Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 2 (2016), pp. 1283-1292

http://www.ripublication.com

Numerical solution of one dimensional contaminant transport equation with variable coefficient (temporal) by using Haar wavelet

A. C. Patel and V. H. Pradhan
Department of Applied Mathematics and Humanities,
S. V. N. I. T, Surat -395007, Gujarat, India.

Abstract

In the present paper Haar wavelet method is implemented on advection-dispersion equation representing one dimensional contaminant transport through a porous medium. Non uniform flow is considered by assuming velocity and dispersion varying with time as an exponentially increasing function. Expressing the Haar wavelets in advection-dispersion equation into Haar series provides the main advantage in the existing method where the simplicity of the Haar wavelet is preserved. The obtained numerical results are compared with the exact solution of advection-dispersion equation with constants coefficients as there are very few analytical solutions with the variable coefficients. The computations are carried out with the aid of the MatLab program. It is concluded that Haar wavelet method is easy, efficient and convenient.

Keywords: Advection-dispersion equation, Haar wavelet, temporally varying coefficients, Peclet number.

Introduction

During the past several years the study was more emphasized towards water supply problems and water supply potential of aquifers. However the studies are now emphasized more in water quality problems in the past 20 years. Consequently the need arises to study the contaminant transport through the subsurface environment. In the earlier study the developing methods where focused to analyze aquifers of high permeability. However the studies are now focused largely on the reactive and non-reactive solute transport in adsorbing and non-adsorbing porous media. An extensive literature is available on the study of transport and dispersion processes with various kinds of contaminants. The deterministic mathematical models used for most of the groundwater models and its computer simulation has gained a technological growth in the solute transport in groundwater system. These types of models are generally governed by the partial differential equations (PDE) based on basic principles and laws. The contaminants being physically, chemically or biologically active the transport of contaminants becomes a subject of great importance in engineering and science. Usually advection-dispersion equation (ADE) describes the contaminant transport through a medium and is a parabolic PDE. This type of equation is broadly classified as ADE with constant and variable coefficients. The velocity and dispersivity are assumed to be constant in the ADE with constant coefficient, whereas velocity and dispersivity vary with either space or time in ADE with variable coefficient.

An extensive literature is available with the exact solutions of ADEin one dimension with constant coefficients [6]. There are very few analytical solutions with the variable coefficients [1-2] and consequently the numerical solutions are compared with the exact solutions of ADE with constant coefficients. The analytical solutions in general are restricted with specific boundary conditions and may not have much practical importance or may be complex. As a result the numerical solutions can be widely used to solve the ADE with non-uniform flow with respect todifferent types of initial and boundary conditions. In the present paper a numerical solution is obtained for the ADE with non-uniform velocity fields and variable dispersion coefficients using Haar wavelet method. Dirichlet boundary conditions at the inflow and out flow ends of the flow system and initial condition in the Gaussian form is considered for various cases.

Model Formulation

We consider the transport of a contaminant through a homogeneous finite aquifer of length L under transient-state flow. Initially it is assumed that the domain is not clean and may contain some contamination. The initial contamination may be assumed to be non-zero constant or in Gaussian form. The inflow and out-flow conditions of the flow system are assumed to be time-dependent. Let $c(x,t)$ be the concentration of contaminants in the aquifer at position x and time t. $v(x,t)$ and $D(x,t)$ are the velocity of the medium transporting the contaminants and the solute dispersion parameter respectively. $D(x,t)$ is assumed to be constant if it is independent of position and time and known as dispersion coefficient. Then the problem with first-order decay can be mathematically formulated as follows:

$$R\frac{\partial c}{\partial t}=\frac{\partial}{\partial x}\left(D(x,t)\frac{\partial c}{\partial x}-v(x,t)c\right)-R\sigma c,\quad 0\le x\le L, 0\le t\le t_0 \tag{1}$$

$$D(t)=D_0V(t)\ ,\ v(t)=v_0V(t) \tag{2}$$

In equation (1) $R = 1 + \frac{\rho_d k_d}{n}$ is the retardation factor. k_d is distribution coefficient. n is the porosity, ρ_d is the density and σ is the decay constant. In equation (2) D_0 is the initial dispersion coefficient and v_0 is the initial velocity.

