Advances in Theoretical and Applied Mathematics
ISSN 0973-4554 Volume 12, Number 2 (2017), pp. 71-79

http://www.ripublication.com

Generalized open and closed sets via fuzzy ideal topological spaces

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Abstract

The aim of this paper is to discuss on generalized open and closed sets in a unified manner. The generalized I – closed sets and FIR - Open sets are discussed via fuzzy ideal topology. We define fuzzy g – I – closed sets, fuzzy g – I – continuous function, FIR- Open Sets and investigate their properties.

Keywords : Fuzzy g – I – closed sets, Fuzzy g – I – continuous function, FIR- Open Sets.
2000 AMS Subject Classification : 54C10, 54C50

1. Introduction and Preliminaries

Fuzzy sets was introduced by Zadeh[16] in 1965 and fuzzy topology by Chang[5]. The theory of fuzzy topological spaces was subsequently developed by several authors by considering the basic topics of general topologyJankovic[7] and several other authors studied Ideal topological spaces. Mahmoud[10] and Sarkar[9] presented some of the ideal concepts in fuzzy and studied many of its properties.

A fuzzy subset A of a fuzzy sets X, denoted by $A \leq X$ is characterized by a membership function. The basic fuzzy sets are the empty set, whole set and the class of all fuzzy subsets of X which is denoted by 0,1 and I^X, respectively. The collection of all fuzzy open sets containing x will be denoted by $\tau(x)$. By (X, τ) we mean a fuzzy topological spaces. A fuzzy set which is a fuzzy point with support $x \in X$ and value $\lambda \in (0,1]$ will be designated by x_λ[15]. Also, for a fuzzy point x_λ and a fuzzy set A we shall write $x_\lambda \in A$ to mean that $\lambda \leq A(x)$. A fuzzy set in (X,τ) is said to be quasicoincident with a fuzzy set B, denoted by AqB, if there exists x∈X such that A(x) + B(x) >1[8] . A fuzzy set V in (X,τ) is called a q-neighb -ourhood (q-nbd, for short) of a fuzzy point x_λ if and only if there exists a fuzzy open set U such that x_λ qU ≤ V[8, 6]. We will denote the set of all q-nbd of x_λ in (X,τ) by N (x_λ) . For a fuzzy set A in *X*, Cl(A), Int(A) and A will respectively denote the closure, interior and complement of *A*.

A non empty collection I of fuzzy subsets of X is called a fuzzy ideal[9,10] if and only if (1) A∈I and B ≤ A, then B∈I (heredity), (2) if A∈I and B∈I, then AVB∈I (finite additivity). The triple (X,τ,I) means fuzzy topological space with a fuzzy ideal I and fuzzy topology τ. For (X,τ,I), the fuzzy local function of A ≤ X with respect to τ and I is denoted by A*(τ,I) (briefly A*)[9]. The fuzzy local function A*(τ,I) of A is the union of all fuzzy points x_λ such that if U∈N(x_λ) and E∈I then there is at least one y∈X for which U(y)+A(y)-1 > E(y) [9]. Fuzzy closure operator of a fuzzy set A in (X,τ,I) is defined as Cl*(A) = AVA* [9]. In

(X,τ,I), the collection τ*(I) means an extension of fuzzy topological space via fuzzy ideal which is constructed by considering the class β = {U-E:U∈τ, E∈I} as a base.

Lemma 1:[9] Let (X, τ, I) be any fuzzy ideal topological space and A, B be fuzzy subsets of X. Then the following properties hold:

1. If A≤ B then A* ≤ B*.
2. A*= Cl(A*) ≤ Cl(A).
3. If U ∈ τ, then U∧A*≤ (U∧ A)*.
4. (A*)*≤ A*.
5. (A∨B)* = A* ∨ B*.

Definition 2:[1] A fuzzy subset A of a fuzzy ideal topological space (X, τ, I) is said to be

1. fuzzy *-dense-in-itself if A ≤ A*.
2. fuzzy *-closed if A* ≤ A.
3. fuzzy *-perfect if A = A*.

2. Fuzzy g – I – closed Sets

Definition 3: A subset A is said to be fuzzy g closed if cl(A)≤ U, whenever A ≤ U and U is open set in (X, τ, I)[4].

