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Exact Solutions of the Slowly Varying Amplitudes of Two Interacting Families for Nonlinearly Coupled Ginzburg-Landau Equations

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Abstract

Exact solutions of nonlinearly coupled Ginzburg-Landau equations are studied. An algorithm for constructing solutions in an analytical way is presented, which involves eventually solving a set of algebraic equations. The ensuing system of equations is solved by means of computer algebra software which permits exact solutions to be obtained. A closed form representation of the solutions is presented and some examples of the numerical solutions are also provided.

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1. Introduction

We study the analytical construction of some exact solutions of a system of nonlinearly coupled physical differential equations, namely, the Complex Ginzburg-Landau Equations (CGLEs) [1]. The CGLEs are intensively studied models of pattern formation in nonlinear dissipative media [2]–[5], with applications to biology, hydrodynamics, nonlinear optics, plasma physics, chemical reaction-diffusion systems [8] and many other fields [13]–[15].

The CGLEs are non-integrable and hence the powerful tools, which work for Schrödinger equation, will not be applicable. There exists a number of analytical methods for finding exact and explicit solutions in the literature [6]–[13]. Recently, we applied the method which involves the use of Hirota operator in finding the solitary and periodic pulses of nonlinear systems [16]–[18]. The Hirota bilinear method is a method for obtaining multi-soliton expressions in integrable nonlinear evolution equations. To extend the use of this method to the CGLEs, we needed a modified bilinear operator which was initiated by Bekki and Nozaki [12]. The aim of this paper is to introduce some analytical techniques, along with the Bekki-Nozaki operator, to produce some exact solutions for the system of nonlinearly coupled CGLEs.

We are ready to present our major results, namely, the analytical construction of some exact solutions of the nonlinearly coupled complex Ginzburg-Landau model. The terminologies of nonlinear optics have been employed for discussion in this paper. Slowly varying amplitudes A and B will typically be governed by the nonlinearly coupled CGLEs:

$$\mathrm{i}A_t + p_1 A_{xx} + \left(q_1|A|^2 + q_2|B|^2\right) A = \mathrm{i}\gamma_1 A, \tag{1}$$

$$\mathrm{i}B_t + p_2 B_{xx} + \left(q_1|B|^2 + q_2|A|^2\right) B = \mathrm{i}\gamma_2 B, \tag{2}$$

with complex dispersion and nonlinearity coefficients $p_{1,2}$ and q_1, and real linear gain coefficients $\gamma_{1,2}$. The parameter q_2 accounts for the nonlinear coupling.

The physical interpretation of the various coefficients can now be explained. The real parts of the coefficients p_1 and p_2 correspond to the group velocity dispersion and the imaginary parts, if any, are associated with the physical effects of bandwidth limited amplification. The real parts of the complex coefficients q_1 and q_2 denote the self- and cross-phase modulations respectively, while the imaginary parts are associated with the nonlinear gain/loss. The linear gain/loss of the optical waveguides is measured by the real coefficients γ_1 and γ_2.

In Sect. 2, the mathematical formulation of the system of the complex governing equations will be presented in detail. The corresponding real parts and imaginary parts of the complex equations will also be given. In Sect. 3–Sect. 5, we shall make use of the package of Groebner basis method in the computer algebra software Maple which permits exact solutions to be obtained. The new exact solutions are produced explicitly.

2. Mathematical formulation

To apply the modified Hirota operator, we make use of the following transformations

$$A = \frac{G\,\exp\left(-\mathrm{i}\Omega_1 t\right)}{f^m}, \qquad B = \frac{H\,\exp\left(-\mathrm{i}\Omega_2 t\right)}{f^n}, \tag{3}$$

where, G and H are complex-valued functions, but f is real-valued, while m and n are complex numbers of the form (in which α and β are real):

$$m = 1 + \mathrm{i}\alpha \qquad \text{and} \qquad n = 1 + \mathrm{i}\beta. \tag{4}$$

We next assume expressions of the forms (in which k and ω are complex):

$$G = g\,\exp\left(kx - \omega t\right), \quad H = h, \quad f = 1 + \exp\left(kx - \omega t\right). \tag{5}$$