Here, we made following assumptions:

a. Fluid is of constant density and viscosity.
b. Solute is subject to first-order nonreactive transformation.
c. No adsorption, $k_d = 0$.

Based on the above assumption, equation (1) reduces to

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(D(x,t)\frac{\partial^2 c}{\partial x^2} - v(x,t)c \right), \quad 0 \le x \le L, 0 \le t \le t_0 \tag{3}$$

Here we considered $V(t) = e^{mt}$ where, m is the flow resistance coefficient.

Initial and Boundary Conditions

We assume that there is some contamination in the flow system initially. Thus in the initial condition the initial concentration of the contaminant can be assumed with non-zero constant value or function of space variable. In the present paper we assume that the initial contaminant profile took on a Gaussian shape. We assume

$$c(x,0) = f(x), 0 \le x \le L \tag{4}$$

To solve equation (3)completely different boundary conditions are associated with it. In general Dirichlet, Neumann and Cauchy boundary conditions can be applied at the inflow and out flow ends of the flow system. The above three conditions are also known as the boundary condition respectively. The value of the concentration is represented by first-type, whereas the gradient and flux are represented by second and third type boundary condition respectively. In the existing paper the first-type boundary condition is implemented at the inflow and out flow of the flow system as the time dependent functions of sufficient smoothness as follows

$$c(0,\text{t}) = g_1(t) \qquad 0 \le t \le t_0 \tag{5}$$

$$c(L,t) = g_2(t) \tag{6}$$

where g_1 and g_2 are the known sufficient smoothness.

Haar Wavelet

A system of square wave is known as Haar wavelet [3, 4, 5] $h_i(X)$ which is orthogonal and orthonormal. In this system $h_0(X)$ is a scaling function, $h_1(X)$ is known as mother wavelet and all other wavelet transform are given by

$h_n(X) = h_1(2^j X - k)$, $n = 2^j + k$, $j \ge 0$, $0 \le k < 2^j$

The Haar wavelet family for $X \in [0,1]$ is defined as follows:

$$h_i(X) = \begin{cases} 1, & X \in \left[\frac{k}{m}, \frac{k+0.5}{m}\right), \\ -1, & X \in \left[\frac{k+0.5}{m}, \frac{k+1}{m}\right), \\ 0, & elsewhere. \end{cases} \tag{7}$$

Here, $m = 2^j$, $j = 0,1,.....,J$(Maximum level of resolution), $k = 0,1,....,m-1$ are known as level of wavelets and translation parameter respectively. In equation(7), the index i is given by $i = m + k + 1$where the minimal value is $i = 2$ and the maximum values $i = 2M = 2^{J+1}$.

Let $$P_i(X) = \int_0^X h_i(X)\,dX, \quad Q_i(X) = \int_0^X P_i(X)\,dX \tag{8}$$

We consider the collocation points given by $X_l = (l - 0.5)/2M$ where $l = 1,2,.....,2M$, through which we get the coefficient matrices H, P, Q with $2M \times 2M$ matrices.
Equation (9) represents the matrix equation for calculating the matrix P of order m [13]

$$P_{(m)} = \frac{1}{2M}\begin{bmatrix} 2MP_{(m/2)} & H_{(m/2)} \\ H^{-1}_{(m/2)} & O \end{bmatrix} \tag{9}$$

in which O is a null matrix of order $\frac{m}{2} \times \frac{m}{2}$,

$$H_{m\times m} \square \left[h_m(x_0), h_m(x_1), ..., h_m(x_{m-1})\right] \tag{10}$$

and $\frac{i}{m} \le x < i + \frac{1}{m}$ and $H^{-1}_{m\times m} = \frac{1}{m} H^T_{m\times m} diag(\mathrm{r})$

Once $P_{(m)}$ and $H_{(m)}$are calculated the same can be used for solving various differential equations.

