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Novel existence criteria for periodic solutions of a prescribed mean curvature Rayleigh equation

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Abstract

By employing Mawhin's continuation theorem, the existence results for periodic solutions of the following prescribed mean curvature Rayleigh equation with a deviating argument

$$\left(\frac{x'(t)}{\sqrt{1+x'^2(t)}}\right)' + f(x'(t)) + \beta(t)g(x(t-\tau(t))) = e(t)$$

are obtained under suitable assumptions.

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1. Introduction

In recent years, the problem of periodic solutions of functional differential equations with a deviating argument has been studied extensively. We refer the readers to [1-5] for details. In [1], Lu and Gui discussed the existence of periodic solutions for Rayleigh type p-Laplacian equation

$$(\varphi_p(x'(t)))' + f(x'(t)) + g(x(t-\tau(t))) = e(t), \tag{1.1}$$

where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$ and $f, g \in C(R, R)$, the growth condition imposed on $g(x)$ is as follows

$$-l \leq g'(x) \leq 0, \forall x \in R. \tag{1.2}$$

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Cheung and Ren [2] studied the Liénard type equation

$$(\varphi_p(x'(t)))' + f(x(t))x'(t) + g(x(t-\tau(t))) = e(t), \tag{1.3}$$

where $g(x)$ satisfies the Lipschitz condition

$$|g(u_1) - g(u_2)| \leq l|u_1 - u_2|, \ \forall u_1, u_2 \in R. \tag{1.4}$$

It's interesting to consider the existence of periodic solution of prescribed mean curvature equations appeared in geometry and physics (see [6–8]). Feng [9] studied the periodic solution for prescribed mean curvature Liénard equation with a deviating argument

$$\Big(\frac{x'(t)}{\sqrt{1+x'^2(t)}}\Big)' + f(x(t))x'(t) + g(t, x(t-\tau(t))) = e(t), \tag{1.5}$$

under the condition that

$$|f(x)| \geq \gamma, \gamma > 0 \text{ and } |g(t,x_1) - g(t,x_2)| \leq l|x_1 - x_2|, \ \forall t \in R, x_1, x_2 \in R. \tag{1.6}$$

In this paper, we study the following prescribed mean Rayleigh equation with a deviating argument

$$\Big(\frac{x'(t)}{\sqrt{1+x'^2(t)}}\Big)' + f(x'(t)) + \beta(t)g(x(t-\tau(t))) = e(t), \tag{1.7}$$

where $f, g, e, \beta, \tau \in C(R,R)$, $\tau(t+T) \equiv \tau(t)$, $e(t) \equiv e(t+T)$, $\beta(t+T) = \beta(t)$, $e(t) \not\equiv$ constant, $\beta(t) > 0$, and $\beta_0 = \max_{t\in[0,T]} \beta(t)$. As far as we know, the results on this question are not rich. The purpose of this paper is to establish a criteria to guarantee the existence of T-periodic solution. Our methods and results are different from the corresponding ones of [9].

2. Preliminaries

Let X and Y be real Banach spaces and $L : D(L) \subset X \to Y$ be a Fredholm operator with index zero, that is, $X = \text{Ker}L \oplus X_1$ and $Y = \text{Im}L \oplus Y_1$. Furthermore, let $P : X \to \text{Ker}L$ and $Q : Y \to Y_1$ be the continuous projectors. Clearly, $\text{Ker}L \cap (D(L) \cap X_1) = \{0\}$, thus the restriction $L_P = L|_{D(L)\cap X_1}$ is invertible. Denote by K the inverse of L_P.

Let Ω be an open bounded subset of X with $D(L) \cap \Omega \neq 0$. A map $N : \bar{\Omega} \to Y$ is called L-compact in $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and the operator $K(I-P)N : \bar{\Omega} \to X$ is compact. The following Mawhin's continuation theorem is well known.

Lemma 2.1. (Gaines and Mawhin [10]) Suppose that X and Y are two Banach spaces, and $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero. Furthermore, $\Omega \subset X$ is an open bounded set and $N : \bar{\Omega} \to Y$ is L-compact in $\bar{\Omega}$. If all the following conditions hold:

(1) $Lx \neq \lambda Nx, \quad \forall x \in \partial\Omega \cap D(L), \lambda \in (0, 1)$;

(2) $Nx \notin \mathrm{Im}L, \quad \forall x \in \partial\Omega \cap \mathrm{Ker}L$; and

(3) $\deg\{JQN, \Omega \cap \mathrm{Ker}L, 0\} \neq 0$, where $J : \mathrm{Im}Q \to \mathrm{Ker}L$ is an isomorphism.