Using the basic principles in simplifying the modified Hirota derivatives of exponential functions, we can eventually deduce the four complex-valued equations [17]:

$$\mathrm{i}\omega m + q_2 hh^* \left[\sigma_1 - 1 - \frac{(1+\sigma_1)(m-2)}{m+1}\right] = 0, \tag{6}$$

$$\mathrm{i}\omega n + q_1 hh^* \left[\sigma_2 - 1 - \frac{(1+\sigma_2)\, n}{n+1}\right] = 0, \tag{7}$$

$$-p_1\, m(m+1)\, k^2 = (1+\sigma_1)\, q_2\, hh^*, \tag{8}$$

$$-p_2\, n(n+1)\, k^2 = (1+\sigma_2)\, q_1\, hh^*. \tag{9}$$

The parameters, Ω_1, Ω_2 (angular frequencies of the envelope), g (amplitude of one waveguide) and γ_1, γ_2 (linear gain/loss) in Eqs (1)–(3) are determined by the following auxiliary conditions:

$$\begin{aligned} &-\mathrm{i}\omega + p_1 k^2 + \Omega_1 - \mathrm{i}\gamma_1 + q_2\, |h|^2 = 0, \\ &\Omega_2 - \mathrm{i}\gamma_2 + q_1 |h|^2 = 0, \\ &|g|^2 = \sqrt{\sigma_1\sigma_2}|h|^2. \end{aligned} \tag{10}$$

By the nonlinear system of four equations (6)–(9), the six real unknowns (all nonzero) are given by $(\omega, \alpha, \beta, \sigma_1, \sigma_2, hh^*)$, $hh^* > 0$. Besides, (q_1, q_2, p_1, p_2) are some complex parameters which are supposed to be given. Note that q_1 is given and q_2 is given by

$$\sigma_1 (q_2)^2 = \sigma_2 (q_1)^2. \tag{11}$$

The above condition implies that σ_1 and σ_2 must be of the same sign but q_1 and q_2 could be of the different signs. Denote that

$$q_1 := q_r + \mathrm{i} q_i, \qquad p_1 := p_{1r} + \mathrm{i}\, p_{1i}, \qquad p_2 := p_{2r} + \mathrm{i}\, p_{2i} \tag{12}$$

and

$$\sigma_1 := \varepsilon_1 s_1^2, \qquad \sigma_2 := \varepsilon_1 s_2^2 \qquad (s_1, s_2 > 0, \varepsilon_1^2 = 1). \tag{13}$$

Hence, (11) implies that

$$q_2 = \pm\sqrt{\frac{\sigma_2}{\sigma_1}}\, q_1 = \varepsilon_2\, \frac{s_2}{s_1}\, q_1, \qquad \varepsilon_2^2 = 1. \tag{14}$$

Separating the real and imaginary parts of the four deduced complex equations in (6)–(9) gives the following eight real equations:

$$
\begin{aligned}
3\varepsilon_1\varepsilon_2\, hh^*\, s_1^2 s_2\, q_r - \varepsilon_2\, hh^*\, s_2\,(q_r - 2\alpha q_i) - 3\omega\alpha s_1 &= 0, & (15)\\
3\varepsilon_1\varepsilon_2\, hh^*\, s_1^2 s_2\, q_i - \varepsilon_2\, hh^*\, s_2\,(q_i + 2\alpha q_r) + \omega s_1(2-\alpha^2) &= 0, & (16)\\
\varepsilon_1\, hh^*\, s_2^2\, q_r + hh^*\,(2\beta q_i - 3q_r) - 3\omega\beta &= 0, & (17)\\
\varepsilon_1\, hh^*\, s_2^2\, q_i - hh^*\,(2\beta q_r + 3q_i) + \omega(2-\beta^2) &= 0, & (18)\\
-\varepsilon_1\varepsilon_2\, hh^*\, s_1^2 s_2\, q_r - \varepsilon_2\, hh^*\, s_2\, q_r + k^2\, s_1\left(p_{1r}\alpha^2 + 3p_{1i}\alpha - 2p_{1r}\right) &= 0, & (19)\\
-\varepsilon_1\varepsilon_2\, hh^*\, s_1^2 s_2\, q_i - \varepsilon_2\, hh^*\, s_2\, q_i + k^2\, s_1\left(p_{1i}\alpha^2 - 3p_{1r}\alpha - 2p_{1i}\right) &= 0, & (20)\\
-\varepsilon_1\, hh^*\, s_2^2\, q_r - hh^*\, q_r + k^2\left(p_{2r}\beta^2 + 3p_{2i}\beta - 2p_{2r}\right) &= 0, & (21)\\
-\varepsilon_1\, hh^*\, s_2^2\, q_i - hh^*\, q_i + k^2\left(p_{2i}\beta^2 - 3p_{2r}\beta - 2p_{2i}\right) &= 0, & (22)
\end{aligned}
$$

