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Some Expansion Formulae for the $\overline{H}$-Function

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Abstract

In the present paper, the authors have established two expansion formula of $\overline{H}$ -Function.

Key words: $\overline{H}$ -Function, Expansion Formula, Gamma Function

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INTRODUCTION

The $\overline{H}$ -function occurring in the paper will be defined and represented as follows:

$$\overline{H}_{P,Q}^{M,N}[z]=\overline{H}_{P,Q}^{M,N}\left[z\left|\begin{matrix}(a_j;\alpha_j;A_j)_{1,N},(a_j;\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\overline{\phi}(\xi)z^{\xi}d\xi \tag{1.1}$$

where
$$\overline{\phi}(\xi)=\frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\left\{\Gamma(1-a_j+\alpha_j\xi)\right\}^{A_j}}{\prod_{j=M+1}^{Q}\left\{\Gamma(1-b_j+\beta_j\xi)\right\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)} \tag{1.2}$$

Which contains fractional powers of the gamma functions. Here, and throughout the paper $a_j\,(j=1,...,p)$ and $b_j\,(j=1,...,Q)$ are complex parameters, $\alpha_j \geq 0(j=1,...,P), \beta_j \geq 0(j=1,...,Q)$ (not all zero simultaneously) and exponents $A_j\,(j=1,...,N)$ and $B_j\,(j=N+1,...,Q)$ can take on non integer values.

The following sufficient condition for the absolute convergence of the defining integral for the $\overline{H}$-function given by equation (1.1) have been given by (Buschman and Srivastava[1]).

$$\Omega \equiv \sum_{j=1}^{M}\left|\beta_j\right| + \sum_{j=1}^{N}\left|A_j\alpha_j\right| - \sum_{j=M+1}^{Q}\left|\beta_j B_j\right| - \sum_{j=N+1}^{P}\left|\alpha_j\right| > 0 \qquad (1.3)$$

and $\left|\arg(z)\right| < \frac{1}{2}\pi\,\Omega$ (1.4)

If we take $A_j = 1(j=1,...,N),\ B_j = 1(j=M+1,...,Q)$ in (1.1), the function $\overline{H}_{P,Q}^{M,N}$ reduces to the Fox's H-function [2].

We shall use the following notation:

$$A^* = \left(a_j,\alpha_j;A_j\right)_{1,N}, \left(a_j,\alpha_j\right)_{N+1,P}, B^* = \left(b_j,\beta_j\right)_{1,M}, \left(b_j,\beta_j;B_j\right)_{M+1,Q} \quad ,$$

2. EXPANSION FORMULA

First Formula

$$\overline{H}_{p,q}^{m,n}\left[\eta\omega\left|\begin{matrix}A^*\\B^*\end{matrix}\right.\right] = \eta^{\frac{(a_1-1)}{\alpha_1}} \sum_{r=0}^{\infty} \frac{\left[1-\eta^{1/\alpha_1}\right]^r}{r!} \overline{H}_{p,q}^{m,n}\left[\omega\left|\begin{matrix}(-r+a_1,\alpha_1;1),(a_j,\alpha_j;A_j)_{2,n},(a_j,\alpha_j)_{n+1,p}\\B^*\end{matrix}\right.\right] \qquad (2.1)$$

Where $\eta > 0, \mathrm{Re}(\eta^{1/\alpha}) > \frac{1}{2}; \arg(\eta\omega) = \alpha_1 \arg(\eta^{1/\alpha_1}) + \arg\omega$ and $|\arg(\eta^{1/\beta_1})| < \frac{\pi}{2}$.