Definition 4: A subset A is said to be fuzzy g – I – closed if A*≤ U, whenever A ≤ U and U is open set in (X, τ, I). The complement of fuzzy g – I – closed is fuzzy g – I – open.

Theorem 5: If A is fuzzy open set and fuzzy g – I – closed then A is τ* - closed. Proof : Let A be fuzzy open set and fuzzy g – I – closed, then A*≤ A and hence A is τ* - closed.

Theorem 6: If A and B are fuzzy g – I – closed sets in fuzzy topological space (X, τ, I), then A ∨ B is also fuzzy g – I – closed set.
Proof : Let A and B be fuzzy g – I – closed sets such that A ∨ B ≤ U, then A≤ U and B≤ U. Therefore A*≤ U and B*≤ U (since A and B be fuzzy g – I – closed) , hence (A∨B)* = A* ∨ B* ≤ U.

Theorem 7: For a fuzzy topological space (X, τ, I) with ideal I the following properties hold:

1. Every fuzzy g – closed is fuzzy g – I – closed except when I = {ϕ}
2. Every closed set is fuzzy g – I – closed.

Remark 8: The converse of the above result is not true
Example 9: Consider τ = {0,1,A,H,K} on X = {a,b,c,d} with I = P(X)

$$A(x) = \begin{cases} 1 \ if \ x = a \\ 0 \ otherwise \end{cases}$$

$$H(x) = \begin{cases} 1 \ if \ x = b, c \\ 0 \ otherwise \end{cases}$$

$$K(x) = \begin{cases} 1 \ if \ x = a, b, c \\ 0 \ otherwise \end{cases}$$

In this fuzzy topological space,

$$A(x) = \begin{cases} 1 \ if \ x = a \\ 0 \ otherwise \end{cases}$$

is not a fuzzy g – closed set but it is a fuzzy g – I – closed set.
Example 10: Consider τ = {0,1,B} on X = {a,b,c} with I = P(X)

$$B(x) = \begin{cases} 1 \ if \ x = b \\ 0 \ otherwise \end{cases}$$

Then $C(x) = \begin{cases} 1 \ if \ x = c \\ 0 \ otherwise \end{cases}$

is not a fuzzy closed set but it is a fuzzy g – I – closed set.

Theorem 11: A is fuzzy g – I – closed if and only if AqB implies A*qB for every fuzzy closed set B.

Proof : Let A be fuzzy g – I – closed. Let B be fuzzy closed in X, such that, AqB. Then A ≤ 1 – B where 1 – B is fuzzy open set. Therefore A* ≤ 1 – B and hence A*qB. Conversly, let D be fuzzy open set such that A ≤ D, then, Aq(1-D) where 1-D is fuzzy closed. Therefore, A*q(1-D) and A*≤ D.

Theorem 12: If A is fuzzy g – I – closed in (X, τ, I) and A ≤ B ≤ A* where B is *-dense, then B is fuzzy g – I – closed in (X, τ, I).
Proof : Since A is fuzzy g – I – closed, A*≤ U whenever A ≤ U and U is fuzzy open set in (X, τ). To prove B is fuzzy g – I – closed. Let B ≤ V where V is fuzzy open set in (X, τ). Since A ≤ B ≤ V and A is fuzzy g – I – closed, A*≤ V implies B ≤ A*≤ V. Since B is *-dense, B*≤ B ≤ V. Therefore B is fuzzy g – I – closed.

Theorem 13: Let (Y, τ_y) be a subspace of (X, τ, I) and A be a fuzzy set of Y. If A is fuzzy g – I – closed in X, then A is fuzzy g – I – closed in Y.
Proof : Let A be fuzzy g – I – closed in X, then, A*≤ U whenever A ≤ U and U is fuzzy open set in X. Suppose A ≤ V and V is fuzzy open set in Y. But V is fuzzy open set in Y implies V is fuzzy open set in X. Therefore, A*≤ V and hence A is fuzzy g – I – closed in Y.