Method of Solution

Case I: We consider $D = D_0 e^{mt}$ (varying temporally), assuming v constant then equation (3) reduces to

$$\frac{\partial C}{\partial t} = D_0 e^{mt} \frac{\partial^2 C}{\partial x^2} - v_0 \frac{\partial C}{\partial x} \tag{11}$$

Using the dimensionless variable $X = \frac{x}{L}$ and $T = \frac{v_0 t}{L}$ equation(11) becomes

$$\frac{\partial C}{\partial T}=\frac{e^{m'T}}{Pe}\frac{\partial^2 C}{\partial X^2}-\frac{\partial C}{\partial X}\ ,0\le \mathrm{X}\le 1,0\le \mathrm{T}\le \mathrm{T}_0 \tag{12}$$

where $Pe=\frac{v_0 L}{D_0}$ is the Peclet number and $m'=\frac{mL}{v_0}$ is dimensionless constant.

with initial and boundary condition

$$C(X,0)=e^{-\frac{(X-2)^2}{8}} \tag{13}$$

$$C(0,T)=\sqrt{\frac{20}{20+T}}\ e^{-\frac{2(5+2T)^2}{5(T+20)}} \tag{14}$$

$$C(1,T)=\sqrt{\frac{20}{20+T}}\ e^{-\frac{(5+4T)^2}{10(T+20)}} \tag{15}$$

Dividing the time interval into N equal parts, $\dot{C}''(X,T)$ in terms of Haar wavelets is expressed as

$$\dot{C}''(X,T)=\sum_{i=1}^{2M} a_i h_i(X) \tag{16}$$

where($\cdot$) and (') represents differentiation with respect to time and space variable respectively, $a^T_{(m)}$ is the row vector in the subinterval $T\in[T_s,T_{s+1}]$ is constant. In equation(16), T and X are integrated from T_s to T and 0 to X respectively, the variable $C''(X,T)$, $C'(X,T)$ $C(X,T)$ and $\dot{C}(X,T)$ can be successively obtained.

$$C''(X,T)=(T-T_s)\sum_{i=1}^{2M} a_i h_i(X)+C''(X,T_s) \tag{17}$$

$$C'(X,T)=(T-T_s)\sum_{i=1}^{2M} a_i P_i(X)+C'(X,T_s)-C'(0,T_s)+C'(0,T) \tag{18}$$

$$\begin{aligned}C(X,T)=&(T-T_s)\sum_{i=1}^{2M} a_i Q_i(X)+C(X,T_s)-C(0,T_s)+C(0,T)\\&+X\left[C'(0,T)-C'(0,T_s)\right]\end{aligned} \tag{19}$$

$$\dot{C}(X,T)=\sum_{i=1}^{2M} a_i Q_i(X)+\dot{C}(0,T)+X\left[\dot{C}'(0,T)\right] \tag{20}$$

Setting $X=1$ in equation (19)and(20), we have

$$C'(0,T)-C'(0,T_s)=-(T-T_s)\sum_{i=1}^{2M} a_i Q_i(1)+u(1,T)-C(1,T_s)-C(0,T)+C(0,T_s) \tag{21}$$

$$\dot{C}'(0,T)=-\sum_{i=1}^{2M} a_i Q_i(1)-\dot{C}(0,T)+\dot{C}(1,T) \tag{22}$$

Substituting equation (21)and (22)into equation (17)-(20)and discretizing X to X_l and T to T_{s+1} we get

$$\sum_{i=1}^{2M} a_i Q_i\left(X_l\right)+\dot{C}\left(0,T_{s+1}\right)+X_l\left[-\sum_{i=1}^{2M} a_i Q_i\left(1\right)+\dot{C}\left(1,T_{s+1}\right)-\dot{C}\left(0,T_{s+1}\right)\right]$$
$$=\frac{e^{m'T_{s+1}}}{Pe}C''\left(X_l,T_{s+1}\right)-C'\left(X_l,T_{s+1}\right) \quad (23)$$

Using equation(23)theHaar coefficients $a_{(2M)}^{T}$ can be successively obtained.

