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On duality of sequences of measurable amounts of the jug problem

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Abstract

Let $M(m, n)$ be the integer sequence of all non-zero measurable amounts less than $m + n$ that are obtainable by two unmarked jugs of capacities m and n units. We introduce the concept of dual sequences which describe the correspondence relation between $M(m_1, n_1)$ and $M(m_2, n_2)$, where $0 < m_1 < n_1$, $0 < m_2 < n_2$, $m_1 + n_1 = m_2 + n_2$ and $gcd(m_1, n_1) = gcd(m_2, n_2) = 1$. Some illustrative examples are provided.

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1. Introduction

The jug problem is a classic problem in Mathematics, Computer Sciences and some related disciplines [1, 2, 3, 4]. The problem can be stated like this:

Given two unmarked jugs with capacities m and n units. How can we measure exactly d $(< m + n)$ *units of water by the jugs?*

There are a number of approaches proposed to solve this problem, such as the search methods [2, 7] and the methods of heuristics [5, 6]. In [8, 9, 10], two simple algorithms were proposed to compute the integer sequence $M(m, n)$ of all measurable amounts less than $m+n$ and hence the feasible solutions of the jug problem. In this paper, we introduce the concept of dual sequences and the related results, which describe the interesting correspondence relation between $M(m_1, n_1)$ and $M(m_2, n_2)$, where $0 < m_1 < n_1$, $0 < m_2 < n_2, m_1 + n_1 = m_2 + n_2$ and $gcd(m_1, n_1) = gcd(m_2, n_2) = 1$. It provides us a convenient way to locate the position of any measurable amount $d \in M(m_1, n_1)$ via $M(m_2, n_2)$, or vice versa;as well as the understanding of the properties of the measurable amounts of the jug problem concerned. Some illustrative examples are provided.

2. Algorithms For Computing the Measurable Amounts

Let us recall the two algorithms introduced in [8, 9, 10] for generating the integer sequences of measurable amounts less than $m + n$ below.

Algorithm 1:
Input: Integers m, n, where $0 < m < n$.
Output: An integer sequence S_1 of all distinct measurable amounts less than $m + n$ obtainable by filling the m units jug first.
Procedure:
Step 1. Initialize a dummy variable $k = 0$.
Step 2. While $k < n$ do $k := m + k$.
Step 3. If $k \geq n$, then $k := k - n$.
Step 4. If $k = 0$, then return the sequence of k obtained and terminate. Otherwise, repeat the steps 2-4.

Algorithm 2:
Input: Integers m, n, where $0 < m < n$.
Output: An integer sequence S_2 of all distinct measurable amounts less than $m + n$ obtainable by filling the n units jug first.
Procedure:
Step 1. Initialize a dummy variable $k = 0$.
Step 2. If $k \neq 0$, then $k := k + n$.
Step 3. While $k \geq m$ do $k := k - m$.
Step 4. If $k = 0$, then return the sequence of k obtained and terminate. Otherwise, repeat the steps 2-4.

Here are two important results related to these algorithms (see [10]).

Theorem 2.1. The total number of distinct measurable amounts less than $m + n$ obtainable by the jugs is equal to $(m + n)/k$, where $k = gcd(m, n)$.

Theorem 2.2. The integer sequence generated by Algorithm 1 is the same as that generated by Algorithm 2 in the reverse order. The additions and subtractions involved in Algorithm 1 can be reversed to become subtractions and additions involved in Algorithm 2.

By Theorem 2.1, we can deduce the following result.

Corollary 2.3. If $gcd(m, n) = 1$, then we can re-arrange the sequence of positive measurable amounts generated by Algorithm 1 (or Algorithm 2) in ascending order to become $\{1, 2, \cdots, m + n - 1\}$.

Proof. If $gcd(m, n) = 1$ and d is a positive integer less than $m+n$, then the Diophantine equation $mx + ny = d$ is solvable for any d. We can apply Algorithm 1 (or Algorithm 2) to generate any measurable amount $d \in \{1, 2, \ldots, m + n - 1\}$. ■

Since the sequence S_1 is same as the sequence S_2 in the reverse order, let us focus on the study of the integer sequences generated by Algorithm 1 from now on.

