Infact, $U \in FmRO(X)$ such that $U \leq W \leq maclU = 1_X \setminus U$ (because $Fm\alpha C(X) = \{0_X, 1_X, 1_X \setminus U\}$). Now $V < W \in FmR\alpha O(X)$. But $maclV = 1_X \setminus U \not\leq W \Rightarrow V$ is not $fgm\alpha$ -closed set in X .

Remark 3.11. It is clear from definitions that union of two $fgmr\alpha$ -closed set is also so, but intersection of two $fgmr\alpha$ -closed sets may not be so as it is seen from the following example.

Example 3.12. Consider Example 3.10 and consider two fuzzy sets A, B defined by $A(c) = 0.6, A(d) = 0.45, B(c) = 0.5, B(d) = 0.7$. Since 1_X is the only fuzzy m -regular α -open set in X containing A and B , A and B are $fgmr\alpha$ -closed sets in X . Let $V = A \wedge B$. Then $V(c) = 0.5, V(d) = 0.45$. By Example 3.10, V is not $fgmr\alpha$ -closed set in X .

Proposition 3.13. Let (X, m) be a fuzzy m -space where m satisfies \mathcal{B} condition and $A \in I^X$. If $A \in FmRO(X)$, $A \in FGmR\alpha C(X)$, then A is both fuzzy m - α -open and fuzzy m - α -closed.

Proof. Since m satisfies \mathcal{B} condition, $A \in FmRO(X) \Rightarrow A \in m \Rightarrow A = mintA \leq mint(mclA) = mint(mcl(mintA)) \Rightarrow A \in Fm\alpha O(X)$.

Since $A \in FGmR\alpha C(X)$ and $A \in FmR\alpha O(X)$ (by Result 3.3), $A \leq A \Rightarrow maclA \leq A \Rightarrow A = maclA \Rightarrow A$ is fuzzy m - α -closed set in X .

Proposition 3.14. Let (X, m) be a fuzzy m -space and $A, B \in I^X$. If $A \in FGmR\alpha C(X)$ and $A \leq B \leq maclA$, then $B \in FGmR\alpha C(X)$.

Proof. Let $A \in FGmR\alpha C(X)$ and $B \leq U \in FmR\alpha O(X)$. Then $A \leq B \leq U$. By hypothesis, $maclA \leq U$. So $A \leq B \leq maclA \leq U$. Now as $macl(maclA) = maclA$, $maclB \leq macl(maclA) = maclA \leq U \Rightarrow B \in FGmR\alpha C(X)$.

But the converse of the above proposition is not necessarily true, as is it seen from the following example.

Example 3.15. Let $X = \{c, d\}$, $m = \{0_X, 1_X, U\}$ where $U(c) = 0.5, U(d) = 0.4$.

Then (X, m) is a fuzzy m -space. Here $Fm\alpha O(X) = FmRO(X) = m$ and $Fm\alpha C(X) = m^c$. Consider two fuzzy sets V and W defined by $V(c) = 0.5, V(d) = 0.6, W(c) = W(d) = 0.6$. We first check whether $V, W \in FGmRaC(X)$ or not. Now $U \leq V \leq maclU = V \Rightarrow V \in FmRaO(X)$. So $V \leq V \in FmRaO(X)$ and $maclV = 1_X \setminus U = V \leq V \Rightarrow V \in FGmRaC(X)$. Again $W < 1_X \in FmRaO(X)$ only and so $maclW \leq 1_X \Rightarrow W \in FGmRaC(X)$. Now $V < W$, but $maclV = 1_X \setminus U \not\leq W \Rightarrow V < W \not\leq maclV$.

Proposition 3.16. Let (X, m) be a fuzzy m -space and $A \in I^X$. If $A \in FmRaO(X)$ and $A \in FGmRaC(X)$, then $A \in Fm\alpha C(X)$.

Proof. Since $A \in FmRaO(X)$, $A \leq A \Rightarrow maclA \leq A$ (as $A \in FGmRaC(X)$) $\Rightarrow maclA = A \Rightarrow A \in Fm\alpha C(X)$.