Case II: We consider $v=v_0e^{mt}$ (varying temporally), assuming D constant in equation (3)we get

$$\frac{\partial C}{\partial t}=D_0\frac{\partial^2 C}{\partial x^2}-v_0e^{mt}\frac{\partial C}{\partial x} \quad (24)$$

Using the dimensionless variable $X=\frac{x}{L}$ and $T=\frac{v_0 t}{L}$ equation(24) reduces to

$$\frac{\partial C}{\partial T}=\frac{1}{Pe}\frac{\partial^2 C}{\partial X^2}-e^{m'T}\frac{\partial C}{\partial X} \qquad 0\le \mathrm{X}\le 1, 0\le \mathrm{T}\le \mathrm{T}_0 \quad (25)$$

where $Pe=\frac{v_0L}{D_0}$ is the Peclet number and $m'=\frac{mL}{v_0}$ is dimensionless constant.

Using initial condition(13)and boundary conditions (14)-(15)and applying Haar wavelet method as discussed in case I to equation(25) we get

$$\sum_{i=1}^{2M} a_i Q_i\left(X_l\right)+\dot{C}\left(0,T_{s+1}\right)+X_l\left[-\sum_{i=1}^{2M} a_i Q_i\left(1\right)+\dot{C}\left(1,T_{s+1}\right)-\dot{C}\left(0,T_{s+1}\right)\right]$$
$$=\frac{1}{Pe}C''\left(X_l,T_{s+1}\right)-e^{m'T_{s+1}}C'\left(X_l,T_{s+1}\right) \quad (26)$$

Using equation(26)the Haar coefficients $a_{(2M)}^{T}$ can be successively obatined.

Case III: We consider, $D=D_0e^{mt}$ and $v=v_0e^{mt}$ (both varying temporally) in equation(3) we get

$$\frac{\partial C}{\partial t}=D_0e^{mt}\frac{\partial^2 C}{\partial x^2}-v_0e^{mt}\frac{\partial C}{\partial x} \quad (27)$$

Using the dimensionless variable $X=\frac{x}{L}$ and $T=\frac{v_0 t}{L}$ equation(27) reduces to

$$\frac{\partial C}{\partial T}=\frac{e^{m'T}}{Pe}\frac{\partial^2 C}{\partial X^2}-e^{m'T}\frac{\partial C}{\partial X} \qquad 0\le \mathrm{X}\le 1, 0\le \mathrm{T}\le \mathrm{T}_0 \quad (28)$$

where $Pe=\frac{v_0L}{D_0}$ is the Peclet number and $m'=\frac{mL}{v_0}$ is dimensionless constant.

Using initial condition (13) and boundary conditions (14)-(15) and similarly applying Haar wavelet method to equation(28) we get

$$\sum_{i=1}^{2M} a_i Q_i\left(X_l\right)+\dot{C}\left(0,T_{s+1}\right)+X_l\left[-\sum_{i=1}^{2M} a_i Q_i\left(1\right)+\dot{C}\left(1,T_{s+1}\right)-\dot{C}\left(0,T_{s+1}\right)\right]$$
$$=\frac{e^{m'T_{s+1}}}{Pe}C''\left(X_l,T_{s+1}\right)-e^{m'T_{s+1}}C'\left(X_l,T_{s+1}\right) \quad (29)$$

Using equation (29) the Haar coefficients $a_{(2M)}^{T}$ can be successively obtained.

Results and Discussion

The numerical results of equation (3)with respect to initial and boundary conditions (13)-(15) are obtained using equation (23) for case (I), equation (26) for case (II) and equation (29) for case (III) using values $D_0 = 0.1$, $v_0 = 0.8$ and $m = 0.1$. The corresponding graphical representations are shown in Figure (1), (2) and(3). Figures reveals that the contaminant concentration decreases with the time and increases with the space variable. This shows that the contamination concentration which is distributed initially in the Gaussian form will vanish after sufficient time with respect to considered time varying boundary conditions at the inflow and out-flow of the flow system. The table (1) depicts a comparative study of ADE with constant coefficient and variable coefficients. Three cases are considered for non-uniform flow. In case (I) dispersivity is varying with time keeping velocity constant, in case (II) velocity is varying with time keeping dispersivity constant and in case (III) the velocity and dispersivity are varying with time. It is found that the contaminant concentration is less case (II) and (III) in comparison to advection dispersion with constant coefficient. However the contaminant concentration is found more in case (I) for exponentially increasing dispersion in uniform flow.