3. Fuzzy g – I – continuous function

Definition 14 : A map f : (X, τ) ⟶ (Y, σ) is said to be fuzzy g – continuous[5] if $f^{-1}(A)$ is fuzzy open in X, for every fuzzy open set A in Y

Definition 15 : A map f : (X, τ, I) ⟶ (Y, σ, J) is said to be fuzzy g – I – continuous if $f^{-1}(A)$ is fuzzy g – I – closed in X, for every fuzzy closed set A in Y.

Definition 16 : A map f : (X, τ, I) ⟶ (Y, σ, J) is said to be fuzzy g – I – irresolute if $f^{-1}(A)$ is fuzzy g – I – closed in X, for every fuzzy g – I – closed set A in Y.

Remark 17: The following properties hold for a function
f : (X, τ, I) ⟶ (Y, σ, J):

1. Every fuzzy g – I – irresolute map is fuzzy g – I – continuous.
2. Every fuzzy continuous map is fuzzy g – I – continuous.
3. Every fuzzy g – continuous map is fuzzy g – I – continuous.

Theorem 18 : A map f : (X, τ, I) ⟶ (Y, σ, J) is said to be fuzzy g – I – continuous if and only if the inverse image of each fuzzy open set of Y is fuzzy g – I – open in X.

Theorem 19: If f : (X, τ,I) ⟶ (Y, σ,J) is fuzzy g – I – continuous, then for every point x_p of X and each A ϵ Y such that f(x_p)qA, there exist a fuzzy g – I – open set B of X such that x_p ϵ B and f(B) ≤ A.
Proof : Let x_p be any point of X. Then f(x_p) is a fuzzy point in Y. Now let A ϵ Y be such that f(x_p)qA. Put f^{-1}(A) = B. Since f is fuzzy g – I – continuous, B is fuzzy g – I – open set B of X and x_p ϵ B. Moreover f(B) = f(f^{-1}(A)) ≤ A.

Theorem 20: If f : (X, τ,I) ⟶ (Y, σ,J) is fuzzy g – I – continuous and g : (Y, σ,J) ⟶ (Z, γ,K) is fuzzy continuous then fog : (X, τ,I) ⟶ (Z, γ,K) is fuzzy g – I – continuous.

Theorem 21: If f : (X, τ,I) ⟶ (Y, σ,J) and g : (Y, σ,J) ⟶ (Z, γ,K) are fuzzy g – I – irresolute then fog : (X, τ,I) ⟶ (Z, γ,K) is fuzzy g – I – irresolute.

4. FIR - Open Sets

Definition 22 :A fuzzy subset A of a fuzzy ideal topological space (X,τ, I) is said to be fuzzy preopen [3] if A≤Int(Cl(A)), fuzzy b-open [14] if A≤Cl(Int(A)) ∨Int(Cl(A))),fuzzy t - I - set [11] if int(A)=Int(Cl*(A)),fuzzy semiopen [2] if A≤Cl(Int(A)), fuzzy pre-I-open [11] if A≤Int(Cl*(A)), fuzzy β-I-open [13] if A≤Cl(Int(Cl*(A))),fuzzy regular open[2] int(A)=Int(Cl*(A)), fuzzy I-open [12] if A ≤Int(A *). The family of all fuzzy preopen, fuzzy regular open is denoted by FPO(X, τ), FRO(X, τ).

Definition 23: A subset A of an ideal topological space (X, τ, I) is called FIR-Open (resp. fuzzy regular I open) if A = int(Cl(A*)) (resp. A = Int(Cl*(A))). The family of all FIR- open sets (resp. fuzzy regular I open) is denoted by FIRO(X, τ) (resp.FRIO(X, τ)).

Theorem 24: If A ≤ X is fuzzy I - open and fuzzy semi - closed then A is FIR- open.
Proof : Suppose A is I - open then A ≤ int(A*) implies A ≤ int(cl(A*)). Since A is semi - closed, we have (X − A) ≤ cl(int(X − A)) implies A ≥ X − cl(int(X − A)) = int(X − int(X −A)) = int(cl(A)) ≥ int(cl(A*)), since A*= cl(A*) ≤ cl(A). Hence A ≥ int(cl(A*)). Thus A is FIR- open.