Proposition 3.17. Let (X, m) be a fuzzy m -space where m satisfies \mathcal{B} condition and $A, B \in I^X$. If $A \in FmRaO(X), A \in FGmRaC(X), B \in Fm\alpha C(X)$, then $A \wedge B \in FGmRaC(X)$.

Proof. By Proposition 3.13, $A \in Fm\alpha C(X)$. Since intersection of two fuzzy m - α -closed sets is also so (as m satisfies \mathcal{B} condition), $A \wedge B \in Fm\alpha C(X) \Rightarrow A \wedge B \in FGmRaC(X)$ as fuzzy m - α -closed set is $fgmr\alpha$ -closed set.

Proposition 3.18. Let (X, m) be a fuzzy m -space and $A \in I^X$. If $A \in m$ and A is fgm -closed set in X , then $A \in FGmRaC(X)$.

Proof. Let $B \in FmRaO(X)$ be such that $A \leq B$. Now $A \in m$ and A is fgm -closed set in X , i.e., $A \leq A \in m \Rightarrow mclA \leq A \leq B \Rightarrow maclA \leq mclA \leq A \leq B \Rightarrow A \in FGmRaC(X)$.

But the converse of the above proposition is not true, in general, as it is seen from the following example.

Example 3.19. Consider Example 3.5. Here $S \in m$. Now $FmRaO(X) = \{0_X, 1_X\}$ and so $S < 1_X \in FmRaO(X)$ only, so $maclS = 1_X \Rightarrow S \in FGmRaC(X)$. Now $S \leq S \in m$, but $mclS = 1_X \not\leq S \Rightarrow S$ is not fgm -closed set in X .

Note 3.20. It is obvious that if in a fuzzy m -space (X, m) , $FmRaO(X) = \{0_X, 1_X\}$, then every fuzzy set in X is $fgmr\alpha$ -closed set in X . But the reverse implication is not necessarily true, as it is seen in the next example.

Example 3.21. Let $X = \{c, d\}$, $m = \{0_X, 1_X, S\}$ where $S(c) = S(d) = 0.5$. Then (X, m) is a fuzzy m -space. Now $FmRO(X) = Fm\alpha C(X) = m$. Clearly $S \in FmRaO(X)$. Let $A \in I^X$ be arbitrary. Then either $A \leq S$ or $A \not\leq S$. If $A \leq S$, then $maclA \leq maclS = S \Rightarrow A \in FGmRaC(X)$. If $A \not\leq S$, then $A < 1_X \in FmRaO(X)$ only $maclA \leq 1_X \Rightarrow A \in FGmRaC(X)$.

Theorem 3.22. Let (X, m) be a fuzzy m -space. Then $FmRaO(X) \subset \{U \in I^X : 1_X \setminus U \in Fm\alpha O(X)\}$ if and only if $A \in FGmRaC(X)$ for every $A \in I^X$.

Proof. Let $FmRaO(X) \subset \{U \in I^X : 1_X \setminus U \in Fm\alpha O(X)\}$. Let $A \in I^X$ be such that $A \leq B \in FmRaO(X)$. Then $B \in \{U \in I^X : 1_X \setminus U \in Fm\alpha O(X)\} \Rightarrow 1_X \setminus B \in Fm\alpha O(X) \Rightarrow B \in Fm\alpha C(X) \Rightarrow maclA \leq maclB = B \Rightarrow A \in FGmRaC(X)$.

Conversely, let $A \in FGmRaC(X)$ for all $A \in I^X$. Let $B \in FmRaO(X)$. Then $B \leq B \Rightarrow maclB \leq B$ (by assumption) $\Rightarrow maclB = B \Rightarrow B \in Fm\alpha C(X) \Rightarrow 1_X \setminus B \in Fm\alpha O(X) \Rightarrow B \in \{U \in I^X : 1_X \setminus U \in Fm\alpha O(X)\}$.

Remark 3.23. It is clear from definitions that the notions $Fm\alpha O(X)$ and $FmRaO(X)$ are independent follow from next two examples.