Then the equation $Lx = Nx$ has a solution in $\overline{\Omega} \cap D(L)$.

To use Mawhin's continuation theorem to study the (1.7), we firstly transform (1.7) into the following form:

$$\begin{cases} x_1'(t) = \dfrac{x_2(t)}{\sqrt{1 - x_2^2(t)}}, \\ x_2'(t) = -f(\dfrac{x_2(t)}{\sqrt{1 - x_2^2(t)}}) - \beta(t)g(x_1(t - \tau(t))) + e(t). \end{cases} \tag{2.1}$$

Denote by $\varphi(x) = \dfrac{x}{\sqrt{1 - x^2}}$, then prescribed mean operator $\dfrac{x}{\sqrt{1 + x^2}}$ is $\varphi^{-1}(x)$. Clearly, if $x(t) = (x_1(t), x_2(t))^\top$ is a T-periodic solution to (2.1), $x_1(t)$ must be a T-periodic solution to (1.7).

Set $C_T = \{\phi : \phi \in C(R, R^2), \phi(t + T) \equiv \phi(t)\}$ with the norm $|\phi|_0 = \max_{t \in [0,T]} |\phi(t)|$, $X = Y = \{x(t) = (x_1(t), x_2(t))^\top \in C(R, R^2) : x(t + T) \equiv x(t)\}$ with the norm $\|x\| = \max\{|x_1|_0, |x_2|_0\}$. Then X and Y are Banach spaces. Let

$$L : D(L) \subset X \longrightarrow Y, \quad Lx = x' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}. \tag{2.2}$$

$$N : X \longrightarrow Y, \quad Nx = \begin{pmatrix} \varphi(x_2(t)) \\ -f(\varphi(x_2(t))) - \beta(t)g(x_1(t - \tau(t))) + e(t) \end{pmatrix}. \tag{2.3}$$

It is easy to see that $\mathrm{Ker}L = R^2, \mathrm{Im}L = \{x : x \in Y, \int_0^T x(s)ds = 0\}$. So L is a Fredholm operator with index zero. Define projectors $P : X \longrightarrow \mathrm{Ker}L$ and $Q : Y \longrightarrow \mathrm{Im}Q$ by

$$Px = \frac{1}{T}\int_0^T x(s)ds, \quad Qy = \frac{1}{T}\int_0^T y(s)ds,$$

and let K be the inverse of $L|_{\mathrm{Ker}P \cap D(L)}$. Obviously $\mathrm{Ker}L = \mathrm{Im}Q = R^2$.

$$(Ky)^{-1}(t) = \int_0^T k(t, s)y(s)ds, \tag{2.4}$$

where

$$k(t, s) = \begin{cases} \dfrac{s}{T}, & 0 \leq s < t \leq T, \\ \dfrac{s - T}{T}, & 0 \leq t \leq s \leq T. \end{cases}$$

From (2.3) and (2.4), one can easily see that N is L-compact on Ω, where Ω is an open bounded subset of X. The following lemma is useful to estimate a prior bounds of periodic solutions of (2.1).

Lemma 2.2. (Lu and Ge [11]) Let $s \in C(R, R)$ with $s(t+T) \equiv s(t)$. Suppose $\max_{t\in[0,T]} |s(t)| \le \gamma$ and $u \in C^1(R, R)$ with $u(t+T) \equiv u(t)$, then

$$\int_0^T |u(t) - u(t - s(t))|^n dt \le 2\gamma^n \int_0^T |u'(t)|^n dt.$$

At the end of this section, we list the basic assumptions which will be used in section 3.

[H$_1$] There is a constant $d > 0$ such that

$$x(g(x) - \frac{e(t)}{\beta(t)}) < 0, \forall x \in R, |x| > d.$$

[H$_2$] There exists an integer m such that $\delta := |\tau(t) - mT|_0 \le T$.

[H$_3$] There are constants $\sigma > 0$ and $n \ge 3$ such that $\forall u \in R, uf(u) \ge \sigma |u|^{n+1}$.