Example 2.4. Given $m = 5, n = 8$. The sequence of distinct measurable amounts obtainable by Algorithm 1 is as follows:

$$0 \underset{+5}{\longrightarrow} 5 \underset{+5}{\longrightarrow} 10 \underset{-8}{\longrightarrow} 2 \underset{+5}{\longrightarrow} 7 \underset{+5}{\longrightarrow} 12 \underset{-8}{\longrightarrow} 4 \underset{+5}{\longrightarrow} 9 \underset{-8}{\longrightarrow} 1 \underset{+5}{\longrightarrow} 6 \underset{+5}{\longrightarrow} 11 \underset{-8}{\longrightarrow} 3 \underset{+5}{\longrightarrow} 8 \underset{-8}{\longrightarrow} 0$$

After re-arranging the sequence of positive integers obtained in ascending order, it becomes $\{1, 2, \ldots, 12\}$. Also, for any measurable amount $d < 13$, the number of additions (say x_0) and subtractions (say y_0) involved provide a solution to the Diophantine equation $5x + 8y = d$, namely $x = x_0$, $y = -y_0$. For instance, when $d = 4$, we have $x_0 = 4$, $y_0 = -2$. Obviously, $(4, -2)$ satisfies $5x + 8y = 4$.

Corollary 2.5. Let $s = m + n$. The general term of the integer sequence $\{a_j\}_{j=0}^{s-1}$ generated by Algorithm 1 satisfies the following equation:

$$a_j \equiv jm \ (mod\ s), \quad j = 0, \ldots, s - 1$$

Proof. According to Algorithm 1, for each j, $a_j \equiv jm \ (mod\ s)$ if $(j - 1)m < n$. If $(j - 1)m \geq n$, then $a_j = (j - 1)m - n = jm - s \equiv jm \ (mod\ s)$. So, the result follows. ■

Remark 2.6. If we replace Algorithm 1 by Algorithm 2 in Corollary 2.5, then the equation becomes $a_j \equiv jn \ (mod\ s)$, $j = 0, \ldots, s - 1$. The proof is similar to the above one.

3. Dual sequences of non-zero measurable amounts

We now introduce the concept of *dual sequences* to describe the correspondence between a pair of integer sequences generated by Algorithm 1. We will use the notation $M(m, n)$ to denote the sequence of non-zero measurable amounts obtainable by Algorithm 1 and the arithmetic operations involved will not be shown explicitly below the arrows.

Example 3.1.

$$M(3, 8): \quad 3 \to 6 \to 9 \to 1 \to 4 \to 7 \to 10 \to 2 \to 5 \to 8$$
$$M(4, 7): \quad 4 \to 8 \to 1 \to 5 \to 9 \to 2 \to 6 \to 10 \to 3 \to 7.$$

In this example, we have $gcd(3, 8) = gcd(4, 7) = 1$ and the length of each sequence is equal to 10 $(= 3 + 8 = 4 + 7)$. The numbers in this pair of integer sequences have the following correspondence relation:

- For the number $3 \in M(3, 8)$, the 3rd term of $M(4, 7)$ is 1, which is the position of 3 in $M(3, 8)$.
- For the number $6 \in M(4, 7)$, the 6th term of $M(3, 8)$ is 7, which is the position of 6 in $M(4, 7)$.

Similar correspondences hold for all the numbers in $M(3, 8)$ and the corresponding numbers in $M(4, 7)$, and vice versa.

Now, let us define the concept of *dual sequence* formally to describe the correspondence relation below.

Definition 3.2. [Dual Sequence] Let m_1, n_1, m_2, n_2 be integers such that $0 < m_1 < n_1$, $0 < m_2 < n_2$, $s = m_1 + n_1 = m_2 + n_2$ and $gcd(m_1, n_1) = gcd(m_2, n_2) = 1$. Suppose $M(m_1, n_1) = \{a_k\}_{k=1}^{s-1}$, $M(m_2, n_2) = \{b_j\}_{j=1}^{s-1}$ is a pair of integer sequences of non-zero measurable amounts obtainable by Algorithm 1. $M(m_1, n_1)$ is called the *dual sequence* of $M(m_2, n_2)$, or vice versa, if the following condition is satisfied by their elements:

$$a_{b_k} = k, \quad \text{where } k = 1, \ldots, s-1$$
$$b_{a_j} = j, \quad \text{where } j = 1, \ldots, s-1$$

The following theorem describes the necessary and sufficient conditions for the existence of dual sequences.