Example 3.24. $Fm\alpha O(X) \not\equiv FmRaO(X)$

Let $X = \{c\}$, $m = \{0_X, 1_X, V\}$ where $V(c) = 0.6$. Then (X, m) is a fuzzy m -space. Let us consider the fuzzy set W in X defined by $W(c) = 0.7$. Then $mint(mcl(mintW)) = mint(mclV) = mint1_X = 1_X \geq W \Rightarrow W \in Fm\alpha O(X)$. Clearly $FmRO(X) = \{0_X, 1_X\}$ and so $W \notin FmRaO(X)$.

Example 3.25. $FmRaO(X) \not\equiv Fm\alpha O(X)$

Let $X = \{c, d\}$, $m = \{0_X, 1_X, U\}$ where $U(c) = 0.5, U(d) = 0.4$. Then (X, m) is a fuzzy m -space. Now $Fm\alpha O(X) = FmRO(X) = m$. Consider the fuzzy set S defined by $S(c) = S(d) = 0.5$. We claim that $S \in FmRaO(X)$ but $S \notin Fm\alpha O(X)$. It is clear that $S \notin Fm\alpha O(X)$. Now $U \in FmRO(X)$ such that $U \leq S \leq maclU = 1_X \setminus U$ and hence $S \in FmRaO(X)$.

Now we introduce a new type of neighbourhood structure of a fuzzy point.

Definition 2.26. A fuzzy set A in a fuzzy m -space (X, m) is called an $fgmr\alpha$ -nbd of a fuzzy point x_t in X if there exists an $fgmr\alpha$ -open set B in X such that $x_t \in B \leq A$.

Definition 2.27. A fuzzy set A in a fuzzy m -space (X, m) is called an $fgmr\alpha$ -nbd of a fuzzy set B in X if there exists an $fgmr\alpha$ -open set U in X such that $B \leq U \leq A$.

Remark 3.27. The next example shows that $fgmr\alpha$ -nbd of a fuzzy point x_t may not be an $fgmr\alpha$ -open set containing x_t .

Example 3.28. Consider Example 3.10. Here V is not an $fgmr\alpha$ -closed set in $X \Rightarrow 1_X \setminus V$ is not $fgmr\alpha$ -open set in X . Consider a fuzzy set S defined by $S(c) = 0.5, S(d) = 0.7$. We first show that S is $fgmr\alpha$ -closed set in X . Here 1_X is the only fuzzy m -regular α -open set in X containing S . So clearly S is $fgmr\alpha$ -closed set in X and so $1_X \setminus S$ is $fgmr\alpha$ -open in X . Consider the fuzzy point $c_{0.5}$. Now $c_{0.5} \in 1_X \setminus S < 1_X \setminus V \Rightarrow 1_X \setminus V$ is an $fgmr\alpha$ -nbd of $c_{0.5}$.

Note 3.29. By Note 3.4, we can say that every fuzzy m -nbd of a fuzzy point x_t is an $fgmr\alpha$ -nbd of it. But the converse may not be true, as it is seen from the following example.

Example 3.30. Consider Example 3.28. Here $1_X \setminus S$ being an $fgmr\alpha$ -open set is $fgmr\alpha$ -nbd of $c_{0.5}$, but $1_X \setminus S$ is not a fuzzy m -nbd of $c_{0.5}$.

Note 3.31. It is clear from definition that an $fgmr\alpha$ -open set is an $fgmr\alpha$ -nbd of

each of its points. But the converse may not be true, in general, as it is seen from the following example.

Example 3.32. Consider Example 3.28. Here $1_X \setminus V$ is not an $fgmr\alpha$ -open set in X . We now show that $1_X \setminus V$ is an $fgmr\alpha$ -nbd of each of its points. The points of $1_X \setminus V$ are either of the form c_t where $0 < t \leq 0.5$ or of the form d_s where $0 < s \leq 0.55$. For the points c_t , $0 < t \leq 0.5$, $1_X \setminus S$ is the $fgmr\alpha$ -open set in X such that $c_t \leq 1_X \setminus S < 1_X \setminus V \Rightarrow 1_X \setminus V$ is an $fgmr\alpha$ -nbd of c_t . For the points d_s , $0 < s \leq 0.55$, consider the fuzzy set T defined by $T(c) = 0.6, T(d) = 0.45$. Then $1_X \in FmR\alpha O(X)$ only containing T and so T is $fgmr\alpha$ -closed in $X \Rightarrow 1_X \setminus T$ is $fgmr\alpha$ -open in X . Now $d_s \in 1_X \setminus T < 1_X \setminus V \Rightarrow 1_X \setminus V$ is an $fgmr\alpha$ -nbd of d_s . Consequently, $1_X \setminus V$ is an $fgmr\alpha$ -nbd of each of its points.