Proof: R.H.S.$= \eta^{\frac{(a_1-1)}{\alpha_1}} \sum_{r=0}^{\infty} \frac{\left[1-\eta^{1/\alpha_1}\right]^r}{r!} \overline{H}_{p,q}^{m,n}\left[\omega\left|\begin{matrix}(-r+a_1,\alpha_1;1),(a_j,\alpha_j;A_j)_{2,n},(a_j,\alpha_j)_{n+1,p}\\B^*\end{matrix}\right.\right]$

$$= \eta^{(a_1-1)/\alpha_1} \sum_{r=0}^{\infty} \frac{\left[1-\eta^{1/\alpha_1}\right]^r}{r!} \frac{1}{2\pi i} \int_L \frac{\Gamma(1+r-a_1+\alpha_1 s)\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=2}^{n}\left\{\Gamma(1-a_j+\alpha_j s)\right\}^{A_j}}{\prod_{j=M+1}^{q}\left\{\Gamma(1-b_j+\beta_j s)\right\}^{B_j}\prod_{j=N+1}^{p}\Gamma(a_j-\alpha_j s)} \omega^s ds$$

Changing the order of integration and summation under the integral sign

$$= \frac{1}{2\pi i} \int_L \bar{\phi}(s)\omega^s \left[\eta^{(a_1-1)/\alpha_1} \sum_{r=0}^{\infty} \frac{\left[1-\eta^{1/\alpha_1}\right]^r}{r!} \Gamma(1+r-a_1+\alpha_1 s) \right] ds$$

Where $$\bar{\phi}(s) = \frac{\prod_{j=2}^{m}\Gamma(b_j-\beta_j s)\prod_{j=2}^{n}\left\{\Gamma(1-a_j+\alpha_j s)\right\}^{A_j}}{\prod_{j=m+1}^{q}\left\{\Gamma(1-b_j+\beta_j s)\right\}^{B_j}\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}$$

$$= \eta^{(a_1-1)/\alpha_1} \sum_{r=0}^{\infty} \frac{\left[1-\eta^{1/\alpha_1}\right]^r}{r!} \frac{1}{2\pi i} \int_L \bar{\phi}(s)\omega^s (1-a_1+\alpha_1 s)_r \Gamma(1-a_1+\alpha_1 s) ds$$

$$= \eta^{(a_1-1)/\alpha_1} \frac{1}{2\pi i} \int_L \bar{\phi}(s)\omega^s \left[1-1+\eta^{1/\alpha_1}\right]^{-1+a_1-\alpha_1 s} \Gamma(1-a_1+\alpha_1 s) ds$$

$$\left[\therefore \sum \frac{x^r}{r!}(a)_r = (1-x)^{-a}\right]$$

$$= \eta^{(a_1-1)/\alpha_1} \frac{1}{2\pi i} \int_L \bar{\phi}(s)\omega^s \eta^{-(a_1-1)/\alpha_1} \eta^s \Gamma(1-a_1+\alpha_1 s) ds$$

$$= \frac{1}{2\pi i} \int_L \bar{\phi}(s)\omega^s \eta^s \Gamma(1-a_1+\alpha_1 s) ds$$

$$= \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\left\{\Gamma(1-a_j+\alpha_j s)\right\}^{A_j}}{\prod_{j=M+1}^{q}\left\{\Gamma(1-b_j+\beta_j s)\right\}^{B_j}\prod_{j=N+1}^{p}\Gamma(a_j-\alpha_j s)} (\omega\eta)^s ds = \text{L.H.S.}$$

Second Formula

$$\overline{H}_{p,q}^{m,n}\left[\eta\omega\left|{}_{B^*}^{A^*}\right.\right]=\eta^{\frac{(a_p-1)}{\alpha_p}}\sum_{r=0}^{\infty}\frac{\left[1-\eta^{1/\alpha_p}\right]^r}{r!}\overline{H}_{p,q}^{m,n}\left[\omega\left|{}_{B^*}^{(a_j,\alpha_j;A_j)_{1,n},(a_j,\alpha_j)_{n+1,p-1},(-r+a_p,\alpha_p)}\right.\right] \quad (2.2)$$

Where $p>n,\operatorname{Re}(\eta^{1/\alpha_p})>\frac{1}{2};\arg(\eta\omega)=\alpha_p\arg(\eta^{1/\alpha_p})+\arg\omega$ and $|\arg(\eta^{1/\beta_1})|<\frac{\pi}{2}$.