[H$_4$] There exist constants $r_1, r_2, r_3 \ge 0$ such that

$$|g(u) - g(v)| \le r_3|u - v|^3 + a_2(v)|u - v|^2 + a_1(v)|u - v|, \quad \forall u, v \in R,$$

where $r_1 = \lim_{|v|\to\infty} \frac{a_1(v)}{|v|^2}, r_2 = \lim_{|v|\to\infty} \frac{a_2(v)}{|v|}$.

3. Main result and the proof

In this section we state the main result and give its proof.

Theorem 3.1. Suppose the assumptions [H$_1$] $-$ [H$_4$] hold. Then (1.7) has at least one T-periodic solution provided one of the following conditions is satisfied:

(A$_1$) $n = 3$ and $\beta_0 2^{\frac{3}{n+1}} r_3 \delta^3 + \beta_0 2^{\frac{1}{n+1}} r_1 \delta T^{n-1} + \beta_0 2^{\frac{2}{n+1}} r_2 \delta^2 T^{n-2} < \sigma$,

(A$_2$) $n > 3$.

Proof. Consider the equation $Lx = \lambda N x$ for $\lambda \in (0, 1)$. Let $\Omega_1 = \{x \in X : Lx = \lambda N x, \lambda \in (0, 1)\}$, then

$$\begin{cases} x_1'(t) = \lambda \dfrac{x_2(t)}{\sqrt{1 - x_2^2(t)}} \triangleq \lambda \varphi(x_2(t)), \\ x_2'(t) = -\lambda f(\lambda \varphi(x_2(t))) - \lambda \beta(t) g(x_1(t - \tau(t))) + \lambda e(t). \end{cases} \tag{3.1}$$

Let $t_0 \in [0, T]$ be the point satisfying $x_1(t_0) = \max_{t\in[0,T]} x_1(t)$. Then $x_1'(t_0) = 0$, which together with the first equation of (3.1) lead to $x_2(t_0) = \varphi^{-1}(\frac{1}{\lambda}x_1'(t_0)) = 0$, for $\lambda \in (0, 1)$. We claim that

$$x_2'(t_0) \leq 0. \tag{3.2}$$

In fact, if $x_2'(t_0) > 0$, then there is a constant $\nu > 0$ such that $x_2'(t) > 0$ for $t \in [t_0, t_0+\nu]$, and $x_2(t) > x_2(t_0) = 0$ for $t \in [t_0, t_0 + \nu]$. Hence, $x_1'(t) = \lambda\varphi(x_2(t)) > 0, t \in [t_0, t_0 + \nu]$, i.e., $x_1(t) > x_1(t_0)$, for $t \in [t_0, t_0 + \nu]$, which contradicts the assumption of $x_1(t_0) = \max_{t\in[0,T]} x_1(t)$. Assumption [H$_3$] together with the continuity of $f(t)$, $f(0) = 0$ holds. From the second equation of (3.1) and $f(0) = 0$, we have

$$-\beta(t_0)g(x_1(t_0 - \tau(t_0))) + e(t_0) \leq 0.$$

From assumption [H$_1$] that

$$x_1(t_0 - \tau(t_0)) \leq d. \tag{3.3}$$

Similarly, if t_1 is the minimum point of $x_1(t)$ on $[0, T]$, we obtain

$$x_1(t_1 - \tau(t_1)) \geq -d. \tag{3.4}$$

It's easy to prove that there is a constant $\xi \in R$ such that $|x_1(\xi)| \leq d$. Since $\xi \in R$ is a constant, there must be an integer p and a point $t^* \in [0, T]$ such that $\xi = pT + t^*$. So $|x_1(t^*)| = |x_1(\xi)| \leq d$, which implies

$$|x_1|_0 \leq d + \int_0^T |x_1'(s)|ds. \tag{3.5}$$

On the other hand, multiplying both sides of the second equation of (3.1) by $x_1'(t)$ and integrating over $[0, T]$. From [H$_3$] we have

$$\begin{aligned} &\lambda \int_0^T |f(\frac{1}{\lambda}x_1'(t))x_1'(t)|dt \\ =\ &\lambda\left|\int_0^T f(\frac{1}{\lambda}x_1'(t))x_1'(t)dt\right| \\ \leq\ &\left|\int_0^T (\varphi^{-1}(\frac{1}{\lambda}x_1'(t)))'x_1'(t)dt\right| + \left|\int_0^T \beta(t)g(x_1(t-\tau(t)))dt\right| + \left|\int_0^T e(t)x_1'(t)dt\right|. \end{aligned} \tag{3.6}$$

Denote $\omega(t) = \varphi^{-1}(x_1'(t)/\lambda)$, then $\int_0^T (\varphi^{-1}(x_1'(t)/\lambda))'x_1'(t)dt = \lambda \int_0^T \varphi(\omega(t))d\omega(t) = 0$.