Result 3.33. Let (X, m) be a fuzzy m -space and $A \in I^X$ be an $fgmr\alpha$ -closed set in X and $x_t \in 1_X \setminus A$. Then there exists an $fgmr\alpha$ -nbd B of x_t such that $B \not\leq A$.

Proof. By hypothesis $1_X \setminus A$ is $fgmr\alpha$ -open in X containing x_t . By Note 3.31, there exists an $fgmr\alpha$ -nbd B of x_t such that $x_t \in B \leq 1_X \setminus A \Rightarrow B \not\leq A$.

Definition 3.34. The set of all $fgmr\alpha$ -nbds of a fuzzy point x_t ($0 < t \leq 1$) in a fuzzy m -space (X, m) is called the $fgmr\alpha$ -nbd system at x_t , to be denoted by $fgmr\alpha-N(x_t)$.

Theorem 3.35. Let (X, m) be a fuzzy m -space and x_t be a fuzzy point in X . Then the following statements hold :

- (a) $fgmr\alpha-N(x_t) \neq \phi$,
- (b) $A \in fgmra-N(x_t) \Rightarrow x_t \in A$,
- (c) $A \in fgmra-N(x_t), A \geq B \Rightarrow B \in fgmra-N(x_t)$,
- (d) $A, B \in fgmra-N(x_t) \Rightarrow A \wedge B \in fgmra-N(x_t)$,
- (e) $A \in fgmra-N(x_t) \Rightarrow$ there exists $B \in fgmra-N(x_t)$ such that $B \leq A$ and $B \in fgmra-N(y_s)$ for every $y_s \in B$.

Proof. (a) Obvious as 1_X is an $fgmr\alpha$ -open set containing any fuzzy point x_t ($0 < t \leq 1$).

(b) and (c) follow from definition of $fgmr\alpha-N(x_t)$.

(d) Follows from Remark 3.11.

(e) Follows from Remark 3.11 and Definition 3.34.

Theorem 3.36. Let x_t be a fuzzy point in a fuzzy m -space (X, m) . Let $fgmr\alpha-N(x_t)$ be a non-empty collection of fuzzy sets in X satisfying the following conditions :

- (a) $A \in fgmra-N(x_t) \Rightarrow x_t \in A$,
- (b) $A, B \in fgmra-N(x_t) \Rightarrow A \wedge B \in fgmra-N(x_t)$.

Let m consist of 0_X and all those non zero fuzzy sets A of X having the property that $x_t \in A \Rightarrow$ there exists a $B \in fgmra-N(x_t)$ such that $x_t \in B \leq A$. Then m is a fuzzy

minimal structure on X satisfying \mathcal{B} condition.

Proof. (1) By hypothesis, $0_X \in m$.

(2) By the given property of m , $1_X \in m$ as 1_X is an $fgmr\alpha$ - $N(x_t)$ for every fuzzy point x_t ($0 < t < 1$) in X by (a).

(3) Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ where $U_\alpha \in m$, for all $\alpha \in \Lambda$. Let $x_t \in \bigvee_{\alpha \in \Lambda} U_\alpha$. Then $x_t \in U_\beta$, for some $\beta \in \Lambda$. By construction of m , there exists $V_\beta \in fgmr\alpha$ - $N(x_t)$ such that $x_t \in V_\beta \leq U_\beta \leq \bigvee_{\alpha \in \Lambda} U_\alpha \Rightarrow \bigvee_{\alpha \in \Lambda} U_\alpha \in m$.

It follows that m is a fuzzy minimal structure on X satisfying \mathcal{B} condition.

4. $fg(m, m_1)r\alpha$ -CONTINUOUS FUNCTION : SOME APPLICATIONS

In this section we first introduce a new type of continuous-like function between two fuzzy m -spaces. Then we establish the mutual relationships between this newly defined function and the functions defined in [3, 4]. Afterwards, we introduce a space in which $fgmr\alpha$ -closed set is fuzzy m -closed set. Lastly applications of this newly defined function are shown here.