where $(\omega, s_1, s_2, hh^*, \alpha, \beta)$ are the real unknowns to be solved with which $s_1, s_2, hh^* > 0$. Eventually we find that $(\omega, s_1, s_2, hh^*, \alpha, \beta, p_{1r}, p_{1i})$ can be explicitly written as expressions in terms of $(q_r, q_i, p_{2r}, p_{2i})$. The actual algebraic manipulations constitute a major undertaking and are accomplished by means of a computer software. The package of Groebner basis method in the computer algebra software Maple permits exact solutions to be obtained.

3. Exact Solutions of the System with $q_r = 0$

The computations for finding the exact solutions of (15)–(22) seem to be larger than expected. Our preliminary work shows that the computations can largely be shortened by considering

$$q_r = 0.$$

Therefore in the following context we will assume that the real part of q_1 is zero and we will try to relax this constraint in the future work. Now with this assumption we can find the following exact solutions of (6)–(9) by first solving the real equations (15)–(22) in Maple:

$$
\begin{cases}
q_2 = q_1 = \mathrm{i}q_i \qquad \text{where } q_i = \text{arbitrary},\\
p_1 = d\left[(-3\alpha) + \mathrm{i}(\alpha^2 - 2)\right] \qquad \text{where } d := \dfrac{(\beta^2+4)\,\omega}{k^2\,(\alpha^2+1)(\alpha^2+4)},\\
p_2 = p_{2r} + \mathrm{i}p_{2i} \qquad \text{with } \hat{p} := \dfrac{p_{2i}}{p_{2r}},\\
\beta^2 + 3\hat{p}\,\beta - 2 = 0, \qquad \alpha^2 = 14 - 9\hat{p}\,\beta,\\
\sigma_2 = \sigma_1 = \dfrac{1}{3}\left(2\beta^2 + 5\right),\\
\left(\dfrac{4\omega}{k^2} - p_{2i}\right)^2 = 8p_{2r}^2 + 9p_{2i}^2, \qquad hh^* = \dfrac{3\omega}{2q_i}.
\end{cases}
\qquad (23)
$$

Some remarks for the above exact solution (23).

I. The procedure of finding some numerical solutions of (23) is:

Step 1: Given the complex parameter p_2 (so that p_{2r} and p_{2i} are known).

Step 2: Solve the (two) values of $\beta_{1,2}$ by the quadratic equation.

Step 3: Find the values of $\alpha^{\pm}$ if β is fixed.

Step 4: Find the values of s_1 and s_2 and hence σ_1 and σ_2.

Step 5: Solve the (two) values of $\omega_{1,2}$ by the quadratic equation.

Step 6: Find the values of p_{1r} and p_{1i}.

Step 7: Find the value of hh^* if ω is fixed.

II. Our calculations show that $s_1^2 = s_2^2 = (\varepsilon_1/3)\,(2\beta^2+5)$ and hence ε_1 must be $+1$ and furthermore (13) shows that both σ_1 and σ_2 must be positive.

III. Since $s_1 = s_2 > 0$ and it follows from (14) that $q_2 = \varepsilon_2 q_1$ ($\varepsilon_2 = \pm 1$). Our calculations show that $p_{1r} = -3\varepsilon_2\,\alpha\omega/d^2$ and $p_{1i} = \varepsilon_2\,(\alpha^2-2)\,\omega/d^2$ as well and finally we can verify that ε_2 must be equal to $+1$ in order to satisfy the original equations. Thus with the assumption $q_r = 0$ we must have $q_2 = q_1$ as a consequence.

IV. The quadratic equation for β gives two distinct real zeros because $\Delta = 9\hat{p}^2+8 > 0$ and the two zeros must be of different signs ($\beta_1\beta_2 < 0$).