Theorem 3.3. Let m, n be integers such that $0 < m < n$, $gcd(m, n) = 1$ and $s = m+n$. If there exists a positive integer m' with $m' < s - m'$ and $gcd(m', s - m') = 1$, then $M(m, n)$ and $M(m', s - m')$ are dual to each other *if and only if* $mm' \equiv 1(mod\ s)$.

Proof. Let $n' = s - m'$, $M(m, n) = \{a_k\}_{k=1}^{s-1}$ and $M(m', n') = \{b_j\}_{j=1}^{s-1}$.
($\Leftarrow$) Since $mm' \equiv 1\ (mod\ s)$,

$$\begin{aligned} a_{b_k} &\equiv b_k \cdot m \quad (mod\ s) \\ &\equiv (km')m \quad (mod\ s) \\ &\equiv k(m'm) \quad (mod\ s) \\ &\equiv k \quad (mod\ s) \end{aligned}$$

Similarly,

$$\begin{aligned} b_{a_j} &\equiv a_j \cdot m' \quad (mod\ s) \\ &\equiv (jm)m' \quad (mod\ s) \\ &\equiv j(mm') \quad (mod\ s) \\ &\equiv j \quad (mod\ s) \end{aligned}$$

Thus, $M(m, n)$ and $M(m', s - m')$ are dual to each other.
($\Rightarrow$) The duality of $M(m, n)$ and $M(m', s - m')$ implies

$$\begin{aligned} k &\equiv a_{b_k} \quad (mod\ s) \\ &\equiv b_k \cdot m \quad (mod\ s) \\ &\equiv km'm \quad (mod\ s), \end{aligned}$$

where $k = 1, \cdots, s - 1$. So, $k(m'm - 1) \equiv 0\ (mod\ s)$ and hence $mm' \equiv 1\ (mod\ s)$. ■

Corollary 3.4. [Uniqueness] Let m, n be integers such that $0 < m < n$, $gcd(m, n) = 1$ and $s = m + n$. If the dual sequence of $M(m, n)$ exists, then it is unique.

Proof. If $M(m', n')$, $M(m'', n'')$ are both dual sequences of $M(m, n)$, then

$$mm' \equiv 1 \equiv mm'' \quad (mod\ s).$$

Since the modular inverse of m is unique, we have $m' = m''$ and $n' = s - m' = s - m'' = n''$. Hence, $M(m', n') = M(m'', n'')$. ■

Corollary 3.5. Let m, n, m', n' be integers such that $0 < m < n$, $0 < m' < n'$, $gcd(m, n) = gcd(m', n') = 1$ and $s = m + n = m' + n'$. If $M(m', n')$ is the dual sequence of $M(m, n)$, then $nn' \equiv 1\ (mod\ s)$.

Proof. $nn' \equiv (s - m) \cdot (s - m') \equiv s^2 - (m' + m)s + mm' \equiv mm' \equiv 1\ (mod\ s)$. ■

Example 3.6. The integer sequences $M(3, 8)$ and $M(4, 7)$ are dual to each other since $3 \cdot 4 \equiv 1(mod\ 11)$, as illustrated in Example 3.1.

Example 3.7. The dual sequence of $M(3, 5)$ is itself since $3 \cdot 3 \equiv 1(mod\ 8)$. We call it a *self-dual* sequence.

Example 3.8. Suppose we want to locate the position of $d = 201$ in $M(123, 491)$. Since $s = 123 + 491 = 614$ and $5 \cdot 123 \equiv 1(mod\ 614)$, so $M(5, 609)$ is the dual sequence of $M(123, 491)$. Since the 201th term of $M(5, 609)$ can be computed easily as follows:

$$5 \cdot 200 - 609 = 391,$$

so $d = 201$ is the 391th term in $M(123, 491)$.

4. Generalization

We now discuss how to generalize the concepts and results described above in this section.

Definition 4.1. Let m_1, n_1, m_2, n_2 be integers such that $0 < m_1 < n_1$, $0 < m_2 < n_2$, $s = m_1 + n_1 = m_2 + n_2$ and $u = gcd(m_1, n_1) = gcd(m_2, n_2) \geq 1$. Suppose

$$M(m_1, n_1) = \{a_k\}_{k=1}^{s/u-1}, \quad M(m_2, n_2) = \{b_j\}_{j=1}^{s/u-1}.$$

The integer sequences $M(m_1, n_1)$ and $M(m_2, n_2)$ are said to be *dual* to each other if the following conditions hold:

$$a_{(b_k)/u} = k \cdot u, \quad b_{(a_j)/u} = j \cdot u.$$

Theorem 4.2. Let m, n be integers such that $0 < m < n$ and $s = m + n$. If there exists a positive integer m' with $m' < s - m'$ and $u = gcd(m, n) = gcd(m', s - m')$, then $M(m, n)$ and $M(m', s - m')$ are dual to each other if and only if $mm' \equiv u^2 \ (mod\ s)$.