Theorem 25: For a subset A ≤ (X, τ, I) the following properties hold:
[1] A = int(A*) for all A ϵ FIRO(X).

[2] If A is *-closed and A ϵ FIRO(X) then int(A) = int(A*).

Proof : (1) Follows from the definition of FIR- open set and the fact that the local function is closed. That is A = int(cl(A*)) = int(A*), since A*= cl(A*).

(2) Since A ϵ FIRO(X) we have A = int(cl(A*)) = int(A*) ≤ int(A) ≤ A [since A is *-closed we have A*≤ A]. Thus int(A) = int(A*).

Lemma 26: If (X, τ, I) is an ideal topological space and A ≤ X is *-dense then A*= cl(A*) = cl*(A) = cl(A).

Theorem 27: Let (X, τ, I) be an ideal topological space and A, B be subsets of X. Then

[1] if A ϵ FIRO(X, τ) and B ϵ τ* then A ∨ B ϵ FPO(X, τ).

[2] if A ϵ FIRO(X, τ) and B is fuzzy open then A ∨ B ϵ FPO(X, τ).

[3] if A ϵ FRO(X, τ) and B ϵ FIRO(X, τ) then A ∨ B ϵ FPO(X, τ).

[4] if A ϵ FRIO(X, τ) is *-dense and B ϵ FIRO(X, τ) then A∨ B ϵ FPO(X, τ).

[5] if A ϵ FIRO(X, τ) and B ϵ FPO(X, τ) then A ∨B ϵ FPO(X, τ).

[6] if A ϵ FIRO(X, τ) and B ϵ FPO(X, τ) then A ∨ B ϵ FBO(X, τ).

Proof : (1) Since A ϵ FIRO(X, τ) and B ϵ τ*, we have

A ∨ B ≤[int(cl(A*)) ∨ int(B*)], by definition

A ∨ B ≤[int(cl(A)) ∨ int(cl(B))], from property of local function

A ∨B ≤[int(cl(A) ∨ (cl(B))]

A ∨ B ≤[int(cl(A ∨ B))], from hypothesis

Hence A U B ϵ FPO(X, τ).

(2) Let A ϵ FIRO(X, τ) and B be fuzzy open, therefore we have

A ∨ B ≤[int(cl(A*)) ∨ int(B)], by definition

A∨B ≤[int(cl(A)) ∨ int(cl(B))], from property of local function

A ∨B ≤[int(cl(A) ∨ (cl(B))]

A ∨ B = [int(cl(A ∨ B))], from hypothesis.

Hence A ∨ B ϵ PO(X, τ).

(3) Let A ϵ FRO(X, τ) and B ϵ FIRO(X, τ), thus we have

A ∨B = [int(cl*(A)) ∨ int(cl(B*))], by definition

A ∨B ≤ [int(cl(A) ∨ (cl(B))], by local function property

A ∨ B = [int(cl(A ∨ B))], from hypothesis

Hence A ∨ B ϵ FPO(X, τ).

(4) Since A ϵ RIO(X, τ) and B ϵ FIRO(X, τ), we use Lemma 26 and obtain that

A ∨B = [int(cl*(A)) ∨ int(cl(B*))], by definition

A ∨ B ≤[int(cl(A)) ∨ int(cl(B*))], by local function property

A ∨ B ≤[int(cl(A) ∨ (cl(B))]

$A \vee B = [int(cl(A \vee B))]$.

Hence A U B ϵ FPO(X, τ).

(5) and (6) are obvious.

Theorem 28: Let (X, τ, I) be an ideal topological space then

[1] If $A \leq W \leq cl(A)$ and A ϵ FIRO(X, τ) then W is β- open.

[2] If $A \leq W \leq cl^*(A)$ and A ϵ FIRO(X, τ) is *-dense then W is β- I - open.

Proof : (1) Let A ϵ FIRO(X,), then we have $A = int(cl(A^*)) \leq int(cl(A))$. Also $A \leq W$ implies $cl(A) \leq cl(W)$ and hence $A \leq int(cl(W))$. Moreover $W \leq cl(A)$ implies $W \leq cl(int(cl(W)))$. Therefore W is β- open.