V. The quadratic equation for ω gives two distinct real zeros because $\Delta = k^4\,(8p_{2r}^2 + 9p_{2i}^2) > 0$ and the two zeros must be of different signs ($\omega_1\omega_2 < 0$). Note also that the value of ω does not depend on α or β.

VI. $hh^* = |h|^2$ must be positive and the positiveness of hh^* will determine the correct sign of q_i (either $q_i < 0$ or $q_i > 0$).

Examples of explicit numerical solutions with

$$q_1 = \mathrm{i}\,q_i\,(q_r = 0) \qquad \text{and} \qquad p_2 = 3+\mathrm{i}.$$

Taking a particular value of p_2 and using the general solution (23), we can deduce the two exact solutions of Eqs (6)–(9) in the following.

(A) With the given values:

$$q_1 := \mathrm{i}\,q_i, \qquad \text{where } q_i > 0 \text{ (arbitrary)},$$

$$p_2 := 3+\mathrm{i},$$

the exact solution of Eqs (6)–(9) is given by

$$\begin{cases} q_2 = \mathrm{i}\, q_i, & p_1 = \mp\dfrac{5\sqrt{5}}{21} + \mathrm{i}\,\dfrac{5}{7}, \\ \alpha = \pm 2\sqrt{5}, & \beta = -2, \\ \sigma_1 = \dfrac{13}{3}, & \sigma_2 = \dfrac{13}{3}, \\ \omega = \dfrac{5k^2}{2}, & hh^* = \dfrac{15k^2}{4q_i}. \end{cases} \tag{24}$$

(B) With the given values:

$$\begin{aligned} q_1 &:= \mathrm{i}\, q_i, \qquad \text{where } q_i < 0 \text{ (arbitrary)}, \\ p_2 &:= 3 + \mathrm{i}, \end{aligned}$$

the exact solution of Eqs (6)–(9) is given by

$$\begin{cases} q_2 = \mathrm{i}\, q_i, & p_1 = \pm\dfrac{\sqrt{11}}{6} - \mathrm{i}\,\dfrac{1}{2}, \\ \alpha = \pm\sqrt{11}, & \beta = 1, \\ \sigma_1 = \dfrac{7}{3}, & \sigma_2 = \dfrac{7}{3}, \\ \omega = -2k^2, & hh^* = -\dfrac{3k^2}{q_i}. \end{cases} \tag{25}$$

Examples of explicit numerical solutions with

$$q_2 = q_1 = \mathrm{i}\, q_i (q_r = 0) \qquad \text{and} \qquad p_2 = -9 + 7\mathrm{i}.$$

Taking another particular value of p_2 and using the general solution (23), we can deduce the two exact solutions of Eqs (6)–(9) in the following.

(C) With the given values:

$$\begin{aligned} q_1 &:= \mathrm{i}\, q_i, \qquad \text{where } q_i > 0 \text{ (arbitrary)}, \\ p_2 &:= -9 + 7\mathrm{i}, \end{aligned}$$

the exact solution of Eqs (6)–(9) is given by

$$\begin{cases} q_2 = \mathrm{i}\, q_i, & p_1 = \mp\dfrac{5\sqrt{35}}{18} + \mathrm{i}\,\dfrac{55}{18}, \\ \alpha = \pm\sqrt{35}, & \beta = 3, \\ \sigma_1 = \dfrac{23}{3}, & \sigma_2 = \dfrac{23}{3}, \\ \omega = 10k^2, & hh^* = \dfrac{15k^2}{q_i}. \end{cases} \tag{26}$$

(D) With the given values:

$$\begin{aligned} q_1 &:= \mathrm{i}\, q_i, \qquad \text{where } q_i < 0 \text{ (arbitrary)}, \\ p_2 &:= -9 + 7\mathrm{i}, \end{aligned}$$

the exact solution of Eqs (6)–(9) is given by

$$\begin{cases} q_2 = \mathrm{i}\, q_i, & p_1 = \pm\dfrac{13\sqrt{21}}{31} - \mathrm{i}\,\dfrac{143}{93}, \\ \alpha = \pm\dfrac{2\sqrt{21}}{3}, & \beta = -\dfrac{2}{3}, \\ \sigma_1 = \dfrac{53}{27}, & \sigma_2 = \dfrac{53}{27}, \\ \omega = -\dfrac{13k^2}{2}, & hh^* = -\dfrac{39k^2}{4q_i}. \end{cases} \tag{27}$$