From condition [H$_3$], we get

$$\int_0^T |f(\frac{1}{\lambda}x_1'(t))x_1'(t)|dt \geq \frac{\sigma}{\lambda^n}\int_0^T |x_1'(t)|^{n+1}dt \geq \sigma \int_0^T |x_1'(t)|^{n+1}dt. \tag{3.7}$$

By (3.6), (3.7) and $\int_0^T g(x_1(t))x_1'(t)dt = 0$, we have

$$\begin{aligned} &\sigma \int_0^T |x_1'(t)|^{n+1}dt \\ \leq\ & |\int_0^T \beta(t)g(x_1(t-\tau(t)))x_1'(t)dt| + |\int_0^T e(t)x_1'(t)dt| \\ \leq\ & \beta_0|\int_0^T (g(x_1(t)) - g(x_1(t-\tau(t))))x_1'(t)dt| + |\int_0^T e(t)x_1'(t)dt|. \end{aligned} \tag{3.8}$$

Define

$$\varepsilon = \begin{cases} \dfrac{\sigma - \beta_0 2^{\frac{3}{n+1}} r_3\delta^3 - \beta_0 2^{\frac{1}{n+1}} r_1\delta T^{n-1} - \beta_0 2^{\frac{2}{n+1}} r_2\delta^2 T^{n-2}}{1 + 2(\beta_0 2^{\frac{1}{n+1}}\delta T^{n-1} + \beta_0 2^{\frac{2}{n+1}}\delta^2 T^{n-2})}, & \text{if } (A_1) \text{ holds.} \\ 1, & \text{if } (A_2) \text{ holds.} \end{cases} \tag{3.9}$$

Obviously, when (A_1) holds, ε is a positive constant independent of λ and

$$\beta_0 2^{\frac{3}{n+1}} r_3\delta^3 + \beta_0 2^{\frac{1}{n+1}}(r_1+\varepsilon)\delta T^{n-1} + \beta_0 2^{\frac{2}{n+1}}(r_2+\varepsilon)\delta^2 T^{n-2} < \sigma. \tag{3.10}$$

For such a constant $\varepsilon > 0$, it follows from assumption [H_4] that there is a constant $\rho > d$ such that

$$a_k(y) < (r_k+\varepsilon)|y|^{3-k}, \ \text{for } k = 1, 2\ ,\ y \in R \text{ with } |y| > \rho. \tag{3.11}$$

Let $E_1 = \{t : t \in [0, 1], |x_1(t-\tau(t))| \leq \rho\}$, $E_2 = \{t : t \in [0, 1], |x_1(t-\tau(t))| > \rho\}$, by (3.11) and [$H_4$], we obtain

$$\begin{aligned} & |\int_0^T (g(x_1(t)) - g(x_1(t-\tau(t))))x_1'(t)dt| \\ \leq\ & r_3\int_0^T |x_1(t) - x_1(t-\tau(t))|^3|x_1'(t)|dt \\ & + \sum_{k=1}^{2}\int_{E_1} |x_1(t) - x_1(t-\tau(t))|^k|a_k(x_1(t-\tau(t)))||x_1'(t)|dt \\ & + \sum_{k=1}^{2}\int_{E_2} |x_1(t) - x_1(t-\tau(t))|^k|a_k(x_1(t-\tau(t)))||x_1'(t)|dt \\ \leq\ & r_3\int_0^T |x_1(t) - x_1(t-\tau(t))|^3|x_1'(t)|dt \\ & + \sum_{k=1}^{2} a_{k,\rho}\int_0^T |x_1(t) - x_1(t-\tau(t))|^k|x_1'(t)|dt \\ & + \sum_{k=1}^{2}(r_k+\varepsilon)|x_1|_0^{3-k}\int_0^T |x_1'(t)||x_1(t) - x_1(t-\tau(t))|^k dt, \end{aligned} \tag{3.12}$$

where $a_{k,\rho} = \max_{|u|\le\rho} a_k(u),\ k = 1, 2$. By Hölder inequality and Lemma 2.2

$$\begin{aligned}
&\int_0^T |x_1(t) - x_1(t-\tau(t))|^k |x_1'(t)|dt \\
\le\ & (\int_0^T |x_1(t) - x_1(t-\tau(t)+mT)|^{n+1} dt)^{\frac{k}{n+1}} (\int_0^T |x_1'(t)|^{\frac{n+1}{n+1-k}} dt)^{\frac{n+1-k}{n+1}} \quad (3.13)\\
\le\ & 2^{\frac{k}{n+1}} T^{\frac{n-k}{n+1}} \delta^k (\int_0^T |x_1'(t)|^{n+1} dt)^{\frac{k+1}{n+1}} (k = 1, 2).
\end{aligned}$$