Proof. Let $n' = s - m'$, $M(m, n) = \{a_k\}_{k=1}^{s/u-1}$ and $M(m', n') = \{b_j\}_{j=1}^{s/u-1}$.
($\Leftarrow$) Since $mm' \equiv u^2 \ (mod\ s)$,

$$\begin{aligned} a_{(b_k)/u} &\equiv (b_k)/u \cdot m \quad (mod\ s) \\ &\equiv (km') \cdot m/u \quad (mod\ s) \\ &\equiv (k/u) \cdot (m'm) \quad (mod\ s) \\ &\equiv k \cdot u \quad (mod\ s) \end{aligned}$$

Similarly,

$$\begin{aligned} b_{(a_j)/u} &\equiv (a_j)/u \cdot m' \quad (mod\ s) \\ &\equiv (jm) \cdot m'/u \quad (mod\ s) \\ &\equiv (j/u) \cdot (mm') \quad (mod\ s) \\ &\equiv j \cdot u \quad (mod\ s) \end{aligned}$$

So, $M(m, n)$ and $M(m', s - m')$ are dual to each other.
($\Rightarrow$) The duality of $M(m, n)$ and $M(m', s - m')$ implies

$$\begin{aligned} ku &\equiv a_{(b_k)/u} \quad (mod\ s) \\ &\equiv (b_k) \cdot (m/u) \quad (mod\ s) \\ &\equiv (km') \cdot (m/u) \quad (mod\ s) \\ \Rightarrow ku^2 &\equiv km'm \quad (mod\ s) \end{aligned}$$

where $k = 1, \ldots, s - 1$. So, $k(m'm - u^2) \equiv 0 \ (mod\ s)$ and hence $mm' \equiv u^2 \ (mod\ s)$. ■

Corollary 4.3. [Uniqueness] Let m, n be integers such that $0 < m < n$, $gcd(m, n) = u \geq 1$ and $s = m + n$. If the dual sequence of $M(m, n)$ exists, then it is unique.

Proof. If $M(m', n')$, $M(m'', n'')$ are both dual sequences of $M(m, n)$, then

$$mm' \equiv u^2 \equiv mm'' \quad (mod\ s).$$

Since $m > 0$, $m(m' - m'') \equiv 0 \ (mod\ s)$, $0 < m' < s$ and $0 < m'' < s$, so it implies $m' = m''$. Now, $n' = s - m' = s - m'' = n''$. Hence, $M(m', n') = M(m'', n'')$. ■

Corollary 4.4. Let m, n, m', n' be integers such that $0 < m < n$, $0 < m' < n'$, $gcd(m, n) = gcd(m', n') = u \geq 1$ and $s = m + n = m' + n'$. If $M(m', n')$ is the dual sequence of $M(m, n)$, then $nn' \equiv u^2 \ (mod\ s)$.

Proof. $nn' \equiv (s - m) \cdot (s - m') \equiv s^2 - (m' + m)s + mm' \equiv mm' \equiv u^2 \ (mod\ s)$. ■

Example 4.5. The integer sequences $M(6, 16)$ and $M(8, 14)$ are dual to each other since $gcd(6, 16) = gcd(8, 14) = 2$ and $6 \cdot 8 \equiv 2^2 (mod\ 22)$.

Example 4.6. The dual sequence of $M(9, 15)$ is itself since $gcd(9, 15) = 3$ and $9 \cdot 9 \equiv 3^2 (mod\ 24)$. We call it a *self-dual* sequence.

5. Concluding Remarks

This paper extends the works of [9, 10, 11, 12] on algorithmic soluton of the two jugs problem and the concept of *dual sequence* of non-zero measurable amounts is introduced. The necessary and sufficient conditions for the existence of dual sequences are provided. It helps to fill up the gap of the study of the properties of the sequence of measurable amounts of the two jugs problem, which is often missed in the existing literature of the jug problem. Further extension or generalization of the concepts and results described in this paper for the study of the N-jugs problem, where $N > 2$, will be an interesting topic for further research.

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