4. Exact Solutions of the System with $q_i = 0$

In Section 3 we have seen the simplified case when the assumption $q_r = 0$ was made in order to reduce the computation work. Here in this section we are going to do some similar work that we impose the assumption

$$q_i = 0.$$

We find that the computations for finding the exact solutions of (15)–(22) seem to be largely reduced. Therefore in the following context we will assume that the imaginary part of q_1 is zero and later we will try to relax this constraint and eventually can handle the general case $q = q_r + \mathrm{i}q_i$. Now with this assumption we can find the following exact solutions of (6)–(9) by first solving the real equations (15)–(22) in Maple:

$$\begin{cases} q_1 = q_r \qquad \text{where } q_r = \text{arbitrary}, \\ q_2 = \varepsilon_2 \sqrt{\dfrac{\sigma_2}{\sigma_1}} q_1 \qquad \text{where } \varepsilon_2^2 := 1, \\ p_2 = p_{2r} + \mathrm{i} p_{2i} \qquad \text{with } \tilde{p} := \dfrac{p_{2r}}{p_{2i}}, \\ p_1 = -3\varepsilon_2 d \left[(\alpha^2 - 2) + \mathrm{i}\,(3\alpha) \right], \quad d := \dfrac{\omega \tilde{p}\,(\sigma_1 + 1)}{2k^2\,(\alpha^2+1)(\alpha^2+4)} \sqrt{\dfrac{\sigma_2}{\sigma_1}}, \\ \beta^2 - 3\tilde{p}\beta - 2 = 0, \qquad 5\alpha^4 + (81\tilde{p}^2 - 54\beta\tilde{p} - 8)\,\alpha^2 - 4 = 0, \\ \sigma_1 = -\dfrac{5\alpha^2+2}{3(\alpha^2-2)}, \qquad \sigma_2 = -\dfrac{3(\beta^2+2)}{\beta^2-2}, \\ \omega = k^2\,(\beta p_{2r} + p_{2i})\,, \qquad hh^* = -\dfrac{3\omega\tilde{p}}{2q_r}. \end{cases} \tag{28}$$

Some remarks for the above exact solution.

I. The procedure of finding some numerical solutions of (28) is:

Step 1: Given the complex parameter p_2 (so that p_{2r} and p_{2i} are known).

Step 2: Solve the (two) values of $\beta_{1,2}$ by the quadratic equation.

Step 3: Find the values of $\alpha^{\pm}$ if β is fixed, omit the two complex conjugate values.

Step 4: Find the positive values of s_1 and s_2 and hence σ_1 and σ_2 (σ_1 and σ_2 can be negative).

Step 5: Solve the value of ω if β is fixed.

Step 6: Find the values of p_{1r} and p_{1i}.

Step 7: Find the value of hh^* which must be positive.

II. Note that s_1 and s_2 are both real positive by definition. Our calculations show that their expressions are

$$s_1 = \sqrt{\frac{-\varepsilon_1\,(5\alpha^2+2)}{3(\alpha^2-2)}} \qquad \text{and} \qquad s_2 = \sqrt{\frac{-3\varepsilon_1\,(\beta^2+2)}{\beta^2-2}}.$$

Therefore, one of the values of $\varepsilon_1 = \pm 1$ should be fixed depending on the chosen value of β (see the expression of s_2 in the above). Additionally, (13) shows that $\sigma_1 = \varepsilon_1 s_1^2$ and $\sigma_2 = \varepsilon_1 s_2^2$ which must be both positive or both negative depending on the fixed value of ε_1.

III. The values of s_1 and s_2 are generally different (unlike the case when $q_r = 0$ in Section 3). It follows from (14) that $q_2 = \varepsilon_2 (s_2/s_1)\, q_1$ ($\varepsilon_2 = \pm 1$). Our calculations show that ε_2 can be $+1$ or -1 in order to satisfy the original equations. Thus with the assumption $q_i = 0$ we then have $q_2 = \pm(s_2/s_1)\, q_1$ as a consequence.

IV. The quadratic equation for β gives two distinct real zeros because $\Delta = 9\hat{p}^2 + 8 > 0$ and the two zeros must be of different signs ($\beta_1 \beta_2 < 0$).