Substituting (3.13) into (3.12), we have

$$\begin{aligned}
&|\int_0^T (g(x_1(t)) - g(x_1(t-\tau(t))))x_1'(t)dt| \le 2^{\frac{3}{n+1}} r_3 \delta^3 T^{\frac{n-3}{n+1}} (\int_0^T |x_1'(t)|^{n+1} dt)^{\frac{4}{n+1}} \\
&+ \sum_{k=1}^{2} a_{k,\rho} 2^{\frac{k}{n+1}} \delta^k T^{\frac{n-k}{n+1}} (\int_0^T |x_1'(t)|^{n+1} dt)^{\frac{k+1}{n+1}} \quad (3.14)\\
&+ \sum_{k=1}^{2} (r_k + \varepsilon) \delta^k 2^{\frac{k}{n+1}} T^{\frac{n-k}{n+1}} |x_1|_0^{3-k} (\int_0^T |x_1'(t)|^{n+1} dt)^{\frac{k+1}{n+1}}.
\end{aligned}$$

In view of $f(0) = 0$, (3.1) and $e(t) \not\equiv$ constant, $\int_0^T |x_1'(s)|ds > 0$ holds. By the knowledge of mathematical inequalities, there is a constant $h > 0$ such that

$$(1 + x)^3 \le 1 + 4x, \quad x \in (0, h]. \quad (3.15)$$

Next we will prove if one of the conditions of (A_1) and (A_2) holds, there must exist a constant $M_1 > 0$ independent of λ such that $|x_1|_0 \le M_1$.

If $\frac{d}{\int_0^T |x'(s)|ds} > h$, then $\int_0^T |x'(s)|ds < \frac{d}{h}$, from (3.5) we have

$$|x_1|_0 \le d + \frac{d}{h}. \quad (3.16)$$

If $\frac{d}{\int_0^T |x'(s)|ds} \le h$, then from (3.5), (3.15) and Hölder inequality,

$$\begin{aligned}
|x_1|_0^{3-k} \le\ & (d + \int_0^T |x'(t)|dt)^{3-k} = (\int_0^T |x_1'(t)|dt)^{3-k} (1 + \frac{d}{\int_0^T |x_1'(t)|dt})^{3-k} \\
\le\ & T^{\frac{3n-nk}{n+1}} (\int_0^T |x_1'(t)|^{n+1} dt)^{\frac{3-k}{n+1}} + 4dT^{\frac{n(2-k)}{n+1}} (\int_0^T |x_1'(t)|^{n+1} dt)^{\frac{2-k}{n+1}}. \quad (3.17)
\end{aligned}$$

So

$$\begin{aligned} & T^{\frac{n-k}{n+1}}|x_1|_0^{3-k}(\int_0^T |x_1'(t)|^{n+1}dt)^{\frac{k+1}{n+1}} \\ \leq \ & T^{\frac{4n-k-nk}{n+1}}(\int_0^T |x_1'(t)|^{n+1}dt)^{\frac{4}{n+1}} + 4dT^{\frac{3n-nk-k}{n+1}}(\int_0^T |x_1'(t)|^{n+1}dt)^{\frac{3}{n+1}}. \quad (3.18) \end{aligned}$$