(2) Suppose A ϵ FIRO(X, τ), then we have $A = int(cl(A^*)) = int(cl^*(A))$, since A is *-dense and therefore satisfies Lemma 26. Also $A \leq W$ implies $cl^*(A) \leq cl^*(W)$ and hence $A \leq int(cl^*(W))$. Moreover $W \leq cl^*(A)$ implies $W \leq cl^*(int(cl^*(W)))$.

Theorem 29: For a subset $A \leq (X, \tau, I)$ that is *-perfect we have

[1] A is fuzzy R -I - open if and only if it is fuzzy open.

[2] A is fuzzy R -I - open if and only if it is fuzzy regular open.

Proof : (1) Since A is *-perfect, we have $A = A^*$and hence $A = int(cl^*(A))$ if and only if $A= int(A \vee A^*)$ if and only if $A = int(A \vee A)$ if and only if $A= int(A)$.

(2) A is *-perfect implies A is *-dense and hence satifies Lemma 26. Thus $A = int(cl^*(A)) = int(cl(A))$.

Theorem 30: Let (X, τ, I) be an ideal topological space and $A \leq X$. Then

[1] A is fuzzy R -I - open if it is fuzzy pre - I - open (resp. fuzzy open) and a fuzzy t - I - set.

[2] A is a fuzzy t - I - set if it is fuzzy open and fuzzy R - I - open.

Proof : (1) A is pre - I - open implies $A \leq int(cl^*(A))$. Since A is a fuzzy t - I - set, we have $int(cl^*(A)) = int(A) \leq A$. Hence $A = int(cl^*(A))$. Therefore A is fuzzy R - I - open.The proof is obvious for an fuzzy open set.

(2) A is fuzzy open and fuzzy R - I - open implies $int(A) = A = int(cl^*(A))$. Hence A is a fuzzy t- I - set.

Theorem 31: For an ideal space (X, τ, I) if $A \leq X$ is *-dense then the following are equivalent :

[1] A is fuzzy FIR- open.

[2] A is fuzzy R - I - open.

[3] A is fuzzy regular open.

Proof : (1) implies (2): A is FIR- open implies $A = int(cl(A^*)) = int(cl^*(A))$ [by Lemma 26]. Hence A is fuzzy R - I - open.(2) implies (3) A is fuzzy R - I - open implies A = int(cl*(A)) = int(cl(A)) [by Lemma 26]. (3) implies (1) also follows using Lemma 26.

Theorem 32: For an ideal space (X, τ,I), if A ≤ X is *-perfect, then A is FIR- open, if and only if, it is fuzzy regular open.
Proof : Suppose A is FIR- open, then , A = int(cl(A*)) , A = int(cl(A)), (since A is *-perfect), A is fuzzy regular open.

Theorem 33: For a space (X, τ, I) and A ≤X, we have : [1] If I = ϕ, then A is FIR- open, if and only if, it is fuzzy regular open.[2] If I = P(X), then A is FIR- open if and only if A = ϕ.
Proof : (1) Let I = ϕ then A*(I) = cl(A) for all A ≤X. Let A be fuzzy regular open, then A = int(cl(A)) implies A = int(A*) implies A = int(cl(A*)) implies A is FIR- open.
(2) Let I = P(X) then A*(I) = ϕ. Since A is IR*- open if and only if A = int(cl(A*)) , A = int(ϕ) = ϕ; we obtain the result.

Theorem 34: For an ideal space (X, τ, I) if A ϵ FIRO(X, τ) and B ϵ FIRO(X, τ) then A × B ϵ IR*O(X × Y) if A*× B*= (A × B)*, where X × Y is the product space.
Proof : Since A ϵFIRO(X, τ) and B ϵ FIRO(X, τ), we have
A × B = [int(cl(A*)) × int(cl(B*))], by definition
A × B = [int(A*) × int(B*)], by property of local function
A × B = int(A*× B*)
A × B = int(A × B)*, from hypothesis
A × B = int(cl(A × B)*), by assumption.
Hence A × B ϵ FIRO(X × Y).

ACKNOWLEDGEMENT
The author is grateful to the university grant commission for providing the financial support.

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