Proof: R.H.S.= $\eta^{\frac{(a_p-1)}{\alpha_p}}\sum_{r=0}^{\infty}\frac{\left[1-\eta^{1/\alpha_p}\right]^r}{r!}\overline{H}_{p,q}^{m,n}\left[\omega\left|{}_{B^*}^{(a_j,\alpha_j;A_j)_{1,n},(a_j,\alpha_j)_{n+1,p-1},(-r+a_p,\alpha_p)}\right.\right]$

=

$$\eta^{(a_p-1)/\alpha_p}\sum_{r=0}^{\infty}\frac{\left[1-\eta^{1/\alpha_p}\right]^r}{r!}\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=2}^{n}\left\{\Gamma(1-a_j+\alpha_j s)\right\}^{A_j}}{\Gamma(-r+a_p-\alpha_p s)\prod_{j=M+1}^{q}\left\{\Gamma(1-b_j+\beta_j s)\right\}^{B_j}\prod_{j=N+1}^{p}\Gamma(a_j-\alpha_j s)}\omega^s ds$$

Changing the order of integration and summation under the integral sign

$$=\eta^{(a_p-1)/\alpha_p}\sum_{r=0}^{\infty}\frac{\left[1-\eta^{1/\alpha_p}\right]^r}{r!}\frac{1}{2\pi i}\int_L\overline{\phi}(s)\omega^s\frac{1}{\Gamma(-r+a_p-\alpha_p s)}ds$$

Where $\overline{\phi}(s)=\dfrac{\prod_{j=2}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\left\{\Gamma(1-a_j+\alpha_j s)\right\}^{A_j}}{\prod_{j=m+1}^{q}\left\{\Gamma(1-b_j+\beta_j s)\right\}^{B_j}\prod_{j=n+1}^{p-1}\Gamma(a_j-\alpha_j s)}$

$$=\eta^{(a_p-1)/\alpha_p}\frac{1}{2\pi i}\int_L\overline{\phi}(s)\omega^s\sum_{r=0}^{\infty}\frac{\left[1-\eta^{1/\alpha_p}\right]^r}{r!}\times\frac{1}{(a_p-\alpha_p s)_r\Gamma(a_p-\alpha_p s)}ds$$

$$=\eta^{(a_p-1)/\alpha_p}\frac{1}{2\pi i}\int_L\overline{\phi}(s)\omega^s\frac{1}{\Gamma(a_p-\alpha_p s)}\sum_{r=0}^{\infty}\frac{\left[\eta^{1/\alpha_p}-1\right]^r}{r!}\frac{(1-a_p+\alpha_p s)_r}{(-1)^r}\ ds$$

$$\left[\therefore\sum\frac{x^r}{r!}(a)_r=(1-x)^{-a}\right]$$

$$= \eta^{(a_p-1)/\alpha_p} \frac{1}{2\pi i} \int_L \overline{\phi}(s) \omega^s \frac{1}{\Gamma(a_p - \alpha_p s)} \frac{(1-\eta^{-1/\alpha_p})^r}{r!} (1 - a_p + \alpha_p s)_r \; ds$$

$$= \eta^{(a_p-1)/\alpha_p} \frac{1}{2\pi i} \int_L \overline{\phi}(s) \omega^s \frac{1}{\Gamma(a_p - \alpha_p s)} \eta^{-(a_p-1)/\alpha_p} \eta^s \; ds$$

$$= \frac{1}{2\pi i} \int_L \overline{\phi}(s) \omega^s \eta^s \frac{1}{\Gamma(a_p - \alpha_p s)} ds$$

$$= \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j s) \prod_{j=1}^{n} \left\{\Gamma(1 - a_j + \alpha_j s)\right\}^{A_j}}{\prod_{j=M+1}^{q} \left\{\Gamma(1 - b_j + \beta_j s)\right\}^{B_j} \prod_{j=N+1}^{p} \Gamma(a_j - \alpha_j s)} (\omega\eta)^s \, ds = \text{L.H.S.}$$

For $A_j = 1(j = 1, ..., n)$, $B_j = 1(j = m+1, ..., q)$ in (2.1),(2.2), we get the results in terms of Fox's H-function [2].

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