Substituting (3.18) into (3.14), we get

$$\begin{aligned} & |\int_0^T (g(x_1(t)) - g(x_1(t-\tau(t))))x_1'(t)dt| \\ \leq \ & (2^{\frac{3}{n+1}}r_3\delta^3T^{\frac{n-3}{n+1}} + \sum_{k=1}^{2}(r_k+\varepsilon)2^{\frac{k}{n+1}}\delta^kT^{\frac{n-k+3n-nk}{n+1}})(\int_0^T |x_1'(t)|^{n+1}dt)^{\frac{4}{n+1}} \quad (3.19) \\ & +(\sum_{k=1}^{2}4d(r_k+\varepsilon)2^{\frac{k}{n+1}}\delta^kT^{\frac{3n-nk-k}{n+1}} + a_{2,\rho}2^{\frac{2}{n+1}}\delta^2T^{\frac{n-2}{n+1}})(\int_0^T |x_1'(t)|^{n+1}dt)^{\frac{3}{n+1}} \\ & +a_{1,\rho}2^{\frac{1}{n+1}}\delta T^{\frac{n-1}{n+1}}(\int_0^T |x_1'(t)|^{n+1}dt)^{\frac{2}{n+1}}. \end{aligned}$$

Substituting (3.19) into (3.8), we have

$$\begin{aligned} & \sigma\int_0^T |x_1'(t)|^{n+1}dt \\ \leq \ & \beta_0(2^{\frac{3}{n+1}}r_3\delta^3T^{\frac{n-3}{n+1}} + \sum_{k=1}^{2}(r_k+\varepsilon)2^{\frac{k}{n+1}}\delta^kT^{\frac{n-k+3n-nk}{n+1}})(\int_0^T |x_1'(t)|^{n+1}dt)^{\frac{4}{n+1}} \\ + \ & \beta_0(a_{2,\rho}2^{\frac{2}{n+1}}\delta^2T^{\frac{n-2}{n+1}} + \sum_{k=1}^{2}4d(r_k+\varepsilon)2^{\frac{k}{n+1}}\delta^kT^{\frac{3n-nk-k}{n+1}})(\int_0^T |x_1'(t)|^{n+1}dt)^{\frac{3}{n+1}} \\ + \ & \beta_0a_{1,\rho}2^{\frac{1}{n+1}}\delta T^{\frac{n-1}{n+1}}(\int_0^T |x_1'(t)|^{n+1}dt)^{\frac{2}{n+1}} \\ + \ & (\int_0^T |e(t)|^{\frac{n+1}{n}}dt)^{\frac{n}{n+1}}(\int_0^T |x_1'(t)|^{n+1}dt)^{\frac{1}{n+1}}. \quad (3.20) \end{aligned}$$

If (A_1) holds, i.e., $n = 3$, $\frac{4}{n+1} = 1$, from (3.20), we have

$$\begin{aligned}
&(\sigma - \beta_0(2^{\frac{3}{n+1}} r_3 \delta^3 T^{\frac{n-3}{n+1}} + \sum_{k=1}^{2} (r_k + \varepsilon) 2^{\frac{k}{n+1}} \delta^k T^{\frac{n-k+3n-nk}{n+1}})) \int_0^T |x_1'(t)|^{n+1} dt \\
\leq\ & \beta_0 (a_{2,\rho} 2^{\frac{2}{n+1}} \delta^2 T^{\frac{n-2}{n+1}} + \sum_{k=1}^{2} 4d(r_k + \varepsilon) 2^{\frac{k}{n+1}} \delta^k T^{\frac{3n-nk-k}{n+1}})(\int_0^T |x_1'(t)|^{n+1} dt)^{\frac{3}{n+1}} \\
+\ & \beta_0 a_{1,\rho} 2^{\frac{1}{n+1}} \delta T^{\frac{n-1}{n+1}} (\int_0^T |x_1'(t)|^{n+1} dt)^{\frac{2}{n+1}} \\
+\ & (\int_0^T |e(t)|^{\frac{n+1}{n}} dt)^{\frac{n}{n+1}} (\int_0^T |x_1'(t)|^{n+1} dt)^{\frac{1}{n+1}}. \qquad (3.21)
\end{aligned}$$

By (3.10),

$$\sigma - \beta_0(2^{\frac{3}{n+1}} r_3 \delta^3 T^{\frac{n-3}{n+1}} + \sum_{k=1}^{2} (r_k + \varepsilon) 2^{\frac{k}{n+1}} \delta^k T^{\frac{n-k+3n-nk}{n+1}}) > 0.$$