V. The quadratic equation for α^2 gives two zeros of different signs. The negative zero of α^2 will be omitted since α must be real. The positive zero of α^2 gives the two values of $\alpha^{\pm}$. We denote that $\alpha = \varepsilon_3 \sqrt{\alpha^2}$, $\varepsilon_3^2 = 1$ in our calculations.

VI. The positiveness of hh^* will determine the correct sign of the (arbitrary) q_r.

Examples of explicit numerical solutions with

$$q_1 = q_r \; (q_i = 0) \qquad \text{and} \qquad p_2 = 1 + 3\mathrm{i}$$

Taking a particular value of p_2 and using the general solution (28) we can deduce the two exact solutions of Eqs (6)–(9) in the following.

(E) With the given values:

$$\begin{aligned} q_1 &:= q_r, \qquad \text{where } q_r < 0 \text{ (arbitrary)}, \\ p_2 &:= 1 + 3\mathrm{i}. \end{aligned}$$

the exact solution of Eqs (6)–(9) is given by

$$\begin{cases} q_2 = \pm\dfrac{9\sqrt{5}}{5}\, q_r, & p_1 = \pm\sqrt{5} + \dfrac{5}{3}\mathrm{i}, \\ \alpha = \mp\dfrac{\sqrt{5}}{5}, & \beta = -1, \\ \sigma_1 = \dfrac{5}{9}, & \sigma_2 = 9, \\ \omega = 2k^2, & hh^* = -\dfrac{k^2}{q_r}. \end{cases} \tag{29}$$

(F) With the given values:

$$\begin{aligned} q_1 &:= q_r, \qquad \text{where } q_r < 0 \text{ (arbitrary)}, \\ p_2 &:= 1 + 3\mathrm{i}. \end{aligned}$$

the exact solution of Eqs (6)–(9) is given by

$$\begin{cases} q_2 = \pm\dfrac{\sqrt{10+\sqrt{145}}\,(\sqrt{145}-10)}{5}q_r, \\ p_1 = \pm\dfrac{5\sqrt{10+\sqrt{145}}\left(5\sqrt{145}-59\right)}{72} + \mathrm{i}\,\dfrac{5\left(15-\sqrt{145}\right)}{24}, \\ \alpha = \pm\dfrac{\sqrt{350+30\sqrt{145}}}{10}, \qquad \beta = 2, \\ \sigma_1 = -\dfrac{10+\sqrt{145}}{9}, \qquad \sigma_2 = -9, \\ \omega = 5k^2, \qquad hh^* = -\dfrac{5k^2}{2q_r}. \end{cases} \tag{30}$$

Examples of explicit numerical solutions with

$$q_1 = q_r\,(q_i = 0) \qquad \text{and} \qquad p_2 \;=\; -7+9\mathrm{i}$$

Taking a particular value of p_2 and using the general solution (28) we can deduce the two exact solutions of Eqs (6)–(9) in the following.

(G) With the given values:

$$q_1 \;:=\; q_r, \qquad \text{where } q_r \;>\; 0 \text{ (arbitrary)},$$

$$p_2 \;:=\; -7+9\mathrm{i}.$$

the exact solution of Eqs (6)–(9) is given by

$$\begin{cases} q_2 = \pm\dfrac{3\sqrt{190+3\sqrt{7305}}\,(3\sqrt{7305}-190)}{2695}q_r, \\ p_1 = \mp\dfrac{5\sqrt{190+3\sqrt{7305}}\,(95\sqrt{7305}-7993)}{6622} + \mathrm{i}\,\dfrac{15\left(95-\sqrt{7305}\right)}{86}, \\ \alpha = \mp\dfrac{\sqrt{850+10\sqrt{7305}}}{10}, \qquad \beta = -3, \\ \sigma_1 = -\dfrac{190+3\sqrt{7305}}{231}, \qquad \sigma_2 = -\dfrac{33}{7}, \\ \omega = 30k^2, \qquad hh^* = \dfrac{35k^2}{q_r}. \end{cases} \tag{31}$$