Table 1 Comparison of one dimensional contaminant transport equation with constant coefficients and variable coefficients (Temporal) at *T =0. 5(P =8).*

	Constant coefficient [6]	***Temporal*** $D = D_0 e^{m'T}$ ***and*** v ***constant***	**Temporal** D **constant and** $v = v_0 e^{m'T}$	**Temporal** D **and** v **both variable**
X = 0. 1	0. 51339	0. 518061	0. 51063	0. 510692
X = 0. 2	0. 54516	0. 552027	0. 542131	0. 542256
X= 0. 3	0. 57678	0. 585807	0. 573379	0. 573571
X = 0. 4	0. 60809	0. 619211	0. 604212	0. 604471
X = 0. 5	0. 63892	0. 652047	0. 634459	0. 634787
X = 0. 6	0. 66909	0. 684116	0. 663949	0. 664345
X = 0. 7	0. 69844	0. 71522	0. 692509	0. 692973
X = 0. 8	0. 72679	0. 745159	0. 719968	0. 720498
X = 0. 9	0. 75396	0. 773737	0. 746155	0. 74675

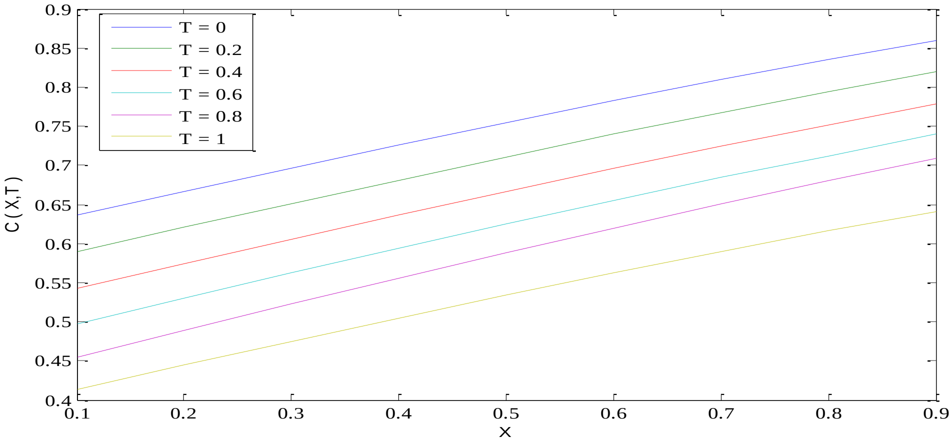

Figure 1: Dimensionless contaminant concentration profilesfor various values of *T* with dispersivity varying with time and velocity as constant

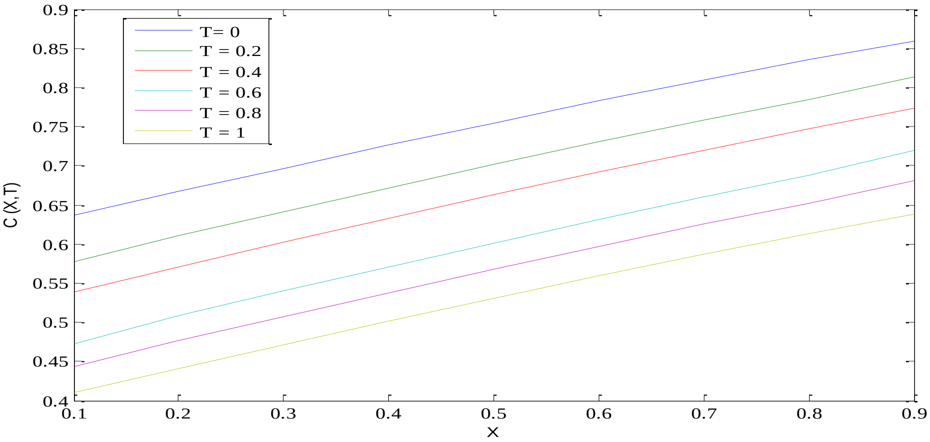

Figure 2: Dimensionless contaminant concentration profiles for various values of *T* with velocity varying with time and dispersivity as constant