Take notice of now $\frac{3}{n+1} < 1$, $\frac{2}{n+1} < 1$, $\frac{1}{n+1} < 1$, by (3.21), there exists a constant R_0 independent of λ such that $\int_0^T |x_1'(t)|^{n+1} dt < R_0$. Then from (3.5) we have

$$|x_1|_0 \leq d + T^{\frac{n}{n+1}} (R_0)^{\frac{1}{n+1}}. \qquad (3.22)$$

If (A_2) holds, i.e., $n > 3$, in view of $\frac{k}{n+1} < 1$, $k = 1, 2, 3, 4$, by (3.20), it is easy to see that there exists a constant R_1 independent of λ such that $\int_0^T |x_1'(t)|^{n+1} dt < R_1$, i.e.,

$$|x_1|_0 \leq d + T^{\frac{n}{n+1}} (R_1)^{\frac{1}{n+1}}. \qquad (3.23)$$

Let

$$M_1 = \max\{d + T^{\frac{n}{n+1}} (R_0)^{\frac{1}{n+1}}, d + T^{\frac{n}{n+1}} (R_1)^{\frac{1}{n+1}}, d + d/h\}.$$

Then it follows from (3.16), (3.22) and (3.23) that in either (A_1) or (A_2), $|x_1|_0 \leq M_1$ always holds.

On the other hand, by the first equation of (3.1), we have $\lambda \int_0^T \varphi(x_2(t)) dt = \int_0^T x_1'(s) ds = 0$, which implies that there is a constant η such that $|x_2(\eta)| = 0$. So we have

$$|x_2(t)| \leq \int_\eta^T |x_2'(s)| ds \leq (\int_0^T |x_2'(s)|^2 ds)^{\frac{1}{2}} T^{\frac{1}{2}}. \qquad (3.24)$$

By the second equation of (3.1), we have

$$\begin{aligned}\int_0^T |x_2'(s)|^2 ds &= |-\lambda\int_0^T \beta(t)g(x_1(t-\tau(t)))x_2'(t)dt + \lambda\int_0^T e(t)x_2'(t)dt| \\ &\le (\beta_0 g_{M_1} T^{\frac{1}{2}} + (\int_0^T |e(t)|^2 dt)^{\frac{1}{2}})(\int_0^T |x_2'(t)|^2 dt)^{\frac{1}{2}},\end{aligned}$$

where $g_{M_1} = \max\limits_{|u|\le M_1} |g(u)|$. Then

$$\int_0^T |x_2'(s)|^2 ds \le (\beta_0 g_{M_1} T^{\frac{1}{2}} + (\int_0^T |e(t)|^2 dt)^{\frac{1}{2}})^2. \tag{3.25}$$

Substituting (3.25) into (3.24), we have

$$|x_2|_0 < d + T^{\frac{1}{2}}(\beta_0 g_{M_1} T^{\frac{1}{2}} + (\int_0^T |e(t)|^2 dt)^{\frac{1}{2}}) := M_2. \tag{3.26}$$

Let $\Omega_2 = \{x : x \in \mathrm{Ker}L, QNx = 0\}$. If $x \in \Omega_2$, then $x \in \ker L$ and $QNx = 0$. It's easy to obtain that $|x_1(t)| \le d,\ x_2(t) = 0 \le d$.

Let $\Omega = \{x : x = (x_1, x_2)^\top \in X, |x_1|_0 < M_1, |x_2|_0 < M_2\}$. Then $\Omega \supset (\Omega_1 \cup \Omega_2)$ and Ω is a bounded open set of X. So (1) and (2) of Lemma 2.1 are satisfied. Next we show that condition (3) of Lemma 2.1 holds. ■

Define a linear isomorphism $J : \mathrm{Im}Q \to \mathrm{Ker}L$ by $J(x_1, x_2) = (x_1, x_2)$, $\Delta_\varepsilon = \{x : x = (x_1, x_2) \in R^2 : |x_1| < M_1, |x_2| < \varepsilon\}$. It's easy to see that there is a sufficiently small $\varepsilon_0 > 0$ such that the equation $QNx = (0, 0)$, i.e.,

$$\begin{cases} \dfrac{x_2(t)}{\sqrt{1 - x_2^2(t)}} = 0 \\ -f(\varphi(x_2(t))) - g(x_1)\dfrac{1}{T}\displaystyle\int_0^T \beta(t)dt + \dfrac{1}{T}\displaystyle\int_0^T e(t)dt = 0, \end{cases} \tag{3.27}$$

has no solution in $\overline{(\Omega \cap \ker L)/\Delta_\varepsilon}$, where $\varepsilon \in (0, \varepsilon_0)$ is an arbitrary constant. So

$$\deg\{JQN, \Omega \cap KerL, 0\} = \deg\{JQN, \Delta_\varepsilon, 0\}.$$

Let

$$QN_0 = \begin{pmatrix} 0 \\ \dfrac{1}{T}\displaystyle\int_0^T (-g(x_1)\beta(t) + e(t))dt \end{pmatrix}.$$