We first recall some definitions from [3, 4] for ready references.

Definition 4.1. A function $h : (X, m) \rightarrow (Y, m_1)$ is said to be

- (a) fuzzy (m, m_1) -continuous [3] if $h^{-1}(V) \in m^c$ for every $V \in m_1^c$,
- (b) fuzzy (m, m_1) -open [3] function if $h(V) \in m_1$ for every $V \in m$,
- (c) $fg(m, m_1)$ -continuous [4] if $h^{-1}(V)$ is fgm -closed set in X for every $V \in m_1^c$.

Let us now introduce the new concept.

Definition 4.2. A function $h : (X, m) \rightarrow (Y, m_1)$ is said to be $fg(m, m_1)r\alpha$ -continuous if $h^{-1}(U) \in FGmR\alpha O(X)$ (resp., $h^{-1}(U) \in FGmR\alpha C(X)$) for all $U \in m_1$ (resp., $U \in m_1^c$).

Remark 4.3.

Clearly fuzzy (m, m_1) -continuous function is $fg(m, m_1)r\alpha$ -continuous. But the converse is not necessarily true, as it is seen from the following example.

Also $fg(m, m_1)$ -continuity and $fg(m, m_1)r\alpha$ -continuity are independent concepts follow from next examples.

Example 4.4. $fg(m, m_1)r\alpha$ -continuity does not imply fuzzy (m, m_1) -continuity

Let $X = \{c, d\}$, $m = \{0_X, 1_X, S\}$, $m_1 = \{0_X, 1_X, T\}$ where $S(c) = 0.5, S(d) = 0.6, T(c) = T(d) = 0.5$. Then (X, m) and (X, m_1) are fuzzy m -spaces. Consider the identity function $i : (X, m) \rightarrow (X, m_1)$. Now $T^c = T \in m_1^c$, but $i^{-1}(T) = T \notin m^c \Rightarrow i$ is not fuzzy (m, m_1) -continuous function. Now $T < 1_X \in FmR\alpha O(X)$ only and so $m\alpha cl T \leq 1_X \Rightarrow T$ is $fgmr\alpha$ -closed in $(X, m) \Rightarrow i$ is $fg(m, m_1)r\alpha$ -continuous function.

Example 4.5. $fg(m, m_1)$ -continuity does not imply $fg(m, m_1)r\alpha$ -continuity

Let $X = \{c, d\}$, $m = \{0_X, 1_X, U\}$, $m_1 = \{0_X, 1_X, V\}$ where $U(c) = 0.5, U(d) = 0.4, V(c) = 0.5, V(d) = 0.55$. Then (X, m) and (X, m_1) are fuzzy m -spaces. Consider the identity function $i : (X, m) \rightarrow (X, m_1)$. Now $1_X \setminus V \in m_1^c, i^{-1}(1_X \setminus V) = 1_X \setminus V < 1_X \in m$ only and so $1_X \setminus V$ is fgm -closed set in $(X, m) \Rightarrow i$ is $fg(m, m_1)$ -continuous function. Now consider the fuzzy set W defined by $W(c) = W(d) = 0.5$. It is already proved in Example 3.10 that $W \in FmR\alpha O(X)$. Now $1_X \setminus V \leq W$, but $m\alpha cl(1_X \setminus V) = 1_X \setminus U \not\leq W \Rightarrow 1_X \setminus V$ is not $fgmr\alpha$ -closed set in $(X, m) \Rightarrow i$ is not $fg(m, m_1)r\alpha$ -continuous function.