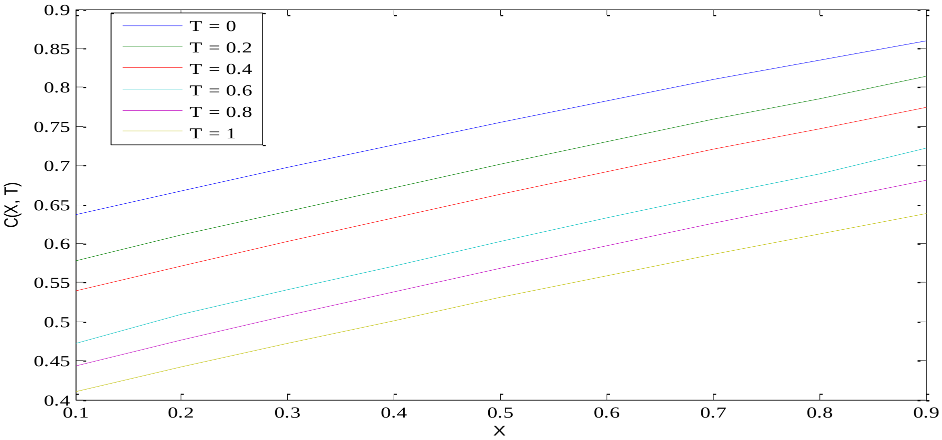

Figure 3: Dimensionless contaminant concentration profile for various values of *T* with dispersivity and velocity both varying with time

List of symbols

R retardation factor
x position
t time
D_0 initial dispersion coefficient
v_0 initial velocity
k_d distribution coefficient
n porosity
σ decay constant
P_e Peclet number
m flow resistance coefficient
m level of wavelets
k translation parameter
J maximum level of resolution
L length
t_0 time duration
X dimensionless space variable
T dimensionless time variable
T_0 dimensionless time duration
m' constant
a_{2M}^T Haar coefficients

Conclusion

In the present paper the ADE is numerically discussed using Haar wavelet method for non-uniform flow. Three cases are considered to observe the effect of dispersion (*D*) and velocity(*v*) varying with time in comparison to ADE with constant coefficients. Exponentially increasing function is considered for *D* and *v* varying with time. From the obtained results it is found that when *D* is varying with time keeping *v* constant, the contaminant concentration will be more in comparison to contaminant concentration for the ADE with constant coefficients. Thus the method is conveniently applied in all the three cases to observe the corresponding effects. It is concluded that the method is easy, efficient and accurate. Since the Haar wavelets are orthonormal the L^2 convergence in (16)is unconditional and the method is always stable [7], the method can be equally appiled to other partial differntial eqautions with various types of initial and boundary conditions.

References

[1] Zoppou C. and Knight J. H. (1999), Analytical solution of a spatially variable coefficient advection-diffusion equation in up to three dimensions, Applied Mathematicalmodeling, 23, 667-685.

[2] Jaiswal D. K., Kumar A., Yadav R. R. (2011), Analytical solution to the one-dimensional advection-diffusion equation with temporally dependent coefficients, Journal of water resources and protection, 3, 76-84.

[3] Chen C. F. and Hsiao C. H. (1997), Haar wavelet method for solving lumped and distributed parameter systems, IEEE Proceeding: Part D, 144, 87-94.

[4] HariharanG. et. al. (2010), Haar wavelet solution for a few reaction diffusion problems, Ph. D thesis, Sastra university, Thanjavur, Tamil Nadu.

[5] LepikU., Hein H. (2014), Haar wavelets with applications, Springer, Switzerland.

[6] Shi Z., Deng Li-Y., Chen Q.-J. (2007), Numerical solution of differential equations by using Haar wavelets, Proc. Int. Conf. Wavelet Ana. An Pattern Recog., 1039-1044.

[7] XuM. (2006), Function approximation methods for optimal control problems, Ph. D. thesis, Saint Louis, Missouri.