(H) With the given values:

$$q_1 := q_r, \qquad \text{where } q_r > 0 \text{ (arbitrary)},$$
$$p_2 := -7 + 9\text{i}.$$

the exact solution of Eqs (6)–(9) is given by

$$\begin{cases} q_2 = \pm \dfrac{3\sqrt{3\sqrt{4841} - 118}\,(3\sqrt{4841} + 118)}{2695} q_r, \\ p_1 = \mp \dfrac{13\sqrt{3\sqrt{4841} - 118}\,(59\sqrt{4841} + 4297)}{94248} + \text{i}\,\dfrac{13\,(\sqrt{4841} + 59)}{408}, \\ \alpha = \pm \dfrac{\sqrt{10\sqrt{4841} - 690}}{10}, \qquad \beta = \dfrac{2}{3}, \\ \sigma_1 = \dfrac{3\sqrt{4841} - 118}{231}, \qquad \sigma_2 = \dfrac{33}{7}, \\ \omega = \dfrac{13k^2}{3}, \qquad hh^* = \dfrac{91k^2}{18q_r}. \end{cases} \tag{32}$$

5. Exact Solutions of the System with $q_1 = q_r + \text{i}q_i$

In Sections 3 and 4 we have seen the simplified cases when the assumptions $q_r = 0$ and $q_i = 0$ were made respectively. Here in this section we are going to relax those assumptions and consider generally

$$q_1 = q_r + \text{i}q_i.$$

By what we have done so far the computations are pretty large that we cannot explicitly write all the expressions of the exact solutions. Some of the coefficient functions are long but we will try to characterize them by showing explicitly the dependent parameters in the functions. Now we write down the following exact solutions of (6)–(9):

$$\begin{cases} q_1 = q_r + \mathrm{i} q_i \qquad \text{where } q_r, q_i \ = \ \text{nonzero and arbitrary}, \\ q_2 = \varepsilon_2 \sqrt{\dfrac{\sigma_2}{\sigma_1}} q_1, \qquad p_2 = p_{2r} + \mathrm{i} p_{2i}, \\ p_1 = -3\varepsilon_2 d \left[(\alpha^2 - 2) + \mathrm{i}\,(3\alpha) \right] \quad \text{where } d \ := \ \dfrac{\omega \tilde{p}\,(\sigma_1 + 1)}{2k^2\,(\alpha^2+1)(\alpha^2+4)} \sqrt{\dfrac{\sigma_2}{\sigma_1}}, \\ \mu\beta^2 + 3\lambda\beta - 2\mu = 0 \quad \text{where } \mu \ = \ \det \begin{bmatrix} p_{2r} & q_r \\ p_{2i} & q_i \end{bmatrix}, \lambda \ = \ \det \begin{bmatrix} p_{2r} & -q_i \\ p_{2i} & q_r \end{bmatrix}, \\ \left(2q_r q_i\,\mu^2\right)\alpha^5 + \left((5q_r^2 - 6q_i^2)\,\mu^2\right)\alpha^4 + \left(-20 q_r q_i\,\mu^2\right)\alpha^3 \\ \qquad + \left(a_2 - 54\beta\, p_{2r} p_{2i}\,(q_r^2 + q_i^2)\right)\alpha^2 \\ \qquad + \left(-4q_r q_i\,\mu^2\right)\alpha + \left(-4q_r^2\mu^2\right) = 0 \quad \text{with } a_2 \ = \ a_2(p_{2r}, p_{2i}, q_r, q_i), \\ \sigma_1 = -\dfrac{2q_i\alpha^3 + 5q_r\alpha^2 - q_i\alpha + 2q_r}{3(q_r\alpha^2 - 3q_i\alpha - 2q_r)}, \sigma_2 = -\dfrac{2q_i\beta^3 + 3q_r\beta^2 + 5q_i\beta + 6q_r}{q_r\beta^2 - 3q_i\beta - 2q_r}, \\ \omega = k^2\left(\beta p_{2r} + p_{2i}\right), \qquad hh^* = -\dfrac{3\omega\tilde{p}}{2q_r}. \end{cases} \tag{33}$$

6. Conclusion

In this paper we presented a method for constructing exact solutions of the local amplitudes of two interacting families for the nonlinearly coupled Ginzburg-Landau equations. Our approach is straightforward via the use of trilinear equations with Bekki-Nozaki modified Hirota operator. Sets of algebraic equations defining the amplitude, phase, wave number and frequency are established, in conjunction with the basic properties of the nonlinear dissipative media, i.e., coefficients of the coupled CGLEs. The close-form representations of the exact solutions are obtained analytically.

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