Example 4.6. $fg(m, m_1)r\alpha$ -continuity does not imply $fg(m, m_1)$ -continuity

Let $X = \{a, b\}$, $m = \{0_X, 1_X, S\}$, $m_1 = \{0_X, 1_X, T\}$ where $S(a) = 0.5, S(b) = 0.6, T(a) = T(b) = 0.5$. Then (X, m) and (X, m_1) are fuzzy m -spaces. Consider the identity function $i : (X, m) \rightarrow (X, m_1)$. Now $T^c = T \in m_1, i^{-1}(T) = T < S \in m$, but $m\alpha cl T = 1_X \not\leq S \Rightarrow T$ is not fgm -closed set in $(X, m) \Rightarrow i$ is not an $fg(m, m_1)$ -continuous function. Again $FmR\alpha O(X) = \{0_X, 1_X\}$ and so clearly T is $fgmr\alpha$ -closed set in $(X, m) \Rightarrow i$ is $fg(m, m_1)r\alpha$ -continuous function.

Remark

4.7.

Composition

of $fg(m, m_1)r\alpha$ -continuous and $fg(m_1, m_2)r\alpha$ -continuous functions need not so, as it is seen from the following example.

Example 4.8. Let $X = \{c, d\}$, $m = \{0_X, 1_X, A, B\}$, $m_1 = \{0_X, 1_X, C\}$, $m_2 = \{0_X, 1_X, D\}$ where $A(c) = 0.5, A(d) = 0.4, B(c) = 0.6, B(d) = 0.5, C(c) = C(d) = 0.6$ and $D(c) = 0.5, D(d) = 0.7$. Then $(X, m), (X, m_1)$ and (X, m_2) are fuzzy m -spaces. Consider two identity functions $i : (X, m) \rightarrow (X, m_1)$ and $i_1 : (X, m_1) \rightarrow (X, m_2)$. Since $Fm_1RO(X) = Fm_2RO(X) = \{0_X, 1_X\}$, every fuzzy set in (X, m_1) and (X, m_2) is $fgm_1r\alpha$ -closed and $fgm_2r\alpha$ -closed sets in (X, m_1) and (X, m_2) respectively. So i_1 is $fg(m_1, m_2)r\alpha$ -continuous function. Now $1_X \setminus C \in m_1^c, i^{-1}(1_X \setminus C) = 1_X \setminus C < A \in FmR\alpha O(X)$ and $m\alpha cl(1_X \setminus C) = 1_X \setminus C < A$ (as $Fm\alpha C(X) = \{0_X, 1_X, 1_X \setminus A, U\}$ where $U \leq 1_X \setminus B) \Rightarrow 1_X \setminus C$ is $fgmr\alpha$ -closed set in $(X, m) \Rightarrow i$ is $fg(m, m_1)r\alpha$ -continuous function. Now we prove that $i_1 \circ i$ is not $fg(m, m_2)r\alpha$ -continuous function. Now $1_X \setminus D \in m_2^c$. But $(i_1 \circ i)^{-1}(1_X \setminus D) = 1_X \setminus D < A \in FmR\alpha O(X)$, but $m\alpha cl(1_X \setminus D) = 1_X \setminus A \not\leq A \Rightarrow 1_X \setminus D$ is not $fgmr\alpha$ -closed in $(X, m) \Rightarrow i_1 \circ i$ is not $fg(m, m_2)r\alpha$ -continuous function.

Definition 4.9. A fuzzy m -space (X, m) is called

- (a) fmT_g -space [4] if every fgm -closed set is fuzzy m -closed,
- (b) $fmT_{gr\alpha}$ -space if every $fgmr\alpha$ -closed set is fuzzy m -closed.

Note 4.10 In fmT_g -space, every fgm -closed set is $fgmr\alpha$ -closed set and in $fmT_{gr\alpha}$ -space, every $fgmr\alpha$ -closed set is fgm -closed set.

Now we can state the following two theorems easily.

Theorem 4.11. Let $h : (X, m) \rightarrow (Y, m_1)$ be an $fg(m, m_1)$ -continuous bijective function where X is fmT_g -space. Then h is $fg(m, m_1)r\alpha$ -continuous function.

Theorem 4.12. Let $h : (X, m) \rightarrow (Y, m_1)$ be an $fg(m, m_1)r\alpha$ -continuous bijective function where X is $fmT_{gr\alpha}$ -space. Then h is $fg(m, m_1)$ -continuous as well as fuzzy (m, m_1) -continuous function.