If $x \in \partial\Delta_\varepsilon$, then

$$\| JQN(x) - JQN_0(x) \| \le \max_{|x_2|\le\varepsilon}\{|f(\varphi(x_2))| + \varphi(x_2)\}.$$

By the continuity of f and the fact $f(0) = 0$, $\| JQN(x) - JQN_0(x) \| \to 0$ as $\varepsilon \to 0$. So if $\varepsilon > 0$ is sufficiently small,

$$\deg\{JQN, \Delta_\varepsilon, 0\} = \deg\{JQN_0, \Delta_\varepsilon, 0\}.$$

In view of $\dim QN_0 = 1$, it follows that

$$\deg\{JQN_0, \Delta_\varepsilon, 0\} = \deg\{JQN_0, \Delta_0, 0\},$$

where $\Delta_0 = \{x : x \in R, |x| < M_0\} \subset R$. By assumption [$H_1$], we have

$$\deg\{JQN, \Omega \cap \mathrm{Ker}L, 0\} = \deg\{JQN_0, \Delta_0, 0\} \neq 0.$$

Therefore, by Lemma 2.1, we see that the equation $Lx = Nx$ has a solution.

If $g \in C^3(R, R)$, $|\frac{g'''(x)}{3!}| \leq r_3, \forall x \in R$, and

$$\lim_{|x|\to\infty} \frac{g^{(k)}(x)}{k!|x|^{3-k}} \leq r_k (k = 1, 2),$$

where $r_1, r_2 \geq 0$ are constants. Then for $\forall x, y \in R$, we have

$$\begin{aligned} |g(x) - g(y)| &= |g'(y)(x-y) + \frac{g''(y)}{2!}(x-y)^2 + \frac{g'''(\xi)}{3!}(x-y)^3| \\ &\leq r_3|x-y|^3 + r_1|y|^2|x-y| + r_2|y||x-y|^2. \end{aligned} \tag{3.28}$$

where ξ is between x and y.

Corollary 3.2. Assume [H_1] − [H_3] and (3.28) hold. Then (1.7) has at least one T-periodic solution if one of the following conditions holds:

(A_1) $n = 3$ and $\sigma - \beta_0 2^{\frac{3}{n+1}} r_3 \delta^3 - \beta_0 2^{\frac{1}{n+1}} r_1 \delta T^{n-1} - \beta_0 2^{\frac{2}{n+1}} r_2 \delta^2 T^{n-2} > 0$;

(A_2) $n > 3$.

4. An example

We give an example to illustrate the application of our result.

Example 4.1. Suppose $f(x) = bx^s$, $g(x) = a_0x^3 + a_1x^2 + a_2x + a_3$, a_0, a_1, a_2, a_3, b are constants, $b \neq 0$, $a_0 \neq 0$, $n = s$, b is odd and $s \geq 3$. Consider the prescribed mean curvature Rayleigh equation with a deviating argument:

$$\left(\frac{x'(t)}{\sqrt{1 + x'^2(t)}}\right)' + f(x'(t)) + (2 - \sin t)g(x(t - \frac{1}{100}\sin t)) = \sin t - 2, \tag{4.1}$$

where $\beta(t) = 2 - \sin t$, $\beta_0 = 3$, $e(t) = \sin t - 2$, $\tau(t) = \dfrac{1}{100}\sin t$.

Corresponding to equation (4.1), it is clear that $[H_1] - [H_3]$ hold with $\sigma = |b|$, $\delta = \dfrac{1}{100}$. Meanwhile, $|\dfrac{g'''(x)}{3!}| \leq |a_0|$, $\lim\limits_{|x|\to\infty} \dfrac{g^{(k)}(x)}{k!|x|^{3-k}} = |a_0|C_3^k$. By Corollary 3.1, (4.1) has at least one 2π-periodic solution if

(A$_1$) $s = 3$ and $3(2^{\frac{s}{s+1}}(1/100)^s|a_0| + \sum\limits_{k=1}^{2} 2^{\frac{k}{1+s}}|a_0|C_s^k(1/100)^k(2\pi)^{3-k}) < |b|$; or

(A$_2$) $s > 3$ holds.

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