Definition 4.13 [3]. A fuzzy m -space (X, m) is said to be fuzzy m -regular if for any fuzzy point x_t and any fuzzy m -closed set A in X with $x_t \notin A$, there exist $U, V \in m$ such that $x_t \in U, A \leq V$ and $U \not\leq V$.

Definition 4.14 [3]. A fuzzy m -space (X, m) is called fuzzy m -normal if for any two fuzzy m -closed sets A, B in X with $A \not\leq B$, there exist fuzzy m -open sets U, V in X such that $A \leq U, B \leq V$ and $U \not\leq V$.

Theorem 4.15. Let $h : (X, m) \rightarrow (Y, m_1)$ be an $fg(m, m_1)r\alpha$ -continuous, fuzzy (m, m_1) -open, bijective function from a fuzzy m -regular, $fmT_{gr\alpha}$ -space X onto Y . Then Y is fuzzy m_1 -regular.

Proof. Let y_t be a fuzzy point in Y and $A \in m_1^c$ such that $y_t \notin A$. Since h is bijective, there exists unique $x \in X$ such that $h(x) = y$. So $h(x_t) \notin A \Rightarrow x_t \notin h^{-1}(A) \in FGmR\alpha C(X)$ (by hypothesis). Since X is $fmT_{gr\alpha}$ -space, $h^{-1}(A) \in m^c$ such that $x_t \notin h^{-1}(A)$. As X is fuzzy m -regular, there exist $U, V \in m$ such that $x_t \in U, h^{-1}(A) \leq V$ and $U \not\leq V$. As h is surjective, $h(x_t)(= y_t) \in h(U), A = h(h^{-1}(A)) \leq h(V)$ and $h(U) \not\leq h(V)$ where $h(U), h(V) \in m_1$ as h is fuzzy (m, m_1) -open function [Indeed, $h(U) \leq h(V) \Rightarrow$ there exists $z \in Y$ such that $[h(U)](z) + [h(V)](z) > 1 \Rightarrow U(h^{-1}(z)) + V(h^{-1}(z)) > 1$ (as h is bijective) $\Rightarrow U \leq V$, a contradiction). Hence Y is fuzzy m_1 -regular space.

Theorem 4.16. Let $h : (X, m) \rightarrow (Y, m_1)$ be an $fg(m, m_1)r\alpha$ -continuous, fuzzy (m, m_1) -open, bijective function from a fuzzy m -normal and $fmT_{gr\alpha}$ -space X onto Y . Then Y is fuzzy m_1 -normal space.

Proof. Let A, B be two fuzzy m_1 -closed sets in Y with $A \not\leq B$. Then $h^{-1}(A), h^{-1}(B) \in FGmR\alpha C(X)$ with $h^{-1}(A) \not\leq h^{-1}(B)$. As X is $fmT_{gr\alpha}$ -space, $h^{-1}(A), h^{-1}(B) \in m^c$ also. As X is fuzzy m -normal space, there exist $U, V \in m$ such that $h^{-1}(A) \leq U, h^{-1}(B) \leq V$ and $U \not\leq V$. As h is surjective, $A = h(h^{-1}(A)) \leq h(U), B = h(h^{-1}(B)) \leq h(V)$ where $h(U), h(V) \in m_1$ as h is a fuzzy (m, m_1) -open function. We claim that $h(U) \not\leq h(V)$. Indeed, $h(U) \leq h(V) \Rightarrow$ there exists $y \in Y$ such that $[h(U)](y) + [h(V)](y) > 1 \Rightarrow U(h^{-1}(y)) + V(h^{-1}(y)) > 1$ (as h is bijective) $\Rightarrow U \leq V$, a contradiction. Consequently, Y is fuzzy m_1 -normal space.

5. CONCLUSION

Here we introduce and study $fgmr\alpha$ -closed set and $fg(m, m_1)r\alpha$ -continuous function between two fuzzy m -spaces. Also some applications of $fg(m, m_1)r\alpha$ -continuous function are shown. Our next approach is to find an idempotent operator by which $fg(m, m_1)r\alpha$ -continuous function will be characterized. Also find different types of generalized version of fuzzy regular and fuzzy normal spaces in fuzzy m -space and wish to establish some condition under which these spaces remain invariant under $fg(m, m_1)r\alpha$ -continuous function.

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